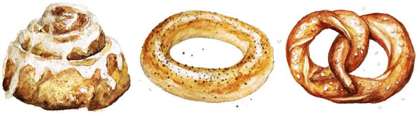
Topology and Knot Theory

Fall 2021 - R. L. Herman



What is Topology?

From Wikipedia

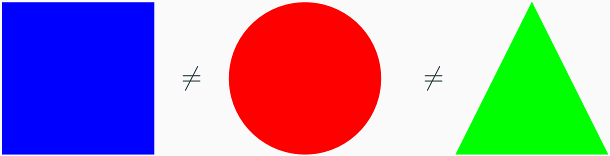
"In mathematics, topology (from the Greek *t*opos, 'place', and *l*ogos, 'study') is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing."

Nobel Prize in Physics 2016

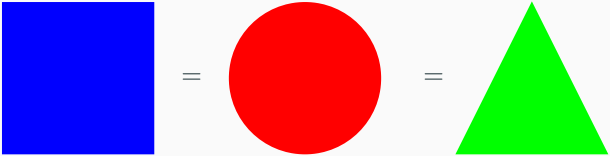
Figure 1: Nobel Prize in Physics 2016

Geometry vs Topology

Geometry



Topology



Types of Topology

General topology (Point Set Topology) Study of basic topological properties derived from properties such as connectivity, compactness, and continuity.

Metric topology Study of distance in different spaces.

Algebraic topology (Combinatorial Topology) Study of topologies using abstract algebra like constructing complex spaces from simpler ones and the search for algebraic invariants to classify topological spaces.

Geometric topology Study of manifolds and their embeddings.

Network topology Study of topology discrete math. Network topologies are graphs consisting of nodes and edges.

Differential Topology Study of manifolds with smoothness at each point to allow calculus.

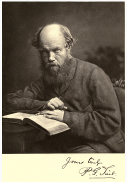
Origins of Topology

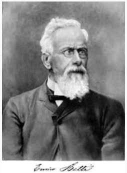
The search for a type of geometry where distance is not relevant.

- Euler - Graphs, Polyhedra
- Gauss, Maxwell - Physics
- Thomson, Tait - Knot Theory
- Riemann - 2D Surfaces in 3D
- Betti - Higher Dimensions
- Klein - Geometry and Groups
- Poincaré - Algebraic Topology
- Noether - Homology Groups









Leonhard Euler (1707-1783) - Königsberg Bridges

- 1736 Correspondences with Carl Gottlieb Ehler (1685-1753)
- Ehler's Letter

  "You would render to me and our friend Kuhn a most valuable service, putting us greatly in your debt, most learned sir, if you would send us the solution, which you know well, to the problem of the **seven Königsberg bridges** together with a proof. It would prove to an outstanding example of the calculus of position [calculi situs] worthy of your great genius. I have added a sketch of the said bridges."

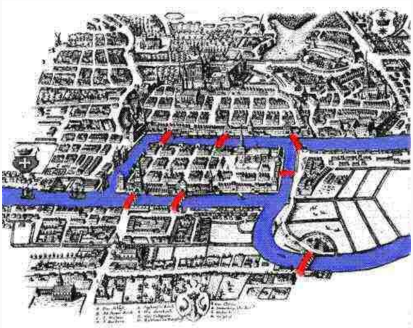
Leonhard Euler (1707-1783) - Königsberg Bridges

- 1736 Correspondences with Carl Gottlieb Ehler (1685-1753)
- Euler's reply

  "Thus you see, most noble sir, how this type of solution bears little relationship to mathematics and I do not understand why you expect a mathematician to produce it rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime most noble sir, you have a assigned this question to the geometry of position but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed this way."
- Based on Liebniz's *geometria situs* and *analysis situs*.
- Geometry of position: concerned only with the determination of position and does not involve using distances.

Königsberg Bridges Problem

Can the seven bridges over the river Preger in the city of Königsberg (formerly in Prussia but now known as Kaliningrad, Russia) all be traversed in a single trip without doubling back and ending where you started?



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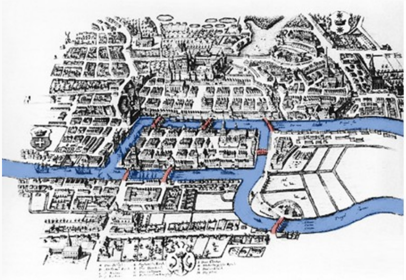

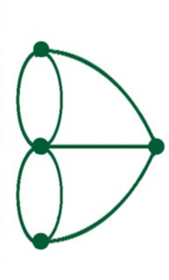

- Euler's result did not depend on the lengths of the bridges or on their distance from one another, but only on connectivity.

Königsberg Bridges Problem

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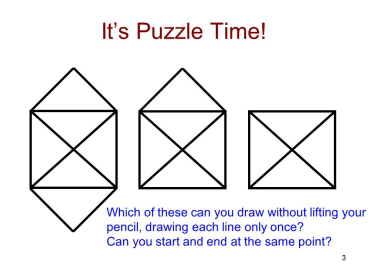

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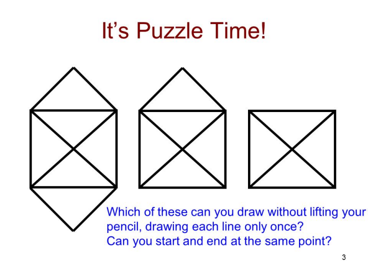


- A connected graph has an Euler cycle $\Leftrightarrow$ every vertex has even degree.

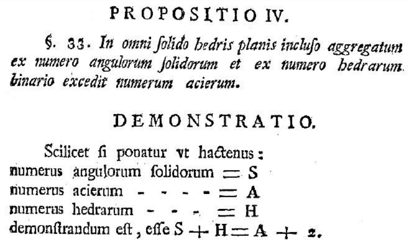
Euler's Polyhedron Formula

- 1750, Euler wrote Christian Goldbach (1690-1764)
- For polyhedron, like Platonic solids,

$$V - E + F = 2.$$

- Published papers in 1752.
- Not known before.
  - Descartes was close (1676).
- Euler characteristic: $\chi = V - E + F$.

Shape	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube/Hexahedron	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12

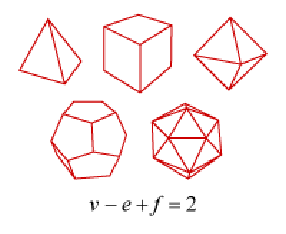


PROPOSITIO IV.

§. 33. *In omni ſolido hedris planis incluſo aggregatum ex numero angulorum ſolidorum et ex numero hedrarum binario excedit numerum acierum.*

DEMONSTRATIO.

Scilicet ſi ponatur vt hactenus:
numerus angulorum ſolidorum $= S$
numerus acierum - - - - $= A$
numerus hedrarum - - - $= H$
demonſtrandum eſt, eſſe $S + H = A + 2$.



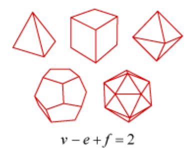
There Are Exactly Five Regular Polyhedra

Proof:

- Let $n = \#$ of sides of each face.
- Let $m = \#$ of faces meeting each vertex.
- $E = \frac{1}{2}Fn$ and $V = \frac{1}{m}Fn$.
- Since $V - E + F = 2$,

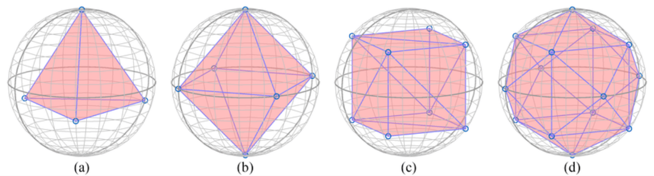
$$F = \frac{4m}{2n - mn + 2m.}$$

- $2n - mn + 2m > 0$ and $n \geq 3$.
  $n = 3 :$ $2n - mn + 2m = 6 - m$.
  So, $m = 3, 4, 5$.
  $n = 4 :$ $2n - mn + 2m = 8 - 2m$.
  So, $m = 3$.
  $n = 5 :$ $2n - mn + 2m = 10 - 3m,\ m = 3$.



Solutions (n, m)
$(3, 3)$ tetrahedron,
$(3, 4)$ octahedron,
$(3, 5)$ icosahedron,
$(4, 3)$ cube,
$(5, 3)$ docecahedron.

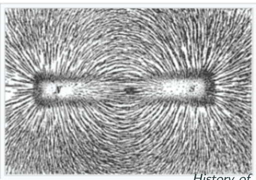
Euler Characteristic of a Sphere



- Inscribe Platonic solids in a sphere.
- a to b: Add 2 vertices, 4 faces, 6 edges. $\Delta\chi = 2 - 6 + 4 = 0$
- Push faces to sphere surface.
- Euler characteristic, $\chi = V - E + F = 2$.

The Birth of Electromagnetism

- 1785, Coulomb's Law.
- 1820, New discoveries:
- Ørsted: Electric current deflects compass.
- Biot-Savart: Currents produce magnetic fields.
- Amperè: Parallel wires carrying currents attract or repel. 1827 *électrodynamiques.*
- 1821, Faraday: Electromagnetic rotation.
- 1831, Electromagnetic induction, Faraday's Law.
- Field lines.



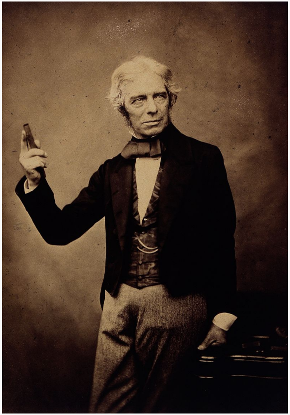
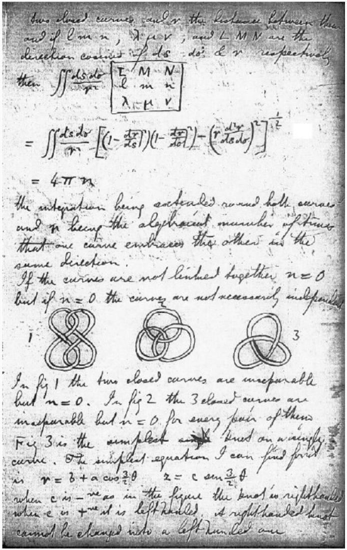


Figure 2: Michael Faraday (1791 – 1867)

C F Gauss (1777-1855) and Wilhelm Weber (1804-1891)

Carl Friedrich Gauss - 1831 - EM Induction.

- Gauss and Weber.
    - First telegraph 1833 to communicate 3 km.
    - Mapped Earth's Magnetic Field.
    - Weber - 1856, $c =$ speed of light.
- Gauss introduced formula for two intertwining curves.



two closed curves and r the distance between them
and if l m n, λ μ ν, and L M N are the
direction cosines of ds dσ & r respectively

$$\text{then} \iint \frac{ds\,d\sigma}{r^2} \begin{vmatrix} L & M & N \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$$

$$= \iint \frac{ds\,d\sigma}{r^2}\left[\left(1-\frac{dr}{ds}^2\right)\left(1-\frac{dr}{d\sigma}^2\right)-\left(r\frac{d^2r}{ds\,d\sigma}\right)^2\right]^{\frac{1}{2}}$$

$$= 4\pi n$$

the integration being extended round both curves
and n being the algebraical number of times
that one curve embraces the other in the
same direction.
If the curves are not linked together n = 0
but if n = 0 the curves are not necessarily independent

1 3

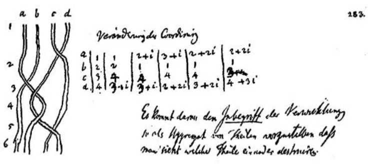
In fig 1 the two closed curves are inseparable
but n = 0. In fig 2 the 3 closed curves are
inseparable but n = 0 for every pair of them
Fig 3 is the simplest knot on a single
curve. The simplest equation I can find for it
is $r = b + a\cos\frac{3}{2}\theta$ $z = c\sin\frac{3}{2}\theta$
when c is −ve as in the figure the knot is righthanded
when c is +ve it is lefthanded. A righthanded knot
cannot be changed into a lefthanded one

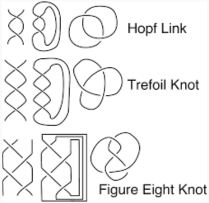
http://mathematiciancfgauss.blogspot.com/2016/11/contribution-of-math-gauss-published-in.html

Gauss' Linking Number and Braids

$$\int\int \frac{(x'-x)\,dydz'-dzdy')+(y'-y)\,(dzdx'-dzdz')+(z'-z)\,(dxdy'-dydx'))}{((x-x')^2+(y-y')^2+(z-z')^2)^{-3/2}}=4m\pi$$

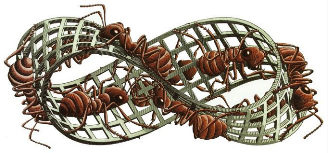
- 1833 Entered in notebook.
- Published in 1867.
- No proof given.
- Possibly from E&M or astronomy. Orbits of Ceres and Pallas 1801
- Studied linked orbits.
- Braids (Artin, 1926) in unpublished notebooks.





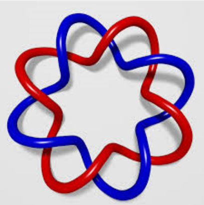
Two of Gauss' Students

Möbius Band - 1865

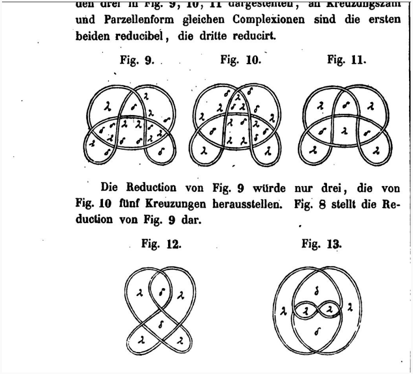


First Use of 'Topologie'

- Johann Listing gave Gauss' *geometria situs* a new name:
- 1847 -*Vorstudien zur Topologie*.
- 1861 - Möbius Band.
- Studied Connectivity and
- Link Invariants.



Inversion (rotation) and perversion (reflection)



den drei in Fig. 9, 10, 11 dargestellten, an Kreuzungszahl und Parzellenform gleichen Complexionen sind die ersten beiden reducibel, die dritte reducirt.

Fig. 9. Fig. 10. Fig. 11.

Die Reduction von Fig. 9 würde nur drei, die von Fig. 10 fünf Kreuzungen herausstellen. Fig. 8 stellt die Reduction von Fig. 9 dar.

Fig. 12. Fig. 13.

How do you distinguish knots and their symmetries?

Hermann Ludwig Ferdinand Helmhotz (1821-1894)

- German Physicist, Physician.
- Mathematics of the eye, theories of vision, perception of sound, electrodynamics, thermodynamics.
- In 1858 Helmholtz wrote on **vortex dynamics**, translated by Tait into English. *On Integrals of the Hydrodynamical Equations, which Express Vortex-motion*

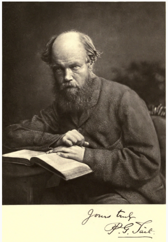
The evolution of a magnetic field $\boldsymbol{B}$ is similar to the evolution of vorticity $\boldsymbol{\omega}$.

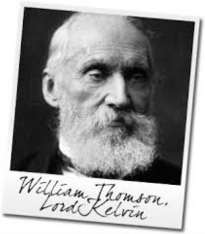
$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \operatorname{curl}(\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

Scottish Physics and Knots

James Clerk Maxwell (1831-1879), Peter Guthrie Tait (1831-1901), and William Thomson (1824-1907)

https://www.gutenberg.org/files/39373/39373-h/39373-h.htm

Maxwell and Tait met at Edinburgh Academy, went to University 1847.

Thomson (22) elected to Glasgow College Chair of Natural Philosophy.

James Clerk Maxwell and Helmholtz's Water Vortices

Maxwell read Gauss' work and referred to the work of Liebniz, Euler, and Vandermonde on *geometria situs.*

Maxwell wrote to Tait about Helmhotz's paper.

Tait's interest in Helmholtz was from recalling reading Hamilton's *Lectures* in 1853 and remembering formulae.

Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen.

(Von Herrn *H. Helmholtz.*)

Es sind bisher Integrale der hydrodynamischen Gleichungen fast nur unter der Voraussetzung gesucht worden, dafs die rechtwinkligen Componenten der Geschwindigkeit jedes Wassertheilchens gleich gesetzt werden können den nach den entsprechenden Richtungen genommenen Differentialquotienten einer bestimmten Function, welche wir das *Geschwindigkeitspotential* nennen wollen. Allerdings hat schon *Lagrange**) nachgewiesen, dafs diese Voraussetzung zulässig ist, so oft die Bewegung der Wassermasse unter dem Einflusse von Kräften entstanden ist und fortgesetzt wird, welche selbst als Differentialquotienten eines *Kräftepotentials* dargestellt werden können, und dafs auch der Einflufs bewegter fester Körper, welche mit der Flüssigkeit in

GLENLAIR
DALBEATTIE,
Nov. 13, 1867.

Dear Tait

If you have any spare copies of your translation of Helmholtz on "Water Twists" I should be obliged if you could send me one.

I set [sic] the Helmholtz dogma to the Senate House in '66, and got it very nearly done by some men, completely as to the calculation, nearly as to the interpretation.

Thomson has set himself to spin the chains of destiny out of a fluid plenum as M. Scott set an eminent person to spin ropes from the sea sand, and I saw you had put your calculus in it too. May you both prosper and disentangle your formulae in proportion as you entangle your worbles. But I fear the simplest indivisible whirl is either two embracing worbles or a worble embracing itself.

For a simple closed worble may be easily split and the parts separated

but two embracing worbles preserve each others solidarity thus

though each may split into many, every one of the one set must embrace every one of the other. So does a knotted one.

yours truly
J. CLERK MAXWELL

Sir William Rowan Hamilton (1805–1865)

This led Tait to work on quaternions.

Hamilton discovered **quaternions** in 1843, an extension of complex numbers: $w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where w, x, y, z are real and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ satisfy the bridge equations.

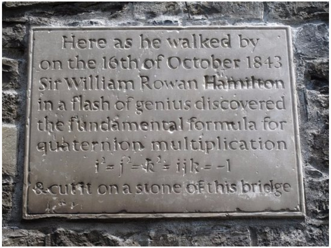


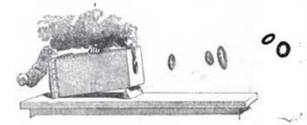
Figure 3: Hamilton carved his equations into the stone of the Brougham Bridge while on a walk.

Back to vortex rings ...

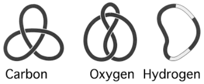
From Vortex Rings to Vortex Atoms

https://i.imgur.com/Y64h8ol.mp4 Movie

Tait experimented with smoke rings in 1867.

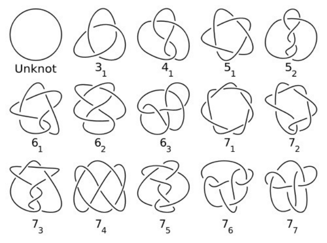


Thomson wrote to Helmholtz ...
later wrote *On vortex atoms*
as vortex rings in the aether.



Tait - Classification of Knots

- Knots up to 7 crossings reduced to 8 different knots.
- Periodic table of knot elements [Mendeleev - 1869].
- Tables up to 10 alternating crossings.
- Aether disproved in 1887 by Michelson Morley Experiment.



Put ideas in envelope for the Royal Society of Edinburgh [Open by 15/10/1987].

Tait Conjectures[1]

1. Reduced alternating diagrams have minimal link crossing number.
2. Any two reduced alternating diagrams of a given knot have equal writhe.
3. The flyping conjecture, which states that the number of crossings is the same for any reduced diagram of an alternating knot.

In 1987 one of Tait's conjectures was found in the envelope.

1,2 proved by Kauffman, Murasugi, and Thistlethwaite 1987.

3 proved by Menasco and Thistlethwaite, 1991 using Jones polynomials, 1984.

Perko Pair 1974

Links to Papers. and Gresham College Lecture about Tait.

[1] From Mathworld

More Knots

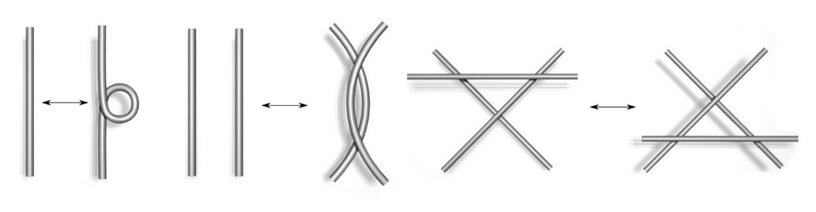
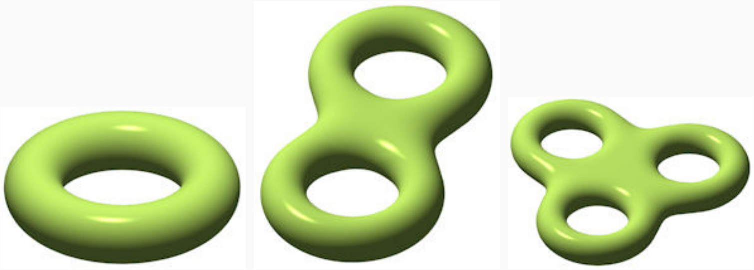


Figure 4: Reidemeister Moves: Untwist, Poke, Slide.

John Conway

Graduate Student Solves Decades-Old Conway Knot Problem

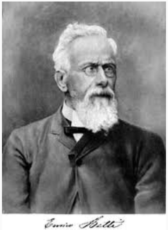
Connection to Surfaces

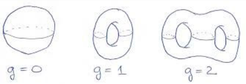


Torus: $V - E + F = 0$
Two-hole Torus: $V - E + F = -2$
Three-hole Torus: $V - E + F = -4$

Bernhard Riemann and Enrico Betti

- Riemann and Betti - connectedness.
- Surfaces with curvature - Manifolds.
- Can we classify surfaces up to a continuous transformation?
- Genus g and Euler Characteristic $\chi = 2 - 2g$.
- Enrico Betti (1823-1892)
- Betti number - maximum number of cuts that can be made without dividing a surface into two separate pieces.





Betti Numbers

- β_0 - number of connected components.
- β_1 - number of handles.
- β_2 - number of voids or. cavities

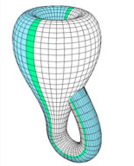
				
β_0	1	1	1	1
β_1	0	1	0	2
β_2	0	0	1	1

Poincaré Polynomial - Generator of Betti numbers.

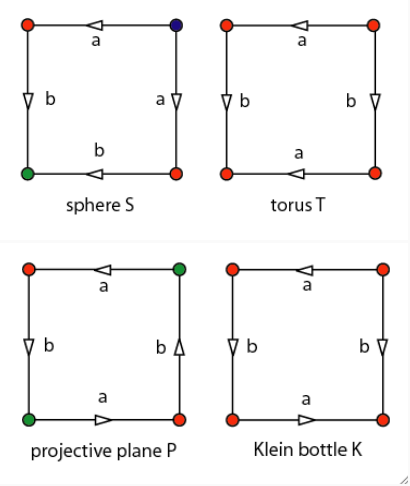
Ex: Torus: $T^2 = S^1 \times S^1$
$p(x) = \beta_0 + \beta_1 x + \beta_2 x^2 = 1 + 2x + x^2 = (1+x)^2.$

Felix Klein (1849-1925)

- 1872 Felix Klein Public Lecture
  - Erlangen Program, Geometry.
  - Symmetry groups $\rightarrow$ invariants.
  - Euclidean Geometry - invariant under translations, rotations.
  - Topology - invariants under continuous transformations.
  - Klein Bottle.



Equivalence Relations



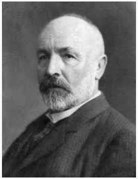
Henri Poincaré (1854-1912)

- 1895 Start of Algebraic Topology.
  - *Analysis Situs*
  - Brings rigor, better Betti Numbers.
- History of Poincaré's Mistakes.
  - 1888 King Oscar II, Sweden, Offered Prize.
  - Judges: Mittag-Leffler, Weierstrass, and Hermite.
  - *Acta Mathematica* - 3 body problem stable.
  - Oops! .... Chaos!
- Topological methods for differential equations.

See Stillwell on Early History of 3-Manifolds

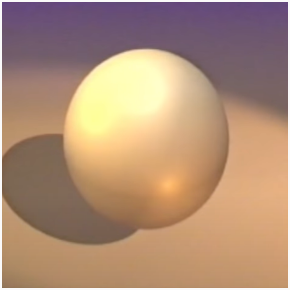
Felix Klein and Georg Cantor

- 1872 Cantor - Open and closed sets.
  - Introduced Set Theory.
  - Infinite and transfinite numbers
  - Cardinality.
- 1902 Hilbert - Neighborhoods.
- 1906 Frechet - Compactness, metric spaces have open and closed sets.
- Riesz 1909 and Hausdorff 1914 - abstract topological spaces.
- 1926 Emmy Noether - Homological groups, corrected Poincaré.




How Do You Turn a Sphere Inside-out?

Sphere Eversion - a continuous deformation, allowing the surface to pass through itself, without puncturing, ripping, creasing, or pinching.



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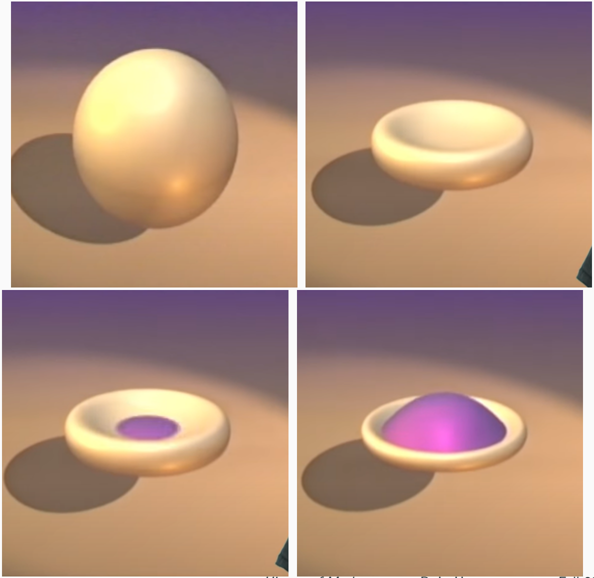

How Do You Turn a Sphere Inside-out?