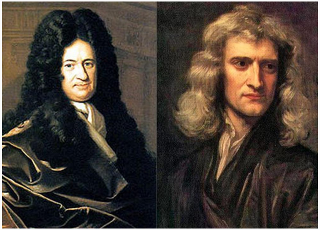
Emergence of Calculus

Fall 2022 - R. L. Herman



The Method of Exhaustion and the Infinite

- Zeno's Paradox of the Arrow

  "If a body moves from A to B, then before it reaches B it passes through the mid-point, say B_1 of AB. Now to move to B_1 it must first reach the mid-point B_2 of AB_1. Continue this argument to see that A must move through an infinite number of distances and so cannot move. " (450 BCE)
- Eudoxus - Method of Exhaustion.
- Archimedes - area of a segment of a parabola is $\frac{4}{3}$ the area of a triangle with the same base and vertex.
- Luca Valerio (1552-1618) published in 1608 *De quadratura parabolae.*
- Kepler (1571-1630): area as sum of lines. Inpsired Cavalieri's indivisibles.

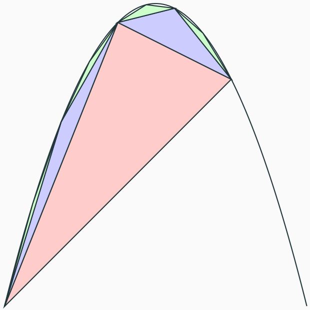


Figure 1: Archimedes: 1st known summation of series.

Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Roberval (1602-1675)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

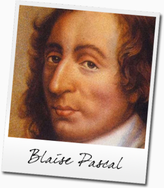
- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Seventeenth Century - French, German, English Mathematics

- Galileo Galilei (1564-1642)
- Johannes Kepler (1571-1630)
- 1590 Vièta, *The Analytic Art*
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)



- Descartes

  philosopher, mathematician
  Discours de la méthode, Marriage of algebra/geometry - analytic geometry
- Pascal

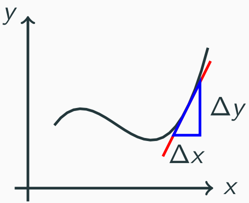
  Wrote math before 16
  Probability theory
  Theology
- Fermat

  Created analytic geometry
  Contributions to Calculus
  Number theory
  Scribbled in Diophantus' *Arithmetica*

Tangents

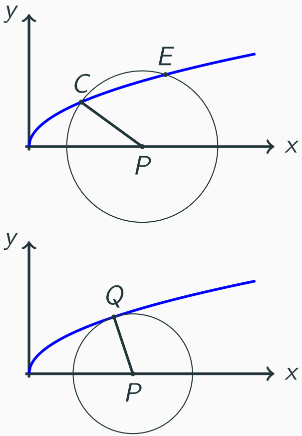
- Pierre de Fermat, René Descartes
- Both studied Apollonius 4-line problem.
- Tangent line approximates the curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter from Descartes to Mersenne
  $dy = f(x + dx) - f(x) = ?dx.$

Figure 2: Fermat and Descartes



Descartes vs Fermat - Analytic Geometry, Tangents, Optics

- Decartes published *La Géométrie* - 1637
- Depicted $ax + by = c$ as a line.
- Introduced x, y.
- Fermat introduced analytic geometry earlier.
- Fermat interested in optimization.
- Fermat: lawyer in Toulouse, Math a hobby.
- Descartes denounced him and challenged him to find tangent to folium, $x^3 + y^3 = 3axy$.
- Descartes' Method of Tangents: Find circles tangent to curves.
- Fermat challenged Descartes to explain refraction. Fermat published in 1662.



Areas Under Curves

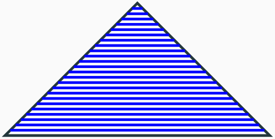
- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - indivisibles

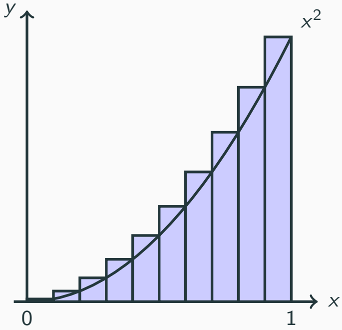
  Fill area with lines.
  But, an infinite number of lines sum to infinity.
- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\,dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

$$A \approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2.$$





Areas Under Curves (cont'd)

Find the sum

$$\begin{aligned} A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\ &= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\ &= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\ &\sim \frac{2N^3}{6N^3} = \frac{1}{3}. \end{aligned}$$

Wallis showed

$$\boxed{\int_0^a x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^a = \frac{a^{n+1}}{n+1}}$$

for $k = 1, 2, \ldots, 9$.

Note:

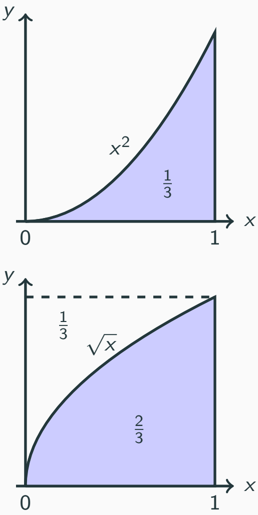
$$\begin{aligned} \sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\ &= N\frac{N+1}{2}. \end{aligned}$$

Figure 3: John Wallis

Integrating Powers, $\int x^k\,dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  Ex: $\int_0^1 \sqrt{x}\,dx$
  Use the symmetry in the figures.
- Areas under x^k, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.

Note: $1^3 + 2^3 + \cdots + k^3 = (1 + 2 + \cdots + k)^3$.



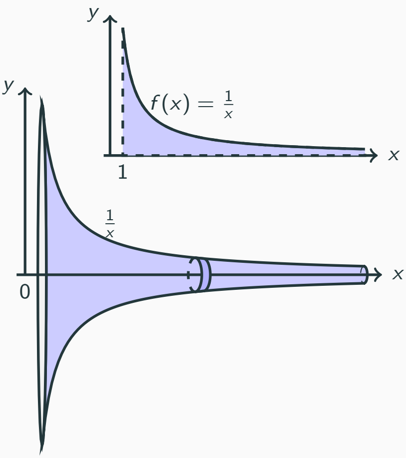
Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

$$\int_1^\infty \frac{1}{x}\, dx = \infty.$$

- 1641 Torricelli's trumpet (Gabriel's horn)

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\, dx = \pi. \\ A &= 2\pi \int_1^\infty \frac{dx}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2} \\ &> 2\pi \int_1^\infty \frac{1}{x}\, dx = \infty. \end{aligned}$$



What? You cannot paint the surface but can fill the trumpet with paint.

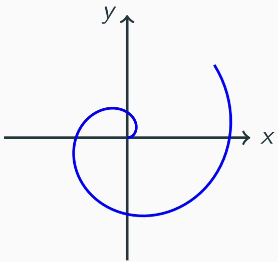
Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

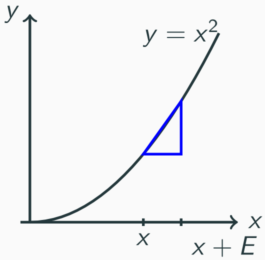
Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x+E)^2 - x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.





John Wallis' (1655) *Arithmetica Infinitorium*

- Combined Descartes' analytic geometry and Cavalieri's indivisibles.
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals - attacked by Thomas Hobbes (1588-1679).
- Formulae for π known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\begin{aligned}
\frac{\pi}{4} &= \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\\
\frac{\pi}{2} &= \left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdot\cdots\\
\frac{4}{\pi} &= 1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cdots}}}
\end{aligned}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola* Written in 1665, 1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$ replaced by x^n.
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

[Note: Here $\log x = \ln x$.]

Note: Geometric series

$$\begin{aligned} 1 + t + t^2 + \cdots &= \frac{1}{1-t}, |t| < 1. \\ 1 - t + t^2 - \cdots &= \frac{1}{1+t}, |t| < 1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

Invert Power Series

We have for $y = \log(1+x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$. Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of y, then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned}
0 &= a_0 - \frac{1}{2}a_0^3 + \frac{1}{3}a_0^3 \\
1 &= a_1\left(a_0^2 - a_0 + 1\right) \\
0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\
0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right).
\end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!}.$$

Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

However,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So, we found the series expansion

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

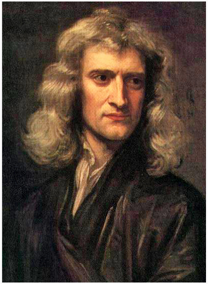


Figure 4: Newton

Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{k=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. Then,

$$\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!}a^2 + \frac{p(p-1)(p-2)}{3!}a^3 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots + \frac{-\frac{1}{2}(-\frac{3}{2})\cdots(\frac{1}{2}-k)}{k!}(-t^2)^k + \cdots \right) \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\frac{x^{2k+1}}{2k+1} + \cdots
\end{aligned}$$

Inverting, Newton found $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$.

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx} \int f(x)\, dx = f(x)$.

Figure 5: Leibniz

Infinite Series

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a+ar+ar^2+\cdots+ar^n+\cdots = \frac{a}{1-r}, |r| < 1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\cdots \\ = & (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ \geq & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, ...

Figure 6: Euler

Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=?$

Figure 7: Jacob and Johann

$$\begin{aligned}
\sum_{n=1}^{N}\frac{1}{n(n+1)} &= \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\left(\frac{1}{N}-\frac{1}{N+1}\right)\\
&= \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots\left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right)\\
&= 1-\frac{1}{N+1}\underset{N\to\infty}{\Longrightarrow}1.
\end{aligned}$$

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies $p(x) = (x - r)q(x)$, $q(x)$ - polynomial

Proof:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi,\ n = 0, \pm 1, \pm 2, \ldots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\cdots \\
&= Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x - \frac{x^3}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + x^5(\cdots) - \cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

What are the next coefficients?

We need the x^5 terms in the expansion

$$\begin{aligned}\sin x &= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\ &= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots\left(1-\frac{x^2}{m^2\pi^2}\right)\cdots\end{aligned}$$

We multiply $\frac{x^2}{m^2\pi^2}$ times the factors $\frac{x^2}{n^2\pi^2}$, $n\neq m$, and summing:

$$x\sum_{m=1}^{\infty}\frac{x^2}{m^2\pi^2}\sum_{n=1,n\neq m}^{\infty}\frac{x^2}{n^2\pi^2}=\frac{x^5}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1,n\neq m}^{\infty}\frac{1}{n^2}.$$

$$\frac{\pi^4}{5!}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2}\left[\zeta(2)-\frac{1}{m^2}\right]=\frac{1}{2}[\zeta(2)^2-\zeta(4)].$$

So, $\zeta(4)\equiv\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{36}-\frac{\pi^4}{60}=\frac{1}{12}\left(\frac{\pi^4}{3}-\frac{\pi^4}{5}\right)=\frac{\pi^4}{90}$.

Another Approach to Obtain $\zeta(4)$

Noting that $\frac{d}{dx}(\ln \sin x) = \cot x$ and using the known series expansion for $x \cot x$ in terms of Bernoulli numbers,

$$
\begin{aligned}
\sin x &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
\ln \sin x &= \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
x \cot x &= 1 - \sum_{n=1}^{\infty} \frac{2x^2}{n^2\pi^2 - x^2} \\
&= 1 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{x^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{x^2}{n^2\pi^2}\right)^k \\
1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} + \dots &= 1 - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{\pi}\right)^{2k+2} \zeta(2k+2) \\
x \cot x &= 1 + \sum_{k=0}^{\infty} (-1)^k B_{2k} (2x)^{2k}
\end{aligned}
$$

Results for the Riemann Zeta Function, $\zeta(s)$

Therefore, we have

$$\begin{aligned}\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots &= \frac{\pi^2}{6},\\ \zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots &= (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},\end{aligned}$$

where B_{2n} are Bernoulli numbers,[1] $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots,$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Euler (1748) - Zeta function can be defined for p prime as

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\\ &= \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1}\cdots\left(1 - \frac{1}{p^s}\right)^{-1}\cdot.\end{aligned}$$

[1] Jacob Bernoulli, 1713, Seki Takakazu, 1712, published posthumously. $B_0 = 1, B_1 = -\frac{1}{2}$.

Georg Friedrich Bernhard Riemann[2] (1826-1866)

- Riemann extended Euler's zeta function, $s \in \mathbb{C}$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

$$\begin{aligned} &\zeta(1) = \infty, \text{ harmonic series} \\ &\zeta(2) = \tfrac{\pi^2}{6} \\ &\zeta(3) \quad \text{irrational, Apery (1981)} \end{aligned}$$

- Zeros

$$\zeta(-2n) = 0, \; n \text{ integer} > 0.$$

Riemann Hypothesis:

$$\zeta(\sigma + it) = 0 \text{ when } \sigma = \tfrac{1}{2}.$$

- Connection to primes?

Figure 8: Bernhard Riemann

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(2s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \\ \zeta(2n) &= \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \\ \zeta(s) &= 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s) \\ \Gamma(s) &= \int_0^\infty x^{s-1} e^{-x}\, dx \end{aligned}$$

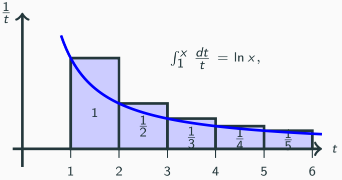
[2] *On the Number of Primes Less Than a Given Magnitude*, 1859

Connection to Primes and Other Tidbits

$$\zeta(s) = \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\cdots\left(1-\frac{1}{p^s}\right)^{-1}\cdots$$
$$= \prod_{p=\text{prime}}\left(1-\frac{1}{p^s}\right)^{-1}$$
$$= \prod_{p=\text{prime}}\left[1+\frac{1}{p^s}+\left(\frac{1}{p^s}\right)^2+\cdots\right]$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant $\gamma \approx 0.577218\ldots$.
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$



$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right) = \gamma.$$

Leonhard Euler (1707-1783)

Euler (at 14) studied under Johann Bernoulli, graduated in 1723.

Went to St. Petersburg in 1727, Berlin in 1741, and back to St. Petersburg in 1766.

By 1730's - lost vision in right eye and blind by 1771.

866 books and papers - 228 after death. *Opera Omnia* - over 25,000 pgs

First appearance of e - letter to Goldbach in 1731.

Euler published *Introductio in Analysin infinitorum* - 1748

Euler's Formula, $e^{ix} = \cos x + i \sin x$.

Euler's Identity, $e^{i\pi} + 1 = 0$.

Euler's constant, γ

Euler's Polyhedral Formula, $V + F = E + 2$.

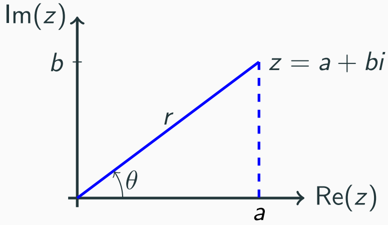
Amicable Numbers

- Recall Greeks knew 220 and 284; i.e., sum of proper factors of 220 = 284 and vice versa.
- Thabit ibn Qurra (836-901) discovered the next amicable pairs, for example 17296, 18416.
- Pierre Fermat rediscovered this pair in 1636.
- René Descartes discovered Qurra's pair 9,363,584 and 9,437,056 in 1638.
- 1747, Euler published [E100] giving 30 amicable pairs.
- By 1750 - Euler found 61 pairs!

Euler's Formula - Exponentiate $i\theta$.

- Complex numbers, polar form.
  $z = a + bi$, $a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta$.

- $\theta = \pi,\ e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\ e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n .$$

- **Example:** $n = 2$

$$\begin{aligned} \cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta. \end{aligned}$$

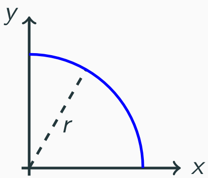
$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta. \end{aligned}$$

Rectification of a Circle

- Rectification = Finding arclengths
- $y = y(x)$

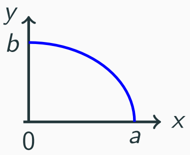
$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

- **Example:** Circle: $x^2 + y^2 = r^2$
  $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$



$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + \frac{x^2}{y^2}}\, dx \\ &= 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r\sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

Arclength of an Ellipse



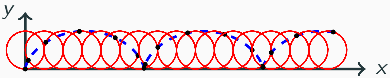
- **Example:** Ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$

$$
\begin{aligned}
y &= \frac{b}{a}\sqrt{a^2 - x^2} \\
y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\
1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\
L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx.
\end{aligned}
$$

Historical Curves

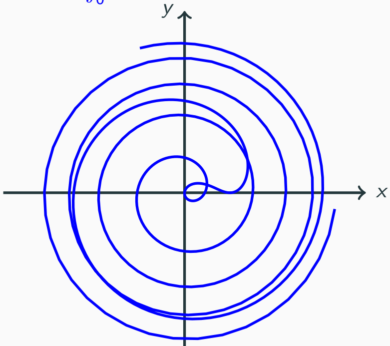
- 1609 - Kepler - Mars' orbit is an ellipse.
- 1659 - Pascal *Dimensions des lignes courbes de toutes les Roulettes.*
- 1658 Proof by Wren published by Wallis in 1659 - proof of the rectification of the cycloid.
- 1676 - Newton - infinite series.
- 1742 - Maclaurin - expansion in eccentricities.
- 1691 - Jacob Bernoulli - parabolic spiral.

Cycloid, Parabolic Spiral, and Lemniscate



$x = r(t - \sin t),\ y = r(1 - \cos t)$

$L = \int_0^{2\pi} r\sqrt{2 - 2\cos t}\, dt = 8r$

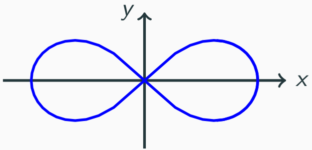


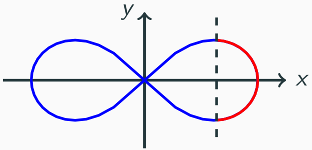
$(a - r)^2 = 2ab\theta$

$s = \int \sqrt{1 + \frac{r^2(a-r)^2}{a^2 b^2}}\, dr$

- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$





Elastica

Sep 1694, Jacob Bernoulli

Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\, dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate, 1718.
- Carl Friedrich Gauss (1777-1855) ∼1800 studied inverse $x = sl(u)$ Doubly periodic

$$\begin{aligned} sl(u + 2\bar{\omega}) &= sl(u) \\ sl(u + 2i\bar{\omega}) &= sl(u) \end{aligned}$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

Addition Theorem for Circle

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2\sin u \cos u \\ &= 2\sin u\sqrt{1-\sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2\int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2\sin u\sqrt{1-\sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Elliptic Integral Addition Theorem for Lemniscate

In 1718 Fagnano found formula for doubling arclength of lemniscate.

He solved differential equation

$$\frac{dt}{\sqrt{1-t^4}} = \frac{2}{dx}\sqrt{1-x^4}, \Rightarrow t = \frac{2x\sqrt{1-x^2}}{1+x^4}.$$

So, if the arclength of lemniscate is

$$\int_0^x \frac{dt}{\sqrt{1-t^4}},$$

then double the arclength is

$$\int_0^{\frac{2x\sqrt{1-x^2}}{1+x^4}} \frac{dt}{\sqrt{1-t^4}}.$$

Led Euler to write extensively on elliptic integrals starting in 1752.

Elliptic Integrals

- Study of Inversions
  Gauss 1790s - $\int \frac{dt}{\sqrt{1-t^3}}$,
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- 1786 (40 yrs later)
  Legendre classified elliptic integrals into 3 cases, book 1825.
  Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1 - k^2 \sin^2\theta}\, d\theta.$$

- Riemann placed in geometric setting - torus.

Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\ldots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots .$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1, 2)$. Start with $a_0 = 1, \quad b_0 = 2,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
$\vdots$	$\vdots$

$$AGM(a, b) = \frac{\pi}{4} \frac{a + b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$