Projective Geometry

Fall 2023 - R. L. Herman

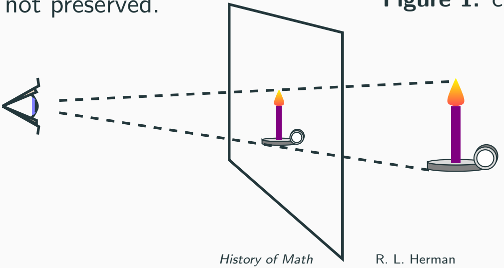
Perspective Drawing

- Art - Perspective Drawing
- Before Renaissance- no illusion of depth and space.
- 13th century Italian masters used shadowing.
- Mathematics of perspective

  Lengths not preserved.

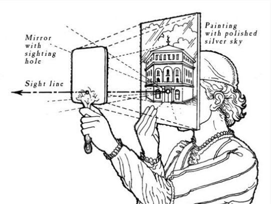
  Angles not preserved.

Figure 1: c.1308-1311



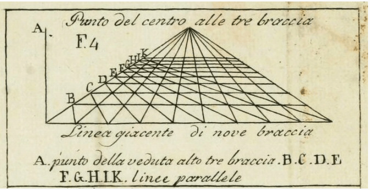
Filippo Brunelleschi (1377-1446) - Architect

- First to describe linear perspective.
- Experimented (1415-1420) using a panel with a grid of squares and a plaque with a hole at eye level.
- Drawings of the Baptistry in Florence, Place San Giovanni and other Florence landmarks.
- His method was studied by Alberti, Da Vinci & della Francesca's *The Perspective of Painting*.



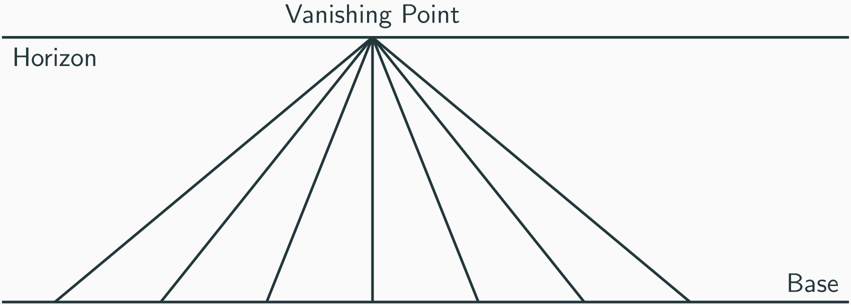
Leon Battista Alberti (1404-1472)

- Alberti's Veil
  Transparent cloth on a frame,
  Good for actual scenes not
  imaginary ones.
- Basic principles:
  1. A straight line in perspective
  remains straight.
  2. Parallel lines either remain
  parallel or converge to a point.
- Drawing a square-tiled floor, solved
  by Alberti (1436).



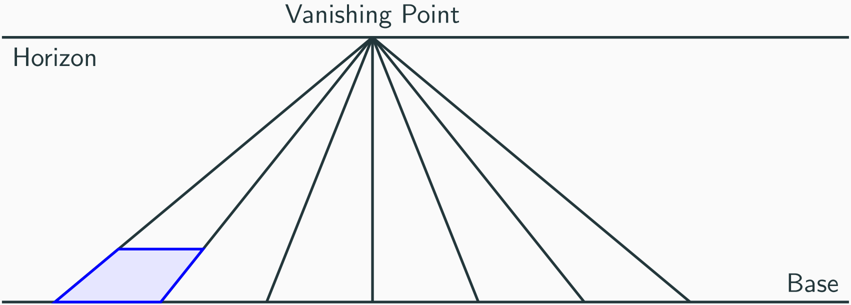
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.



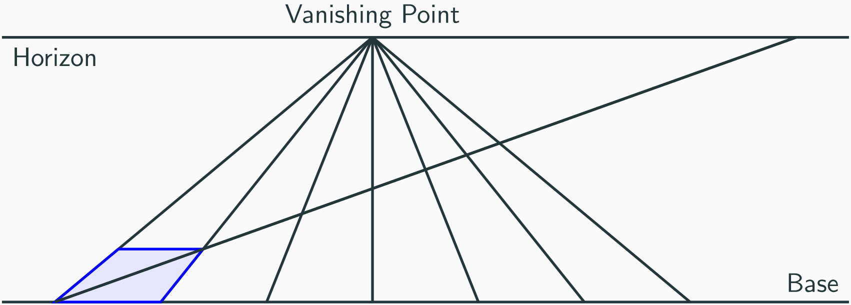
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.



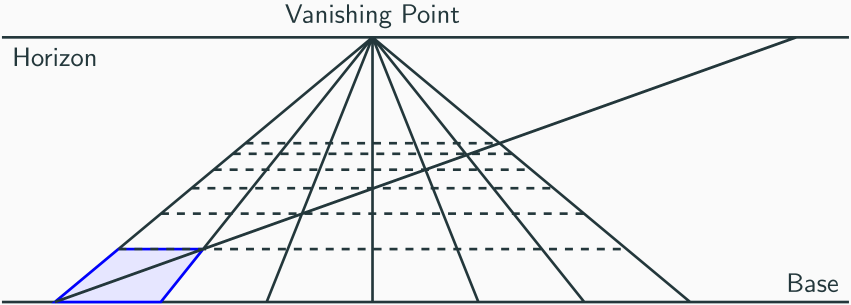
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.



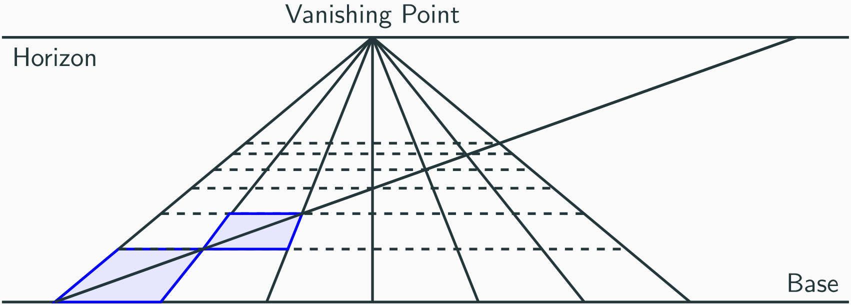
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



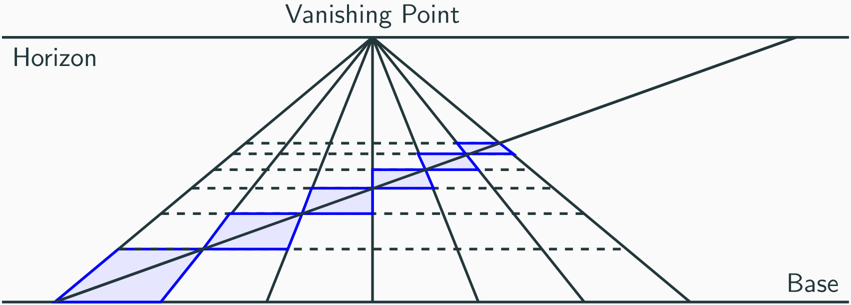
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



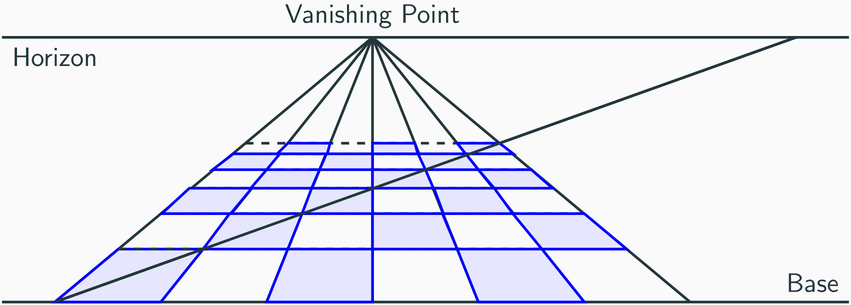
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



Desargues' Projective Geometry[1]

- Mathematics behind Alberti's Veil:
  Family of lines (light rays) through a point (eye) plus a plane (veil).
- Recall **Pappus' Theorem**:
  A_1, A_2, A_3, collinear;
  B_1, B_2, B_3, collinear;
  then, so are C_1, C_2, C_3.
- Blaise Pascal (1623-1662) at 16 generalized to conics.
- Desargues (1640) **Projective Geometry** only relies on a straight edge.
- Note: Piero della Francesca (c. 1415-1492) formalized rules of perspective, mid-1470s.

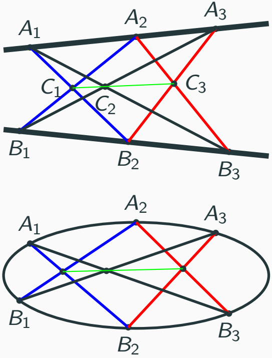
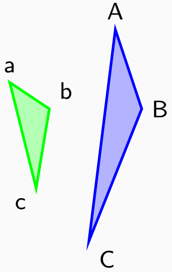


Figure 2: Pappus' and Pascal's Theorems.

[1] Two centuries ahead of his time.

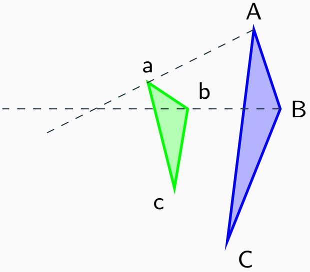
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in **perspective axially** *if and only if they are b) in* **perspective centrally**.



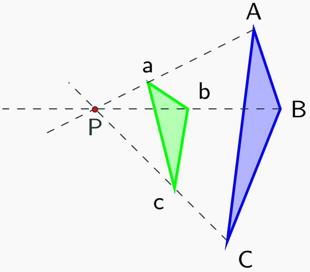
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc
  center of perspectivity P.



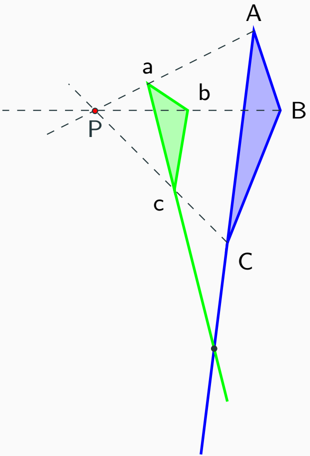
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676). *Two triangles are a) in* ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc **center of perspectivity** P.



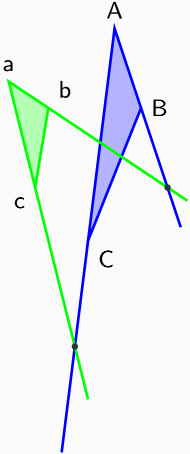
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676). *Two triangles are a) in* ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc **center of perspectivity** P.
- a) Extend pairs, AC-ac, AB-ab, etc.



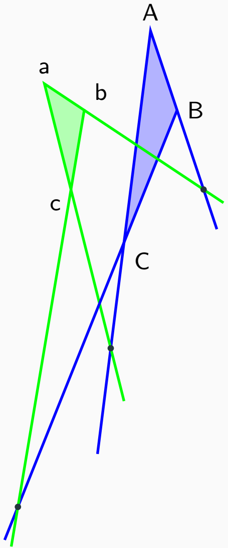
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc
  center of perspectivity P.
- a) Extend pairs, AC-ac, AB-ab, etc.



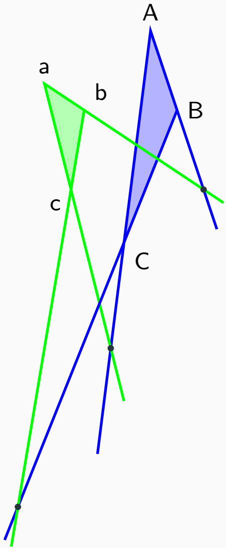
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc
  center of perspectivity P.
- a) Extend pairs, AC-ac, AB-ab, etc.



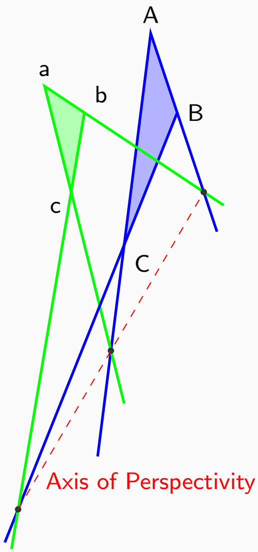
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc
  center of perspectivity P.
- a) Extend pairs, AC-ac, AB-ab, etc.



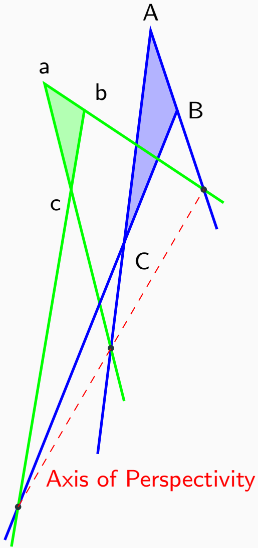
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc
  center of perspectivity P.
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.



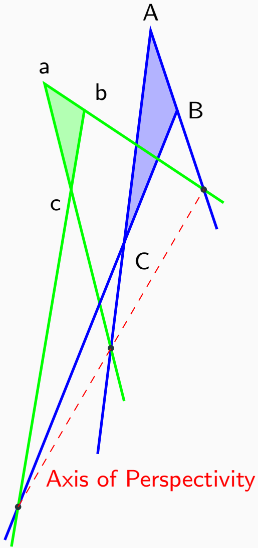
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc
  center of perspectivity P.
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.
- What if two sides are parallel?



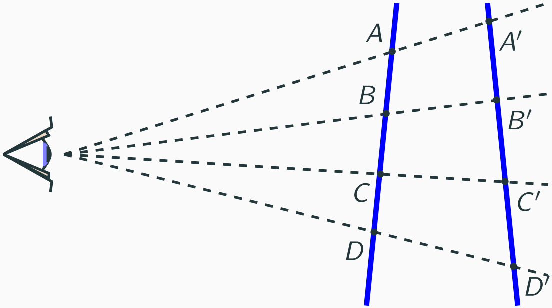
Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally****.*
- b) Extend Aa, Bb, Cc **center of perspectivity** P.
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.
- What if two sides are parallel?
- Need **Projective plane**.



Invariance of the Cross Ratio

Lengths and angles are not preserved under projection.



But, for any four points on a line, $\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$ is invariant. That is,

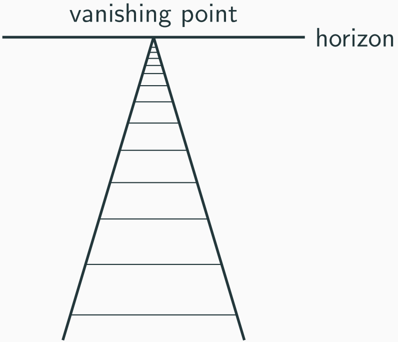
$$\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}}.$$

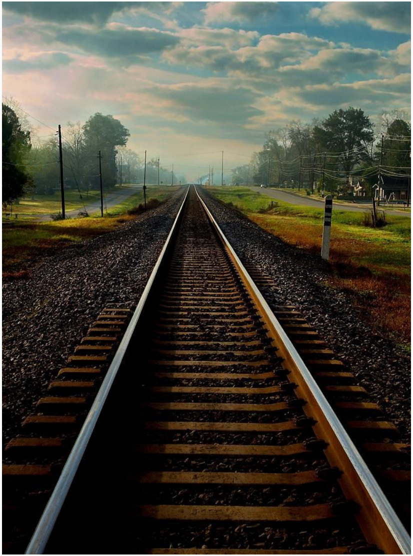
Projective Geometry Rebirth in 1800's.

Perspective

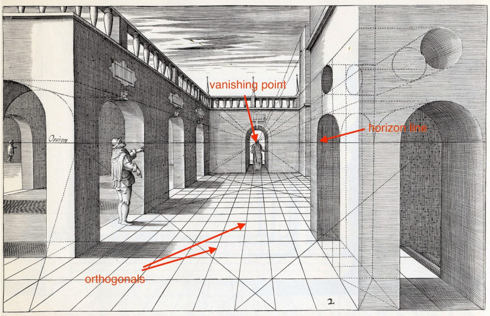
1. Parallel lines meet at a pt.
2. Lines map to lines.
3. Conics map to conics.

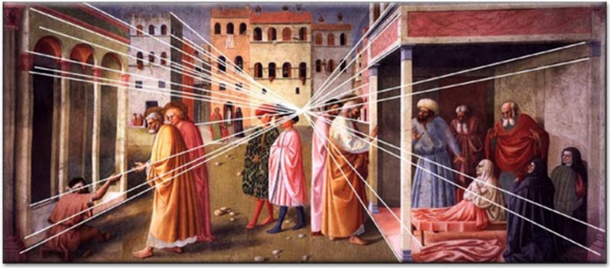
Example: Train tracks.





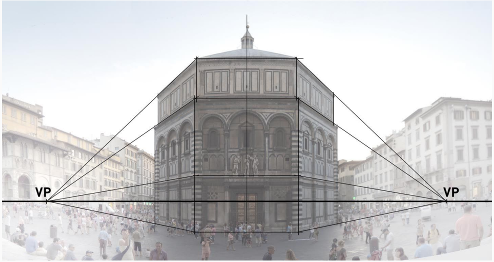
One Point Perspective - Find the Vanishing Points





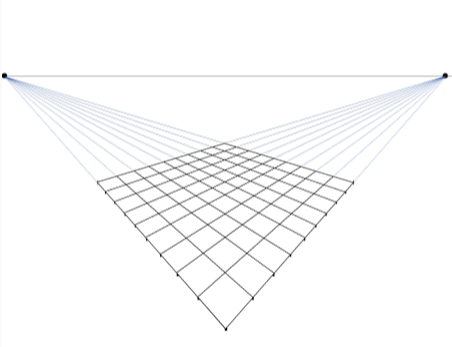
Two Point Perspective - Find the Vanishing Points

Two Point Perspective - Find the Vanishing Points



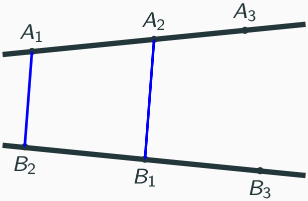
Two Point Perspective Vanishing Point(s)

Two Point Perspective Vanishing Point(s)



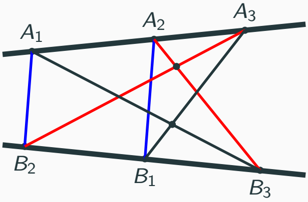
Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



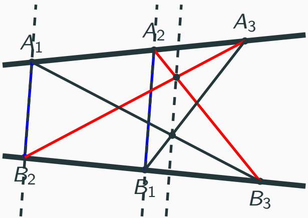
Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



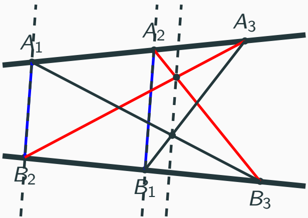
Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



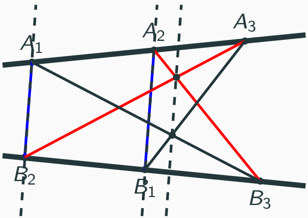
Points at Infinity

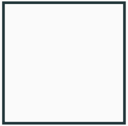
- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.



Points at Infinity

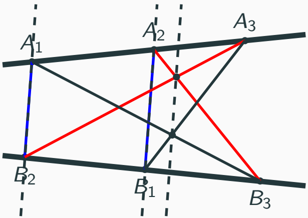
- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane





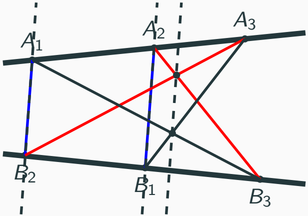
Points at Infinity

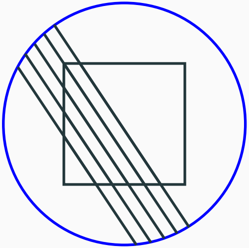
- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane



Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane
- Add parallel lines.
  Where do they go?
- Line at Infinity
- Plane + line at infinity = **Projective Plane**

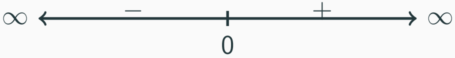




Line at infinity

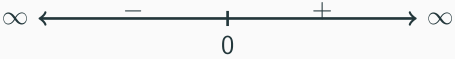
Projective Line

- Consider the real line, $\mathbb{R}$.



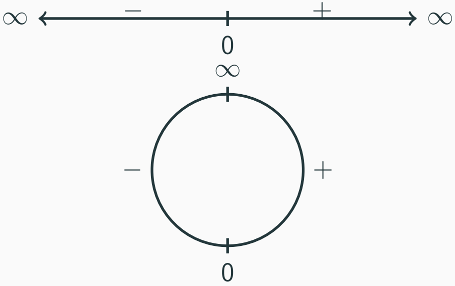
Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.



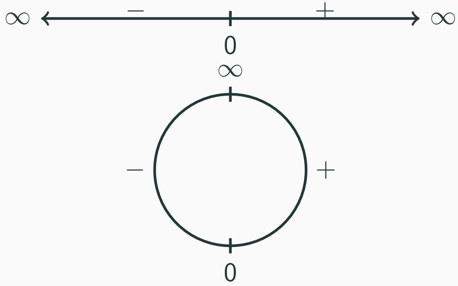
Projective Line

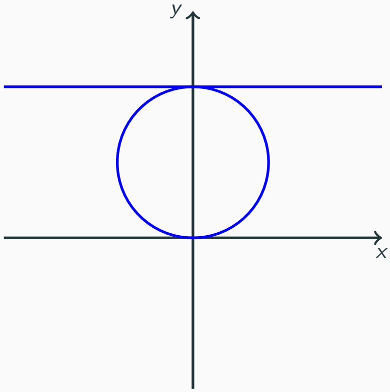
- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.
- Topologically a circle!



Projective Line

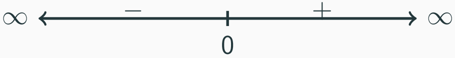
- Consider the real line, $\mathbb{R}$.
- Add point at infinity, real projective line, $\mathbb{RP}^1$.
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.

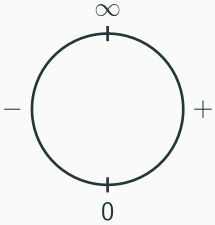


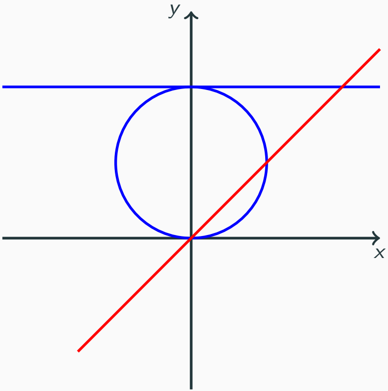


Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.



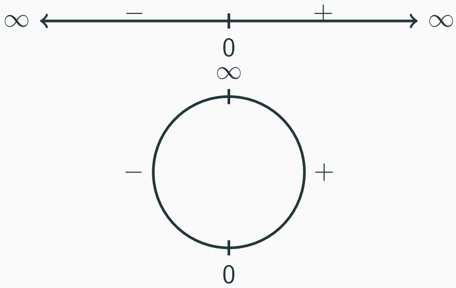


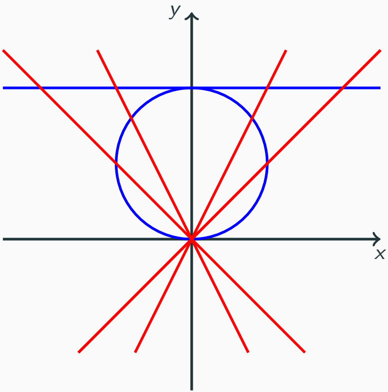


Intersection: $y = b$, $y = mx$:
$x = \frac{b}{m}$.

Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.



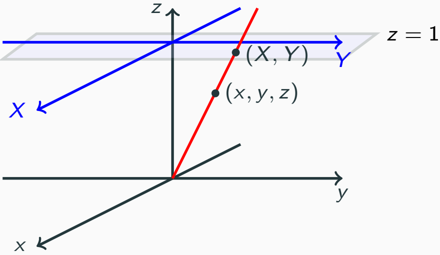


Intersection: $y = b$, $y = mx$:
$x = \frac{b}{m}$.

Homogeneous Coordinates

- Point on line: (x, y, z)
- All points on line map to (X, Y) in the plane.
- (X, Y) are called homogeneous coordinates.
- Points on line are multiples, $(x', y', z') = \lambda(x, y, z)$.
- Point on plane: Let $\lambda = \frac{1}{z}$. Then, $(x', y', z') = (\frac{x}{z}, \frac{y}{z}, 1)$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z}.$$



Curves: Given $Y = f(X)$, find $(x, y, z)-$surface.

- Curve in plane $z = 1$, $Y = X^2$.
- $X = \dfrac{x}{z}$, $Y = \dfrac{y}{z}$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- Multiply by z^2.
- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

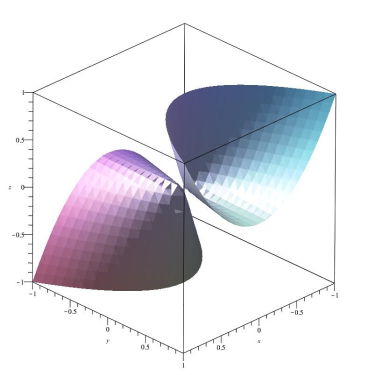


Figure 3: Surface $x^2 = yz$.

Curves: Given $Y = f(X)$, find $(x, y, z)-$surface.

- Curve in plane $z = 1$, $Y = X^2$.
- $X = \dfrac{x}{z}$, $Y = \dfrac{y}{z}$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- Multiply by z^2.
- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

- Slicing with planes, like Alberti's veil, one gets projections of the curve.

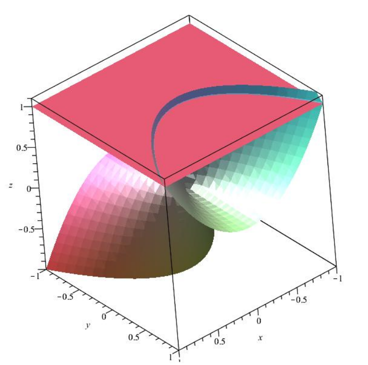


Figure 3: Surface $x^2 = yz$.

Curves: Given $Y = f(X)$, find $(x, y, z)-$surface.

- Curve in plane $z = 1$, $Y = X^2$.
- $X = \frac{x}{z}$, $Y = \frac{y}{z}$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- Multiply by z^2.
- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

- Slicing with planes, like Alberti's veil, one gets projections of the curve.

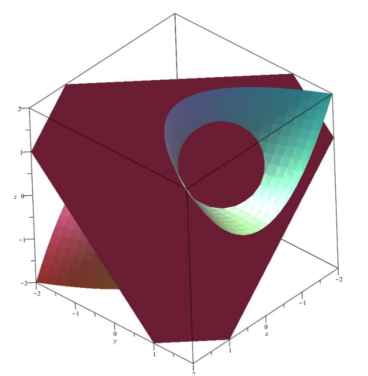


Figure 3: Surface $x^2 = yz$.

Projective Sphere: Extending $\mathbb{RP}^1$.

- Map points on a plane to points on surface of unit sphere, $\mathbb{S}^2$.
- Lines through South Pole uniquely intersect the plane and sphere.
- All points mapped except $(0, 0, 0)$. This point can be mapped to the line at infinity.
- Lines through origin are points of the real projective plane, $\mathbb{RP}^2$.

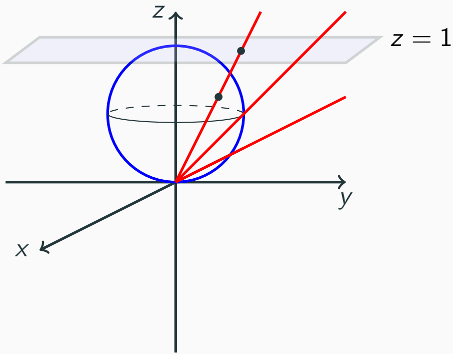
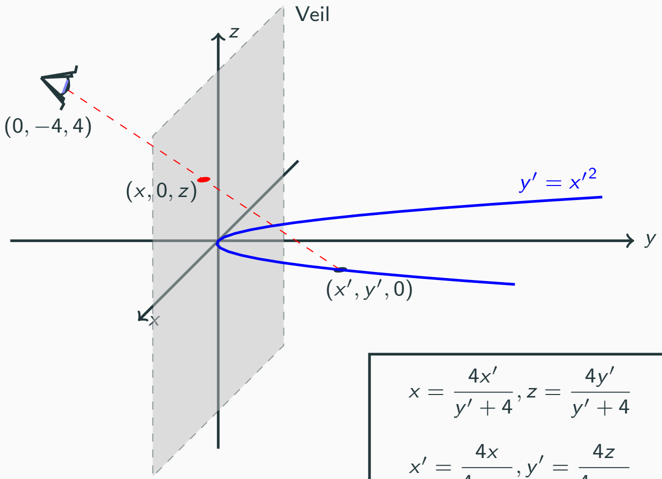


Figure 4: Stereographic Projection

Looking into the Veil - Parabola Projected

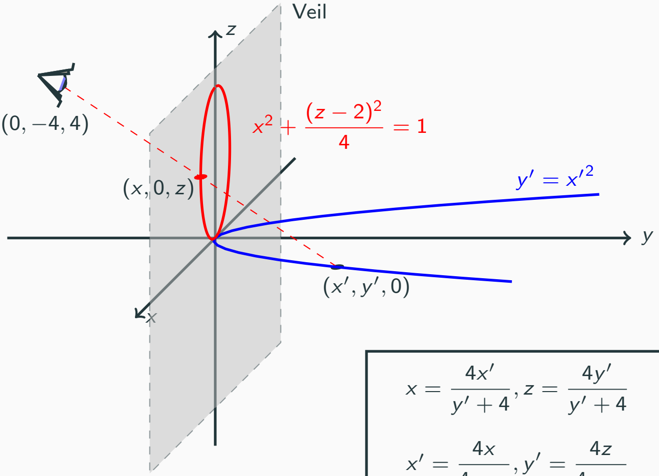


$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

Figure 5: Problems 8.4.2-8.4.4

Looking into the Veil - Parabola Projected

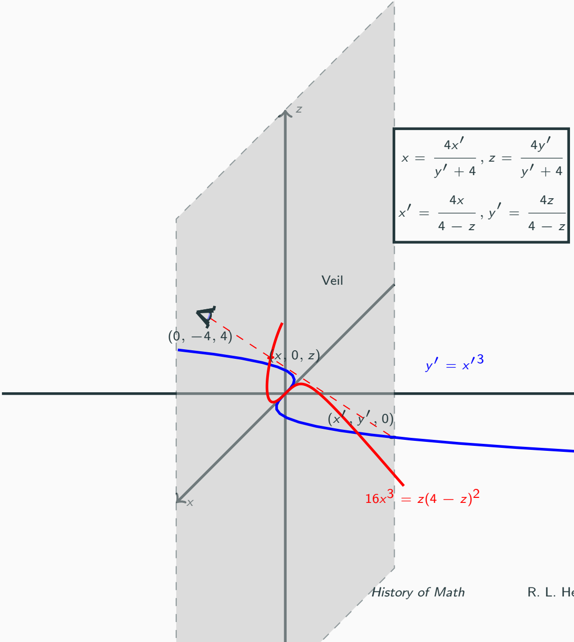


$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

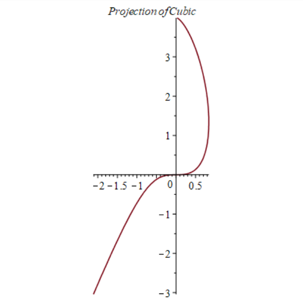
Figure 5: Problems 8.4.2-8.4.4

Viewing A Cubic in the Veil



$$x = \frac{4x'}{y'+4}, \; z = \frac{4y'}{y'+4}$$

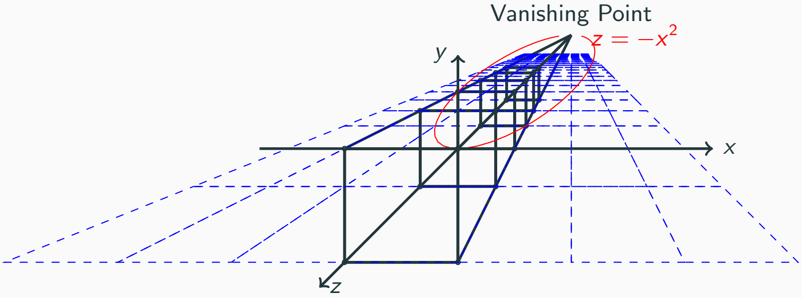
$$x' = \frac{4x}{4-z}, \; y' = \frac{4z}{4-z}$$



Above: Maple plot of projected cubic to the veil.

Perspective Drawing

Looking at conics from a different perspective: The parabola $z = -x^2$ looks like an ellipse.



In the 1600's mathematicians had other mathematics to attend to. So, we return to geometry in the 1800's.