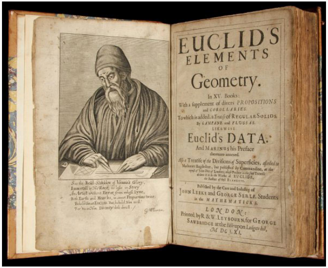
Euclid's Elements

Fall 2022 - R. L. Herman



Euclid of Alexandria (c. 325 - c. 265 BCE)

- Founder of Geometry.
- Active in Alexandria, Egypt during reign of Ptolemy I (323–283 BC).
- *Elements of Geometry*
    - Most famous mathematical work of classical antiquity.
    - World's oldest continuously used mathematical textbook.
    - Geometry, proportion, and number theory.
    - 13 Books.
    - 465 Propositions.
    - 23 Definitions. (point, line, straight line, . . .)
    - 5 Postulates.
    - 5 Axioms.

Figure 1: Euclid.

The Thirteen Books

Book 1 Fundamental propositions of plane geometry.

- Congruent triangles.
- Theorems on parallel lines.
- Sum of the angles of a triangle.
- The Pythagorean theorem.

Book 2 Geometric algebra.

Book 3 Properties of circles.

- Theorems on tangents and inscribed angles.

Book 4 Inscribed and circumscribed regular polygons around circles.

Book 5 Arithmetic theory of proportion.

Book 6 Theory of proportion in plane geometry.

Book 7 Elementary number theory.

- prime numbers, greatest common denominators, etc.

Book 8 Geometric series.

Book 9 Applications and theorems on the infinitude of prime numbers, and the sum of a geometric series.

The Thirteen Books

Book 10 Incommensurable (irrational) magnitudes using the "Method of Exhaustion." [Eudoxus (390-337 BCE), Antiphon (480-411 BCE).]

Book 11 Propositions of three-dimensional geometry.

Book 12 Relative volumes of cones, pyramids, cylinders, and spheres using the Method of Exhaustion.

Book 13 The five Platonic solids.

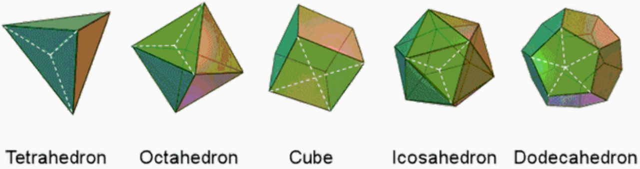


Figure 2: Platonic Solids.

Definitions i

Def 1. A point is that which has no part.

Def 2. A line is breadthless length.

Def 3. The ends of a line are points.

Def 4. A straight line is a line which lies evenly with the points on itself.

Def 5. A surface is that which has length and breadth only.

Def 6. The edges of a surface are lines.

Def 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Def 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Def 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Definitions ii

Def 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Def 11. An obtuse angle is an angle greater than a right angle.

Def 12. An acute angle is an angle less than a right angle.

Def 13. A boundary is that which is an extremity of anything.

Def 14. A figure is that which is contained by any boundary or boundaries.

Def 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Def 16. And the point is called the center of the circle.

Definitions iii

Def 17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Def 18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Def 19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Def 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definitions iv

Def 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Def 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Def 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

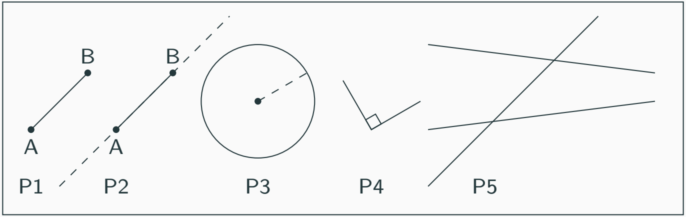
Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



Common Notions

Notion 1. Things which equal the same thing also equal one another.

Notion 2. If equals are added to equals, then the wholes are equal.

Notion 3. If equals are subtracted from equals, then the remainders are equal.

Notion 4. Things which coincide with one another equal one another.

Notion 5. The whole is greater than the part.

Proposition 1

To construct an equilateral triangle on a given finite straight line.

- Start with segment AB.

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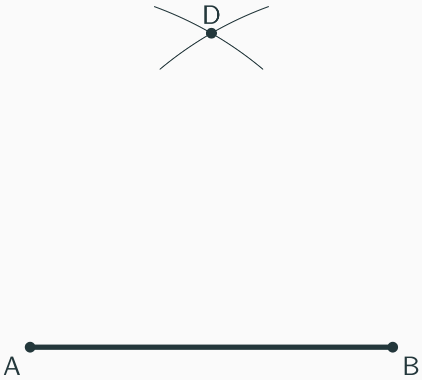
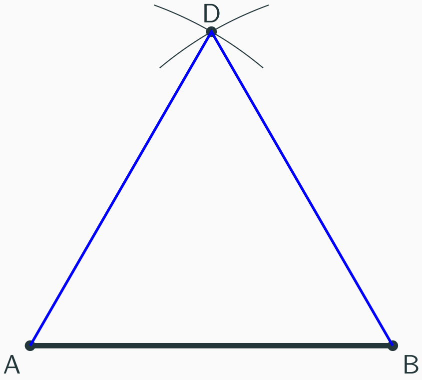


Proposition 1

To construct an equilateral triangle on a given finite straight line.

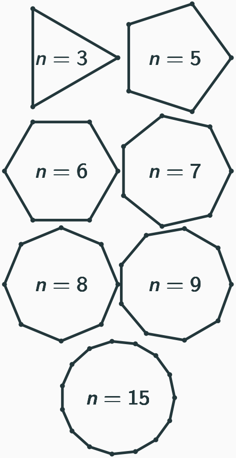
- Start with segment AB.
- Draw circular arcs about A, B of radius AB.
- Draw line segments AD, BD.



Regular Polygons

Construct using straight edge and compass.

- Triangles (Euclid I.1)
- Squares (Euclid I.46)
- Pentagons (Euclid IV.11)
- Hexagons (Euclid IV.15)
- Septagon (heptagon) (no)
- Octagon (Euclid III.30)
- Nonagon (no)
- 15-gon (Euclid IV.16) pentadecagon
- Double the number of sides of a given regular polygon, 8, 10, 12, 16, 20, 24, etc. (Euclid III.30)



Constructible regular n-gons

Is it possible to construct all regular polygons with compass and straightedge? If not, which n-gons (that is polygons with n edges) are constructible and which are not?

- Young C. F. Gauss, 1796: the regular 17-gon is constructible.
- Theory of Gaussian periods in his *Disquisitiones Arithmeticae*. 1801.
- Gave sufficient condition for the constructibility.
- Proof of necessity - Pierre Wantzel in 1837.
- Gauss–Wantzel theorem:
  A regular n-gon can be constructed with compass and straightedge if and only if n is the product of a power of 2 and any number of distinct Fermat primes, p_ℓ:[only 3, 5, 17, 257, 65537.]

$$n = 2^m p_1 p_2 \cdots p_k, \qquad p_\ell = 2^{2^\ell} + 1.$$

Book 13 - Platonic Solids - Regular Polyhedra

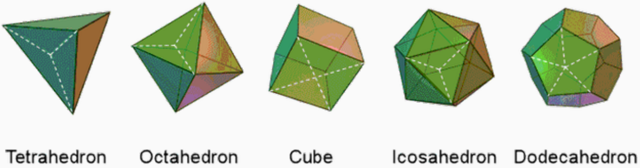


Figure 3: Platonic Solids: Fire, Air, Earth, Water, Universe.

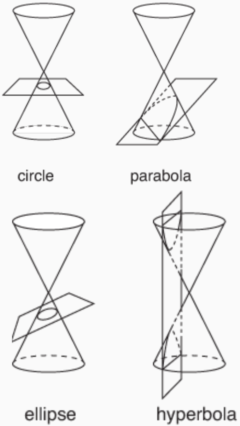
Polyhedron	Faces	Edges	Vertices
Tetrahedron	4 $\Delta's$	6	4
Cube	6 $\square$'s	12	8
Octahedron	8 $\Delta's$	4	6
Dodecahedron	12 Pentagons	30	20
Icosahedron	20 $\Delta's$	30	12

Note: Johannes Kepler (1571-1630) systematized and extended what was known about polyhedra. See *Harmonice Mundi*, 1619. Proposed relationships between six known planets and the Platonic solids.

Conic Sections

Possibly discovered by **Menaechmus**[1] (380–320 BCE) to duplicate cube: Intersect parabola $y = \frac{1}{2}x^2$ and hyperbola $xy = 1$.

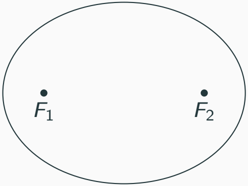
- **Euclid** - four lost books on conics.
- **Archimedes** of Syracuse (287-212 BCE) studied conics, area bounded by a parabola and a chord in *Quadrature of the Parabola*.
- **Apollonius** of Perga (262-190 BCE), eight-volume *Conics*.
  Terms: parabola, ellipse, hyperbola
- **Pappus** of Alexandria (290 – 350) - focus directrix.
- Applied by Kepler (1609), Newton (1687).



[1] To Alexander, *"O king, for travelling through the country there are private roads and royal roads, but in geometry there is one road for all."*

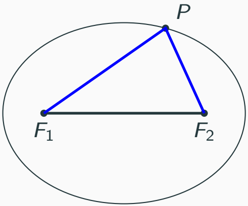
Ellipse

- Focal points: F_1, F_2.



Ellipse

- Focal points: F_1, F_2.
- $\overline{F_1P} + \overline{F_2P} = 2a$.



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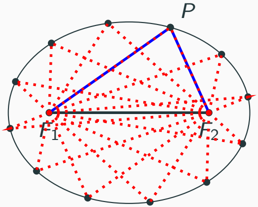
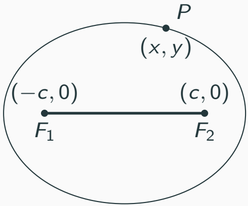


Ellipse

- Focal points: F_1, F_2.
- $\overline{F_1 P} + \overline{F_2 P} = 2a$.
- Algebra leads to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

a, $b =$ semimajor/semiminor axes
with $c = \sqrt{a^2 - b^2}$, $a > b$.



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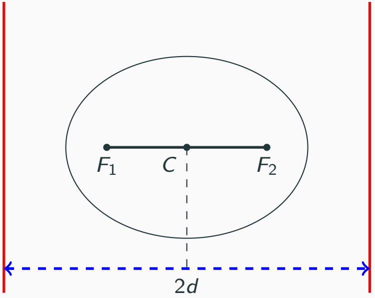
- Directrix $d = \frac{a^2}{c}$,



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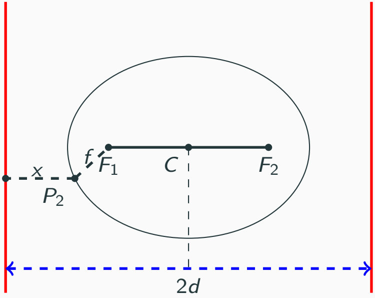
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.



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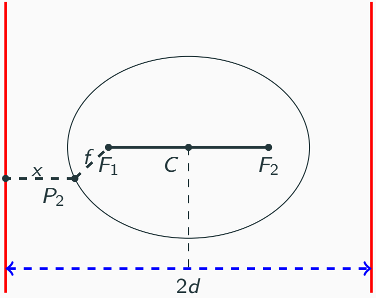
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.
- Eccentricty of Conics:

$\epsilon = 0$, circle.

$0 < \epsilon < 1$, ellipse.

$\epsilon = 1$, parabola.

$\epsilon > 1$, hyperbola.



Dandelin Sphere - Germinal Pierre Dandelin (1794-1847)

- Inscribed spheres tangent to cone and intersecting plane.
- Intersection is a conic.
- Tangent pts to sphere are focal points.
- Used to prove theorems of Apollonius.
  - Conic section is the set points such that the sum of the distances to two fixed points is constant.
  - The distance from the focus is proportional to the distance from a fixed line (directrix).
  - The constant of proportionality is the eccentricity

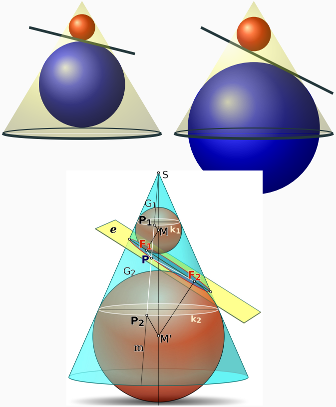
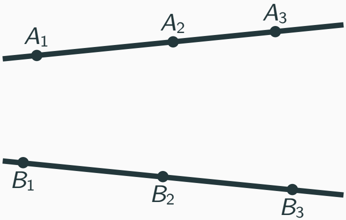


Figure 4: Dandelin Spheres (Paper in 1822)

Other Geometry Gems

- Pappas' Hexagon Theorem (290-350)
  - Connect 6 pts on two lines.
  - A_1-B_2, B_2-A_1, etc.



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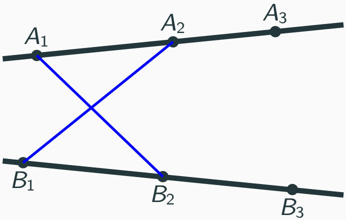

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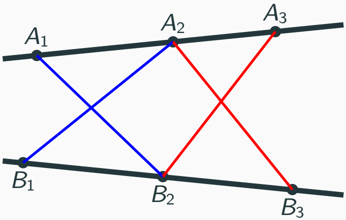

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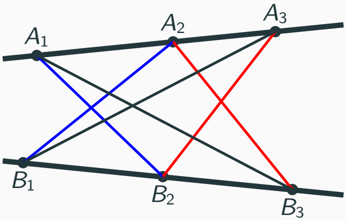

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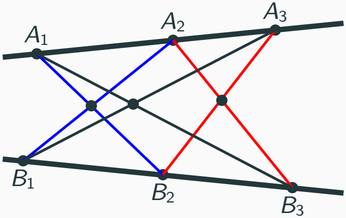
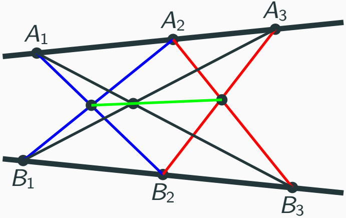


Other Geometry Gems

- Pappas' Hexagon Theorem (290-350)
  - Connect 6 pts on two lines.
  - A_1-B_2, B_2-A_1, etc.
- The three points are collinear.
- The beginning of projective geometry.



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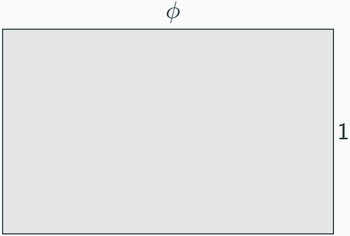

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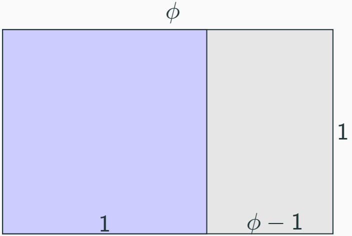
  - Golden Rectangle: $\frac{\phi}{1} = \frac{1}{\phi - 1}$. [Gray region is similar to the large rectangle.]



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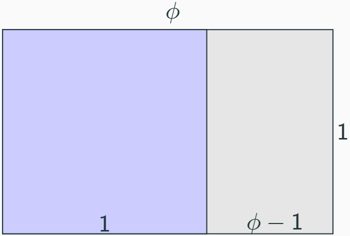
    - Solution $\phi^2 = \phi + 1$:



$$\phi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \approx 1.61803\ldots.$$

The Parthenon

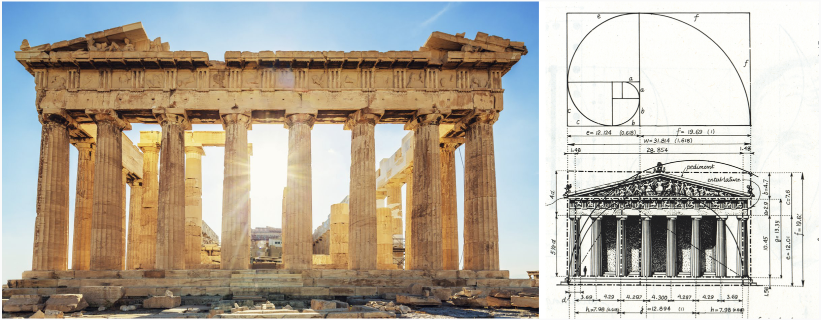


Figure 5: The Parthenon and the Golden Rectangle