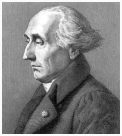
Non-Euclidean Geometry and Group Theory

Fall 2023 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

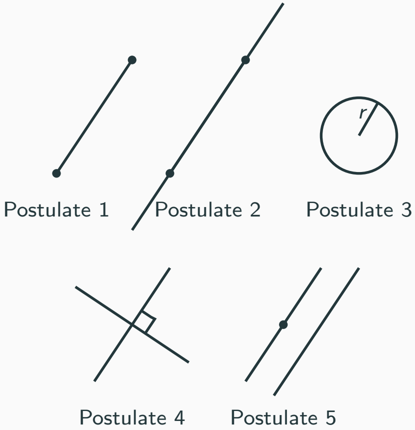


Figure 1: Euclid's 5 Postulates.

Statements of Parallel Axiom in Text

$\mathbf{P}_1$ For each straight line L and point P outside L there is exactly one line through P that does not meet L.

Equivalent statements

> The angle sum of a triangle $= \pi$. - Euclid.
>
> The locus of points equidistant from a straight line is a straight line. - al-Haytham.
>
> Similar triangles of different sizes exist. - Wallis

Saccheri (1733) - provided two alternatives to arrive at proof by contradiction.

$\mathbf{P}_0$ There is not line through P that does not meet L.

$\mathbf{P}_2$ There are at least two lines through P that do not meet L.

Parallel Postulate

- Proclus (410-485) Equivalent postulate. Revived as Playfair axiom.
- William Ludlam (1785):
  Two straight lines, meeting at a point, are not both parallel to a third line.
- John Playfair, *Elements of Geometry* (1795):
  Playfair's axiom: Two straight lines which intersect one another cannot be both parallel to the same straight line.
- Many false attempts to prove based on other four postulates.
- 1663 John Wallis "To each triangle, there exists a similar triangle of arbitrary magnitude."
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines are infinite. Led to contradiction of $\mathbf{P}_1$, almost $\mathbf{P}_2$.
- d'Alembert, 1767 - "The scandal of elementary geometry."

Spherical Geometry

- Lines = geodesics,
  Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
  - General properties of map projections.
  - hyperbolic functions
  - π is irrational
  - optics

$$a + b + c = \pi + \frac{A}{R^2}.$$

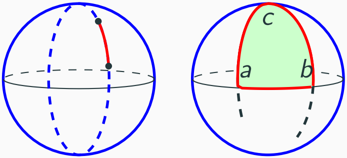


Figure 2: Harriot and Lambert.

Other Geometries

- Ferdinand Karl Schweikart (1780) Astral geometry, sum of three angles of a triangle is less than two right angles.
- Wrote to Gauss, 1818, via student Christian Ludwig Gerling (1788-1864).
- Franz Taurinus (1784-1854), Schweikart's nephew. Proposed geometry on a sphere of imaginary radius, logarithmic-spherical geometry.
- 1826, hyperbolic law of cosines in *Geometriae prima elementa.*
- Wrote to Gauss. after being encouraged, he sent copies of his works with no reply.
- Later he burned copies of his book.

Figure 3: Gerling and Schweikart

Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- Discussed with Farkas Bolyai (1775 - 1856) - told his son no to waste his time.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line.
  Developed trig identities, hyperbolic geometry.

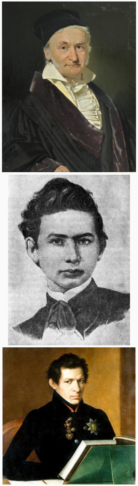
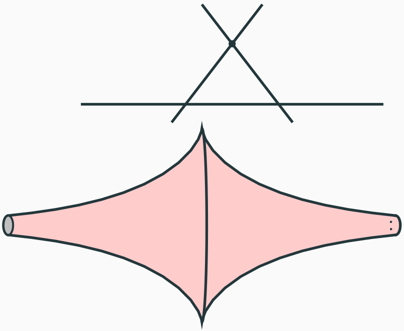


Figure 4: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

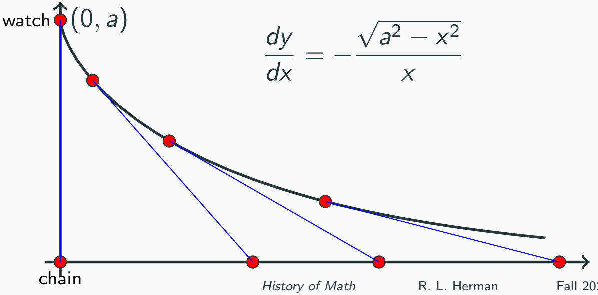
- Georg Friedrich Bernhard Riemann (1826-1866)
  - Published in 1868 Lecture
  - Spherical geometry
  - Riemannian geometry → differential geometry
  - Every line through a point not on a given line meets the line.
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868 using a **tractrix**.



tractrix $(a(t - \tanh t), a \operatorname{sech} t)$

Aside: The Tractrix

- Claude Perrault [brother Charles author of *Cinderella*, *Puss-in-Boots*] in 1693, Paris, placed a watch in the middle of a table and pulled its chain along the edge of the table. What was the curve traced out ?
- Studied by Newton (1676), Huygens (1692) and Leibniz (1693). Euler gave complete theory in 1788. [Am. Math. Monthly, 72(10) (1965), 1065-1071.]
- Huygens coined name from Latin, *tractus*.



$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA$.

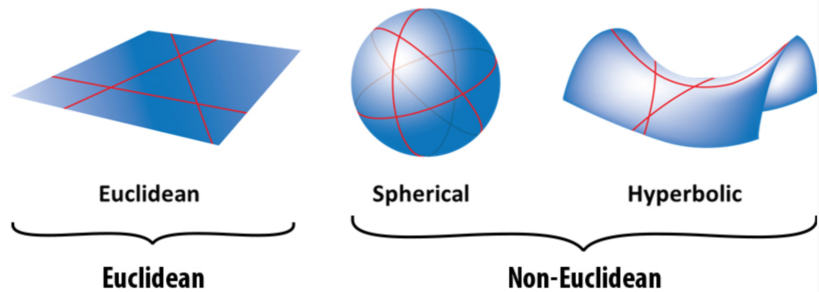


Figure 5: Surfaces of Constant Curvature.

Hyperbolic Geometry

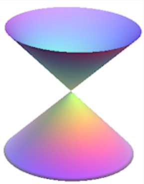
- Sphere

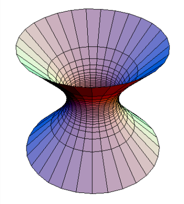
$$x^2 + y^2 + z^2 = \text{const}$$

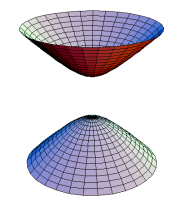
- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$. Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$. Hyperboloid of two sheets.



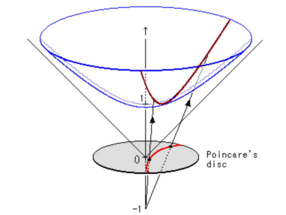


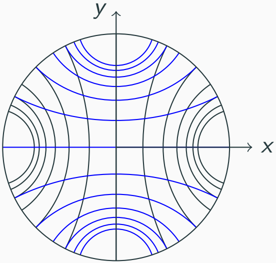


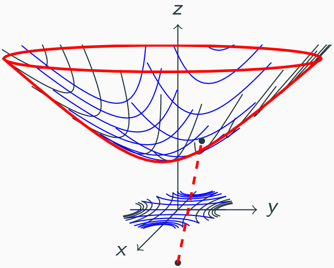
- Poincaré's Disks

$$(x, y, z) = (c \cosh t, \sinh t, \sqrt{1 + c^2} \cosh t)$$

- Stereographic Projection thru $(0, 0, -1)$ to $z = 0$: $(x, y, z) \rightarrow \frac{(x,y)}{1+z}$.
- Hyperbolic geometry.



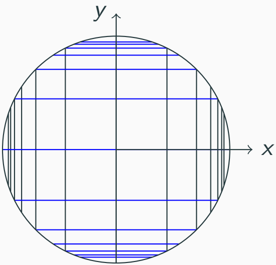


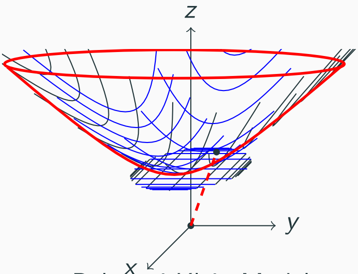


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0,0,0)$ to $z=1:(x,y,z)\to\frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0,0,1)$

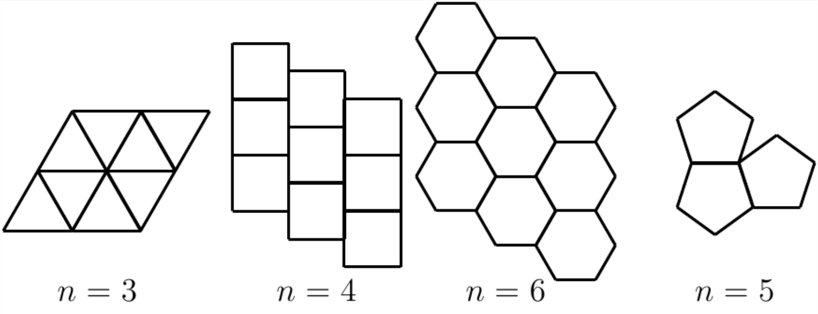




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



Other Tilings

- Johannes Kepler (1571-1630)
  - Studied Tilings
  - *Harmonicae Mundi* (Harmony of the World).
  - Planned in 1599.
  - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
  - 2020 Nobel Prize
  - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
  - and M. C. Escher (1889-1972)
  - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

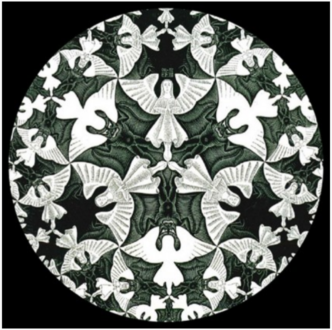


Figure 6: Circle Limit IV

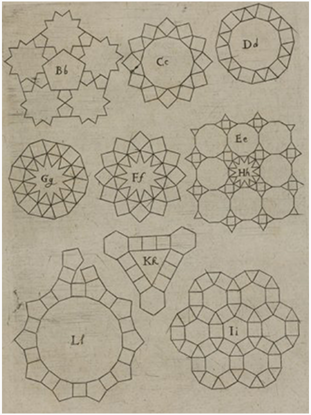
Hyperbolic Tessellations

- Tessellation = cover plane by tiles, no tiles overlap and no space empty.
- Schläfli symbol: $\{n, m\}$,
  $n =$ number of sides on the tile,
  $m =$ number of tiles that meet at a vertex.
  Euclidean: $\frac{1}{n} + \frac{1}{m} = \frac{1}{2}$,
  Hyperbolic: $\frac{1}{n} + \frac{1}{m} < \frac{1}{2}$.

Figure 7: Circle Limits I-IV.

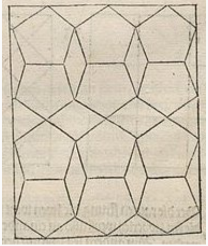
Kepler's Tiling

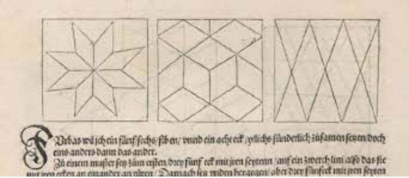
- 1619 Johannes Kepler published the first classification regular polygon tilings, Book II of *Harmonices Mundi*.
- First catalog of the 11 uniform tilings of the plane.
- See online discussion.



Albrecht Dürer's Tilings

- 1525, *Underweysung der Messung mit dem Zirckel und Richtscheyt* (A Course in the Art of Measurement with Compass and Ruler), the Painter's Manual.
- Constructed various curves and regular polygons with a ruler and compass.
- Illustrates three regular tilings (squares, triangles and hexagons), octagon tiling, uniform tiling with a six pointed star pattern, and rhomb tiling.





Aperiodic Tiling

- Non-periodic tiling that does not contain arbitrarily large periodic regions.
- 1964 Robert Berger, 20,426 Wang tiles.Later reduced his set to 104.
- 1966 Hans Läuchli, 40 Wang tiles.
- 1967 Raphael M. Robinson, 104.
- 1968, Donald Knuth, 96.
- 1971, Robinson, 6 tiles.
- 1974 Penrose, 6 tiles. P1.

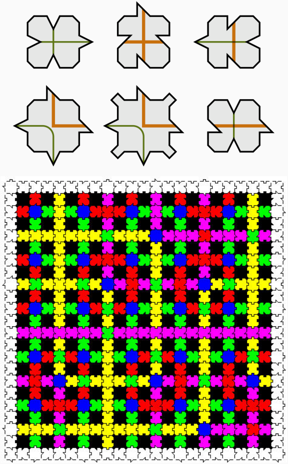
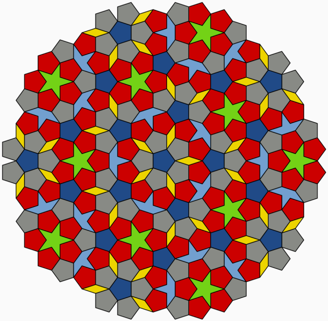


Figure 8: Robinson's tiles.

Penrose Tiling P1

- Penrose's first tiling, 1978.
- Uses pentagons and three other shapes:
  a five-pointed star,
  a "boat"
  and a "diamond".
- Need matching rules specifying how tiles meet each other to give non-periodic tilings.
- There are three different types of matching rules for the pentagonal tiles.
- Treating these as different prototiles gives a set of six.
- Indicate the three different pentagonal tiles using different colors.



Penrose Tiling P2

- Penrose introduced aperiodic tiling with two tiles.
- P2: Used quadrilaterals, “kite” and “dart.” Can be combined to make a rhombus.
- Need matching rules.
- A. Color the vertices and require that adjacent tiles have matching vertices.
- B. Use circular arcs to constrain the placement of tiles. The patterns must match at these edges.

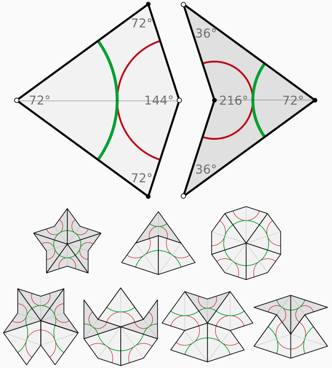


Figure 9: Penrose Kites and Darts. Use to create shown shapes: star, ace, sun, king, jack, queen, deuce.

Penrose Tiling P3

- Rhomus tiles.
- Thin rhombus with angles of 36, 144, 36, and 144 degrees.
- Thick rhombus with angles of 72, 108, 72, and 108 degrees.
- Tiles must be assembled such that the curves on the faces match in color and position across an edge.

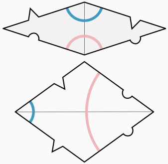


Figure 10: Thin and thick rhombs.

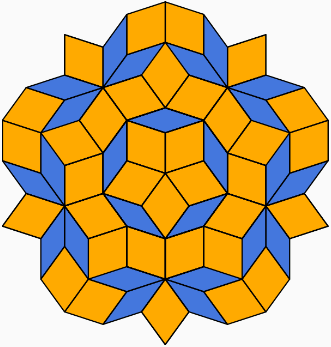
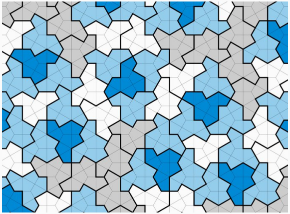
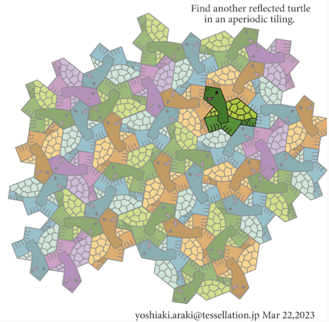


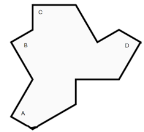
Figure 11: Penrose Rhombus Tiling.

Einstein Tiles

- The search for tiling with one tile.
- David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss, `https://arxiv.org/abs/2303.10798`, March 20, 2023.
- *An Aperiodic Monotile*
- Proved that "the hat" is an aperiodic monotile, called an einstein (one stone).
- Involves the hat and its mirror image, noted by Yoshiaki Araki.

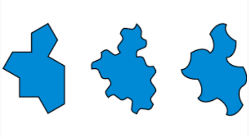


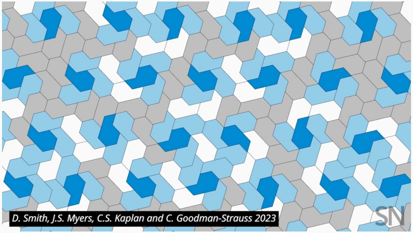




Spectre Tiles

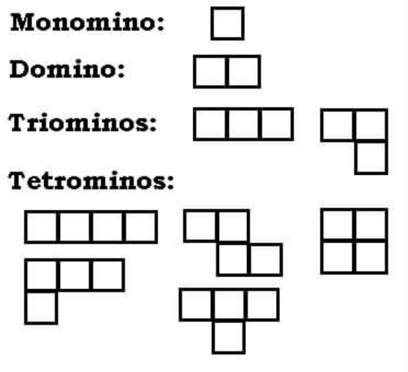
- David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss found a new tile `https://arxiv.org/abs/2305.17743`, May 28, 2023.
- *A Chiral Aperiodic Monotile*
- Is not accompanied by its reflection, a "vampire einstein."

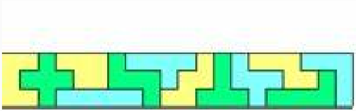
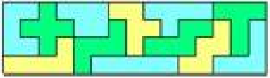
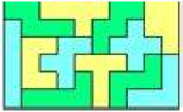


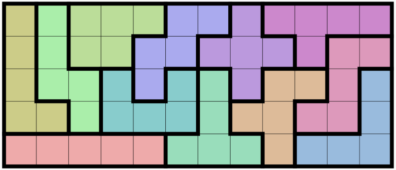


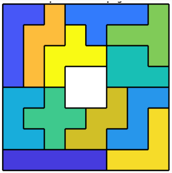
Polyominoes

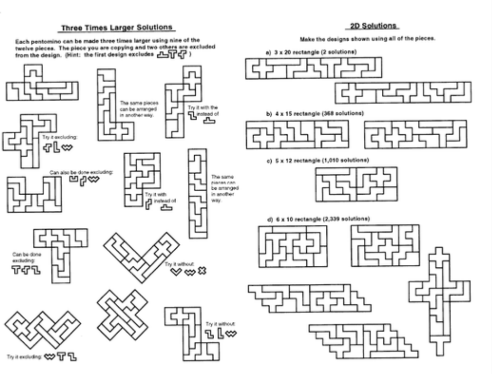
- A plane geometric figure formed by joining one or more equal squares edge to edge.
- Used in puzzles since at least 1907.
- Name *polyomino* invented by Solomon W. Golomb in 1953.
- Some types: domino, triomino, tetromino, pentomino, etc,
- Wikipedia page








Three Times Larger Solutions
Each pentomino can be made three times larger using nine of the twelve pieces. The piece you are copying and two others are excluded from the design. (Hint: the first design excludes)
The same pieces can be arranged in another way.
Try it with the instead of
Try it excluding:
Can also be done excluding:
Try it with instead of
Can be done excluding:
Try it without
Try it excluding:
Try it without
2D Solutions
Make the designs shown using all of the pieces.
a) 3 x 20 rectangle (2 solutions)
b) 4 x 15 rectangle (368 solutions)
c) 5 x 12 rectangle (1,010 solutions)
d) 6 x 10 rectangle (2,339 solutions)

Return to the Quintic

- In the meantime, the search for solving the quintic continues.
- The general quintic cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions.
- Malfatti (1731-1807) was the first to solve a solvable quintic using a resolvent of sixth degree, 1771.
- The general quintic was solved in terms of Jacobi theta functions by Hermite in 1858. See story.
- Our story begins with Gauss and Lagrange.

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
  - closed, identity, inverse, associative,
  plus commutative.
- Proved Fermat's Little Theorem: If p is prime, then for any integer a, $a^{p-1} \equiv 1 \mod(a)$.
- Represented integers as quadratic forms, like Fermat Primes ($4n + 1 = x^2 + y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2 + bxy + cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

Figure 12: List of things named after Gauss

Joseph Louis Lagrange (1736-1813)

- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great (Prussia) sought a great mathematician to replace Euler.
- Lagrange went to Berlin for 20 yrs.
- Invited by Louis XVI to Paris, 1786.
- 1793, Reign of Terror, saved by Lavoisier.
- 1795 - established dept. at École Normal.
- 1797 - established dept. at École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.

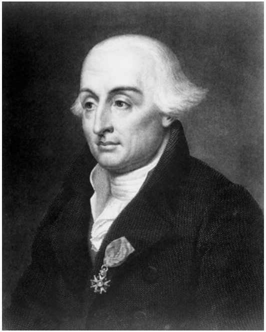
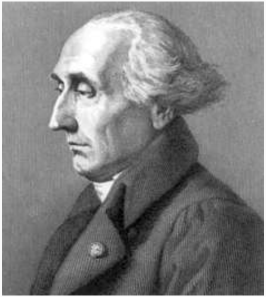


Figure 13: List of things named after Lagrange

Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial, $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots. $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a cube root of unity, $\omega^3 = 1$. Then,

$$\begin{aligned} x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\ x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\ x_3 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2} \end{aligned}$$



Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are
  $x_1 + x_2 + x_3$, $p = x_1x_2x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1x_2 + x_1x_3 + x_2x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - gave proofs that quintic can't be solved. Proofs not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 14: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Jul 1831 - April 29, 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Stayed up all night; wrote letters and note to Auguste Chevalier.
- On May 30, fought in duel and lost.
- Chevalier forwarded papers for publication by Joseph Liouville.

Figure 15: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed Galois' delayed manuscript, published 1846. - introduction of groups and fields.

- Multiplicative group modulo n.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,
  $$a^{p-1} \equiv 1 (\text{mod } p).$$
- Euler's ϕ function:
  $$\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} | (k, n) = 1\}.$$
  $\phi(5) = 4$, $\{1, 2, 3, 4\}$,
  $\phi(8) = 4$, $\{1, 3, 5, 7\}$.
- Group Properties:

  closed, identity, inverse, associative

Examples: Mod 5 and 8.

x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

x	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Symmetry Groups

- Levi ben Gorshun (1321) Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.

Figure 16: Sophus Lie and Emmy Noether.