Greek Mathematics II

Fall 2022 - R. L. Herman

Greek Number Theory

- **Pythagorean Theorem**
  $x^2 + y^2 = z^2$, x, y, z integers.
- **Diophantine Equations**
  Solve $3x + 5y = 1$, x, y integers.
- Euclid
  - Proved **# primes infinite**,
    Book IX, Prop 20.
  - **Perfect Numbers**, Book VII,
    Def 22, Book IX, Prop 36.
    Euclid proves:
    If $2^n - 1$ is prime, then
    $(2^n - 1)\, 2^{n-1}$ is perfect.
    Mersenne prime: $2^n - 1$.
  - **Polygonal Numbers**

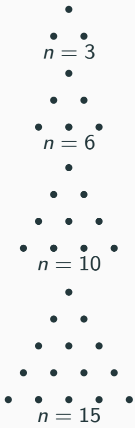

Figure 1: Polygonal Numbers.

Euclidean Algorithm - Book VII, Prop 1

- $\gcd(a, b)$: **Greatest common divisor** of a and b.
- Algorithm

$$\begin{aligned} a_1 &= \max(a, b) - \min(a, b) \\ b_1 &= \min(a, b) \\ &\text{repeat} \end{aligned}$$

Terminates when $a_{i+1} = b_{i+1}$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} a_1 &= 210 - 45 = 165 \\ b_1 &= 45 \end{aligned}$$

Continuing the computation:

a_i	b_i
210	45
165	45
120	45
75	45
30	45
15	30
15	15

We find $\gcd(210, 45) = 15$.

Euclidean Algorithm - Another Approach

Find the greatest common divisor of positive integers, a and b.

- If $a < b$, exchange a and b.
- Divide a by b and get the remainder, r. Thus,

$$a = qb + r.$$

  - If $r \neq 0$, replace a by b and b by r. Repeat the division.
  - If $r = 0$, report $gcd(a, b) = b$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} 210 &= 4 \cdot 45 + 30 \\ 45 &= 1 \cdot 30 + 15 \\ 30 &= 2 \cdot 15 \end{aligned}$$

Thus, $\gcd(210, 45) = 15$.

Pell's Equation (1611-1685)

- $x^2 - Ny^2 = 1$,
  N is a nonsquare integer, and
  x, y are integer solutions.
- Example of a Diophantine equation.
- Related to $\sqrt{2}$: $x^2 - 2y^2 = 0$,
  $y = 1 \Rightarrow x = \sqrt{2}$.
- $x^2 - 2y^2 = 1$,
  If x, y large, then $\frac{x}{y} \approx \sqrt{2}$.
- Known to Pythagoreans,
  Diophantus, and
- Archimedes' Cattle Problem can be reduced to solving Pell's Equation.
- Brahmagupta (598-570) first to solve.

From *The New York Times*
January 18, 1931, p. 54

CATTLE PROBLEM SOLVED

Moreover, Final Conditions Set by Archimedes Can Be Worked Out

To the Editor of The New York Times:

Frank G. Nelson, whose interesting letter regarding his solution of the cattle problem of Archimedes appeared in THE TIMES, would feel flattered if he had the translation of this problem which is possessed by me, for, according to Archimedes, he is no mere "novice in numbers," since no such person could be expected to arrive at a correct solution—as has Mr. Nelson—of the first seven equations presented by the problem.

But Mr. Nelson's conclusion that the final conditions set by the problem cannot be solved is erroneous—at least according to a large number of mathematicians who have worked on it. As far back as 1860, Amthor showed that the total of the cattle would be represented by a number containing 206,545 figures, the printing of which would require about two full pages of THE NEW YORK TIMES. Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number, it is obvious that the world will never have a complete solution, which should relieve the mind of any linotype operator who fears that he might be called on to set it. However, the first thirty-one figures have been computed, as have the last twelve, and the solution, for those who are interested, is

7,760,271 081,800

in which the line of dots represents thirty solved and 206,502 unsolved numbers.

The above solution was worked out by the Hillsboro Mathematical Club of Hillsboro, Ill., which was formed by A. H. Bell in 1889 to labor on the problem. Nearly four years were spent by the three club members on the work, and the results were published in the American Mathematical Monthly in 1895. An interesting summary of the mathematical steps involved in the determination of these enormous numbers—there are ten altogether, each containing 206,544 or 206,545 figures—is to be found in Recreations in Mathematics, by H. E. Licks (Van Nostrand, 1917).

Archimedes was evidently fond of problems involving enormous numbers, as his book "Arenarius" discusses the solution of the problem of determining the number of grains of sand in a sphere the size of the earth. This number is, however, of insignificant size in comparison with that representing the solution of the cattle problem. In fact, it has been calculated that if the cattle represented by this number were reduced to the size of the smallest bacterium, they could not be contained in a sphere having the diameter of the Milky Way, across which astronomers calculate that it takes light, traveling at about 186,000 miles a second, 10,000 years to travel.

NORMAN MERRIMAN.
New York, Jan. 12, 1931.

Pell's Equation General Solution

- $x^2 - ny^2 = 1$,
  n is a nonsquare integer and
  x, y are integer solutions.
- Let $z = x + y\sqrt{n}$, $x, y \in \mathbb{Z}$
  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.

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Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

Then

$$\begin{aligned} x_3 &= x_1x_2 + ny_1y_2, \\ y_3 &= x_1y_2 + x_2y_1. \end{aligned}$$

Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

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- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.
- Example: $x^2 - 3y^2 = 1$
- Guess $(2, 1)$.
  So, $z = 2 + \sqrt{3} = w$.

$$\begin{aligned} zw &= \left(2 + \sqrt{3}\right)^2 \\ &= 7 + 4\sqrt{3}. \end{aligned}$$

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(x_3, y_3) is a solution.

Then, $(7, 4)$ is a solution.

Eudoxus of Cnidus (c.390 – c. 337 BCE)

- Studied under Plato.
- Taught Aristotle.
- Astronomer, Mathematician.
- Theory of Proportions:

  Circles: $A \propto r^2$,
  Spheres: $V \propto r^3$,
  Volume of a pyramid .
  Volume of a cone.
- Studied reals, continuous quantities.
- Method of Exhaustion:
  Due to Antiphon (480–411 BCE).
  Area from a sequence of inscribed polygons.

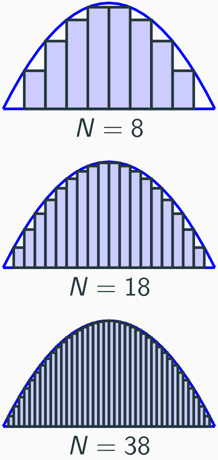
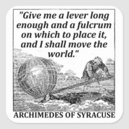
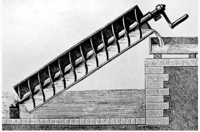


Figure 2: Method of Exhaustion.

Archimedes of Syracuse (287-212 BCE)

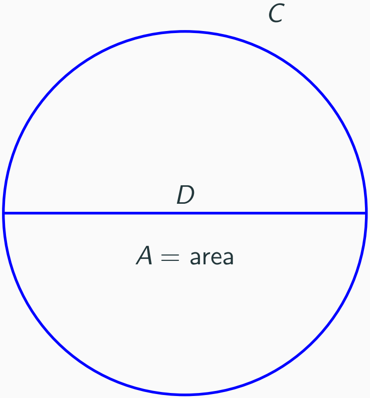
- Went to Alexandria, Egypt
  then, back to Syracuse, Sicily.
- Greatest Mathematician of Antiquity.
- Mathematician, Engineer, Inventor.
  - Archimedean screw, lever, pulley.
- King Heiro II's crown - Eureka.
  Archimedes Principle of Bouyancy.
- According to Plutarch (46-120):
  - Marcellus - Syracuse 212 BCE.
  - Claw of Archimedes.
  - Heat Ray.
  - Prone to intense concentration.
  - Death of Archimedes.





Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion.
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



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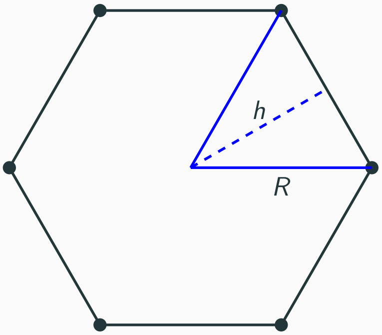
- Regular Polygons (n-gon)
  $A_p = \frac{1}{2}hQ$, $Q =$ Perimeter.



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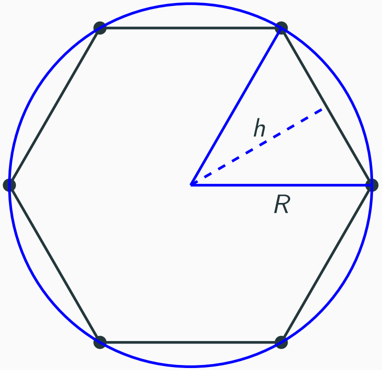
- Inscribed Polygons
  $A_p = \frac{1}{2}anh < \text{area of circle}$.



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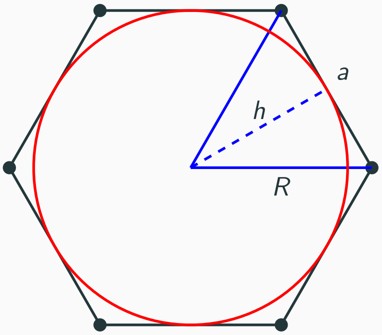
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.



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- Approximation of π,
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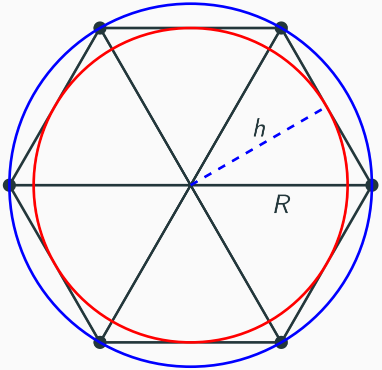
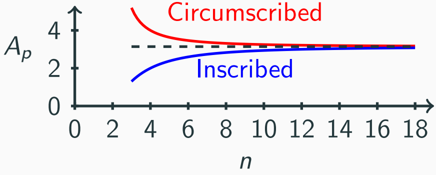


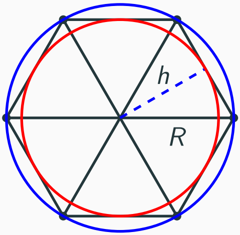
Estimating π

- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$,
- Recall
  $a = 2R \sin \frac{180}{n}$,
  $h = \sqrt{R^2 - \frac{a^2}{4}} = R \cos \frac{180}{n}$,
  $A_p = \frac{1}{2} anh = nhR \sin \frac{180}{n}$.
- Therefore,

$$\frac{n}{2} \sin \frac{360}{n} < \pi < n \tan \frac{180}{n},$$

- Hexagon ($n = 6$),
  $2.598 < \pi < 3.464$.
- Archimedes - up to 96-gon
  $3.1394 < \pi < 3.1427$.





Archimedes' Inscribed and Circumscribed n-gons

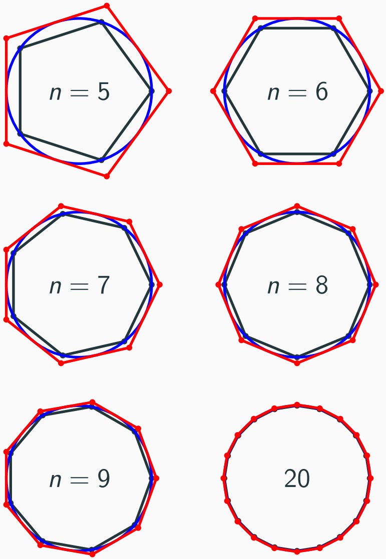
Consider a fixed circle of radius R.

- Inscribed n-gon: $h = R\cos\frac{180}{n}$, $A_i = nhR\sin\frac{180}{n}$.
- Circumscribed n-gon: $r = \frac{R}{\cos\frac{180}{n}}$, $A_c = nHr\sin\frac{180}{n}$.
- Thus,

$$\begin{aligned} A_i &= nR^2\tan\frac{180}{n}, \\ A_c &= \frac{n}{2}R^2\sin\frac{360}{n}. \end{aligned}$$

- This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$

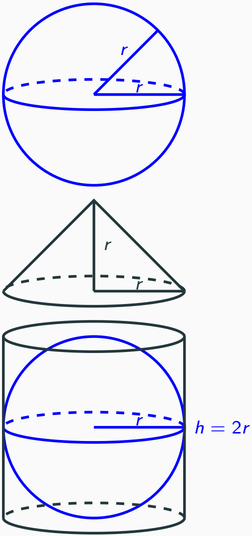


Early Approximations of π: Peripherion $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$

- Bible, $\pi \approx 3$.
- Babylonian $3 + \frac{1}{8}$.
- Egyptians, $\left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.1604938$.
- Sulbasutrakaras ($<$ 800 BCE), 3.08.
- Archimedes (250 BCE) $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
- Aryabhata (499), $\frac{62832}{20000}$.
- Ptolemy (150), 360-gon, 3.14166.
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$.
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$.
- Viete', 393,216-gon, π to 9 places.
- van Ceulen (1540-1610) Dutch, 35 places.
- William Shanks (1873) 527 digits.
- Lambert (1728-1777) - irrationality proof.
- William Jones (1706) introduced π.
- Euler popularized notation.
- See Approximations of π .
- Leibniz-Madhava $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$
- Euler $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$
- Ramanujan $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!}{k!^4} \frac{1103 + 26390k}{396^{4k}}$

On the Sphere and the Cylinder, **Archimedes**

- Spirals, Area of Parabolae.
- Volumes and Surface Areas of 3D Objects.
- A(Sphere) = 4 A(Great circle)
  $A = 4(\pi r^2)$.
- V(Sphere) = 4 V(Cone)
  $V = 4\left(\frac{1}{3}\pi r^3\right)$.
- Sphere inside Cylinder
  Cylinder Area =
  $2\pi r(2r) + 2(\pi r^2) = 6\pi r^2$
  $= \frac{3}{2}$ Sphere Area.
  Volume $= (\pi r^2)h = 2\pi r^3$
  $= \frac{3}{2}$ Sphere Volume.



Archimedes' Manuscripts

- What we know is from 3 books.
- Codex A Lost in 1564.
- Codex B Lost in 1311.
- Codex C Discovered 1906.
  - 4th century Parchment bound.
  - 10th Century Book, Constantinople housed great texts.
  - 1204 4th Crusade destroyed books.
  - 87 Sheets (43.5 goat skins).
  - 1229 Century taken apart, scraped, cut in half, written over with Christian prayer.
  - Moved to Palestine, 400 yrs.

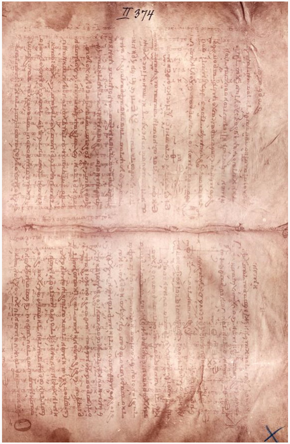


Figure 3: Codex C Page.

The Walters Museum - `http://www.archimedespalimpsest.net`

Palimpsest, http://www.archimedespalimpsest.org/about/

- 1846, It was in Istanbul, leaf removed to Cambridge.
- 1906, Johan Heiberg took pictures and translated.
- 1922, It went missing.
- 1998, Sold for $2,000,000 - Christies of NY auction.
  Moldy, Charred,
- 7 Manuscripts

  The Equilibrium of Planes, Spiral Lines, The Measurement of the Circle, Sphere and Cylinder, On Floating Bodies, The Method of Mechanical Theorems, and the Stomachion.

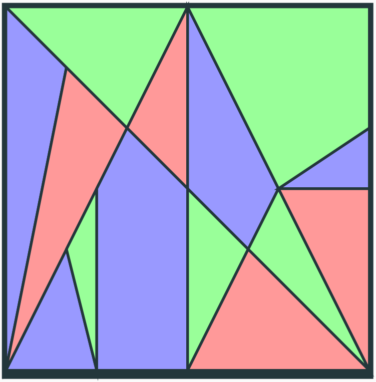


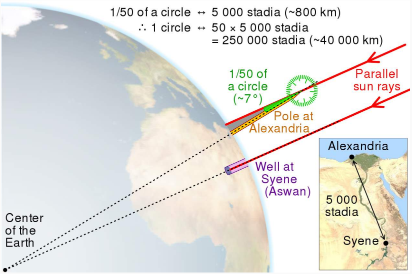
Figure 4: Number of different arrangements of the Stomachion, 17,152.

Eratosthenes of Cyrene (276 - 194 BCE)

- Chief Librarian at Library of Alexandria. (300 BCE-?). Burned 641 by Arabs?
- Sieve of Eratosthenes
- Finding primes:

$$\not{1}, 2, 3, \not{4}, 5, \not{6}, 7, \not{8}, \not{9}, \not{10}, 11, \ldots$$

- Circumference of Earth:
    - Syrene - 1st day summer Sun directly overhead.
    - Alexandria - small shadow.
    - 250,000 stadia.
      1 stade $\approx$ 526.37 ft
      Equals $24,466$ mi.
      Current - $24,860$ mi.



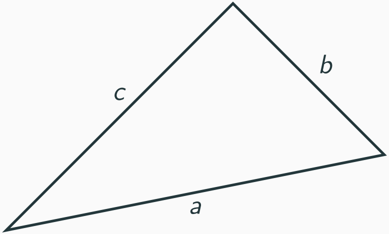
Heron of Alexandria (c. 10-70)

- Inventor.
- Aeolipile,rocket-like reaction engine.
- First-recorded steam engine.
- Hero's wind-powered organ.
- The first vending machine.
- A wind-wheel operating an organ,
- The force pump.
- A syringe-like device.
- Principle of the shortest path of light:
- Standalone fountain.
- A programmable cart.

Heron's Formula

$$A = \sqrt{(s-a)(s-b)(s-c)}$$

$$S = \frac{1}{2}(a+b+c)$$



Last of the Ancient Greek Mathematicians

- Ptolemy (100-170)
  - Astronomy.
  - Copernicus (1300).
  - Heliocentric vs geocentric.
- Diophantus (200's)
  - Equations with integer solutions.
  - Series of 13 books, *Arithmetica*, - algebraic equations.
- Hypatia (370-415)
  - Father - Theon.
  - Martyr.
  - Movie - *Agora*.

Romans - Little contribution to mathematics.

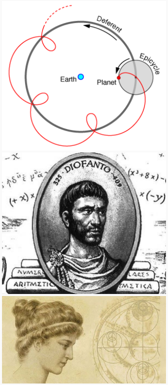
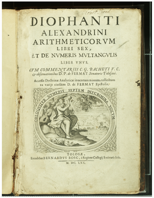


Figure 5: Epicycles, Diophantus, and Hypatia.

Diophantus' Epitaph



"Here lies Diophantus.

God gave him his boyhood one-sixth of his life;

One twelfth more as youth while whiskers grew rife;

And then yet one-seventh 'ere marriage begun.

In five years there came a bouncing new son;

Alas, the dear child of master and sage,

After attaining half the measure of his father's life, chill fate took him.

After consoling his fate by the science of numbers for four years, he ended his life."

- *Metrodorus, Greek Anthology.*

One Last Thing - The Antikythera Mechanism

- Found in a wreck in 1900 near Antikythera.
- Recovered statues and other items.
- Small corroded fragments found.
- Fragment A has 27 gears.
- 100 years later ...
- It is an astronomical calculator.
- Predicts moon's position and phase, solar eclipses, motion of planets, and more.
- Possibly from 1st or 2nd century BCE.
- See March 2021 Paper, Freeth, et al.

Timeline of Greek Mathematicians