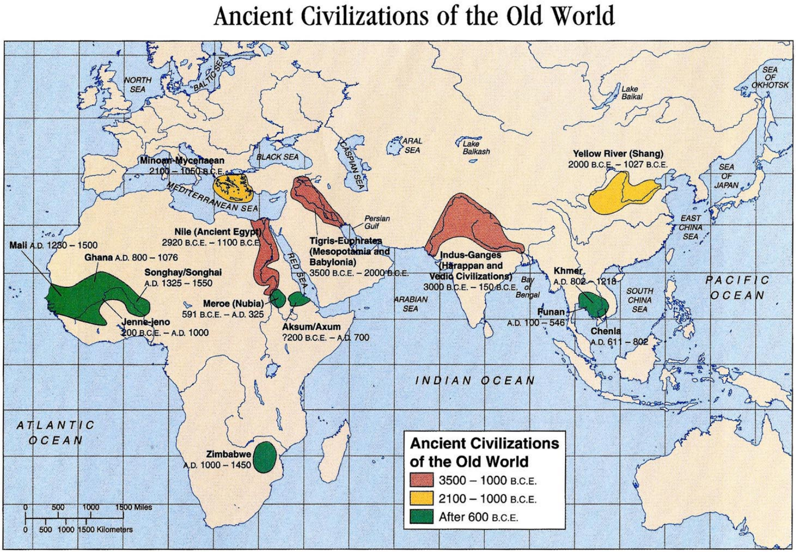
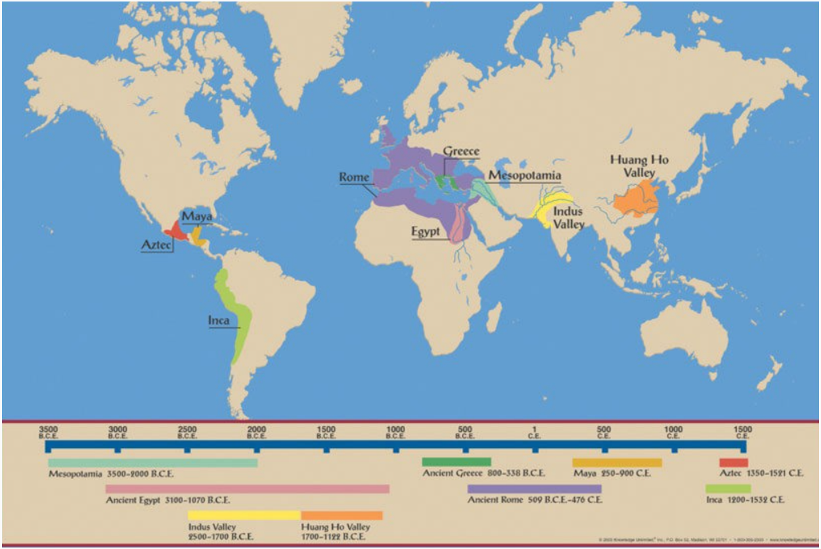
Early Mathematics - Egypt and Mesopotamia

Fall 2022 - R. L. Herman

Maps of Ancient Civilizations



Maps of Ancient Civilizations



Early Civilizations

- Egypt (3150-30 BCE)
- Mesopotamia (3100-539 BCE)
- Indus (3300-1700 BCE)

Arithmetic, Geometry,
No proofs
Problems were practical or recreational

- Greek (640 BCE-415 CE)
- Chinese (1766 BCE-220 CE)
- Indian Mathematics (500-1200)
- Islamic Mathematics (700-1200)
- Mayan Mathematics (250-900)
- Aztec Empire (c.1345-1521)
- Inca Civilization (1400-1560)

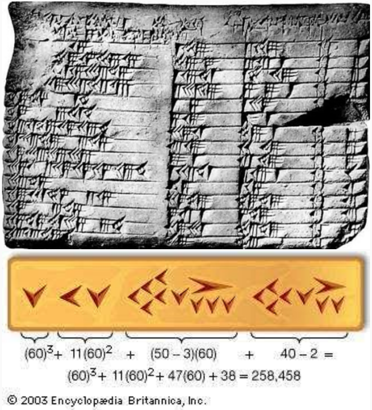
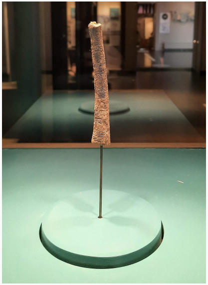


Figure 1: Babylonian tablet - Base 60

Ancient African Mathematics

- Lebombo bone, 43,000-44,200 yrs old.
  Oldest known mathematical artifact,
  29 notches on a baboon's fibula.
  Found in Border Cave, Lebombo Mountains, Swaziland.
- Ishango bone, 20,000 BCE.
  Also baboon bone.
  Ishango, Democratic Republic of Congo.
  Numerical patterns with differing interpretations.

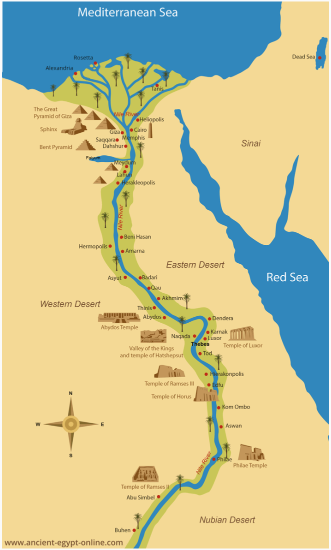
See Blog and Article



See Wikipedia.

Ancient Egypt

- Early Dynastic Period (3150–2686 BCE), writing
- Old Kingdom (2686–2181 BCE) (Great Pyramid of Giza)
- 1st Intermediate Period (2181–2055 BCE)
- Middle Kingdom (2055–1650 BCE), **Reisner Papyri** and **Moscow Papyrus**
- 2nd Intermediate Period (1650–1550 BCE), **Rhind Papyrus**
- New Kingdom (1550–1069 BCE)
- 3rd Intermediate Period (1069–664 BCE)
- Late Period (664–332 BCE)



The Papyri

- Papyri - scrolls.
  - Rhind Papyrus, 1650 BCE.
  - Moscow Papyrus, 1850 BCE.
  - Reisner Papyri, 1950 BCE.
- Reisner Papyri
  - Dr. G.A. Reisner .
  - 1901–04 - southern Egypt.
  - 4 scrolls.
  - Mostly accounts.
- Egyptian Arithmetic.
  - Base-10.
  - heiroglyphic and hieratic numerals.
  - integers, fractions.
  - surveying, building.
  - areas, volumes.

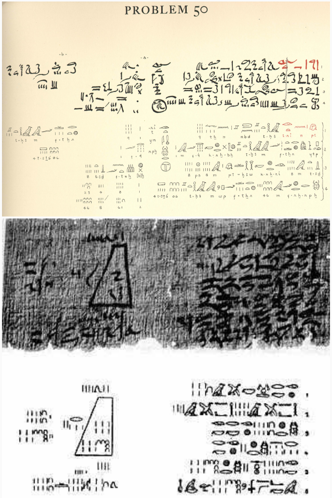
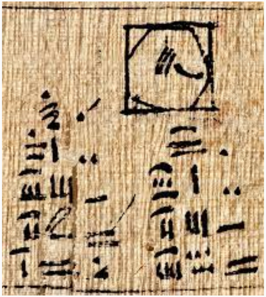


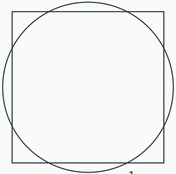
Figure 2: Papyri

The Rhind Papyrus

- Found in Thebes.
- Purchased 1858, by A. Henry Rhind.
- Size: $18' \times 13''$.
- Red and black ink.
- Geometry.
  - Areas, Volumes.
  - Ratios of sides of right triangles.
- Measures - grain.
  - 1 hekat $\approx 29,224$ in$^3 \geq \frac{1}{2}$ peck.
  - 1 ro $= \frac{1}{320}$ hekat.
- Areas of Circles - 48, 50.

$$A = \left(\frac{8}{9}D\right)^2 = \frac{256}{81}r^2 \approx 3.16049r^2.$$





$$A_{circle} = A_{square} - \frac{1}{9}A_{square}$$

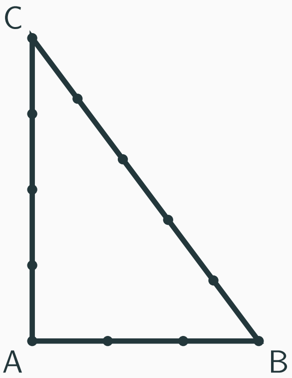
Figure 3: Problem 48

Pythagorean Triples

- Pythagorean Theorem.
- Triples (a, b, c),

$$a^2 + b^2 = c^2.$$

  Examples:
  - 3-4-5.
  - 5-12-13.
- Used to Measure Perimeters.
- Knotted Ropes.
  - Loop with 12 knots.
- Other Units:
  - Finger - 1.9 cm.
  - Palm = 4 fingers - 7.5 cm.
  - Cubit = 7 palms - 52.3 cm.



Egyptian Arithmetic

Multiplication - 17×13.

1	17
2	34
4	68
8	136
13	221

Division - $108/12$.

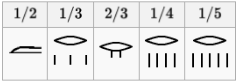
1	12
2	24
4	48
8	96
9	108

Unit fractions:

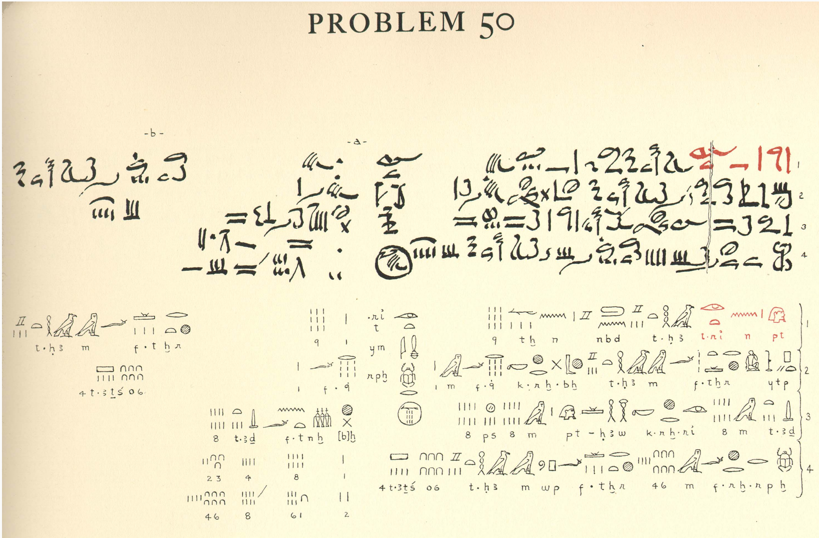
$$\frac{4}{7} = \frac{1}{2} + \frac{1}{14}.$$

Counting and Unit Fractions

Hieroglyphics for some fractions[5]



Rhind Papyrus - Problem 50



Rhind Papyrus - Problem 50

Problem 50

tp n ỉr·t ꜣḥ·t dbn n ẖt·w[1] 9 pty rḫt · f m ꜣḥ·t
Example of making a field round of khet 9. *What is the amount of it in area?*

ẖb · ḫr · k 9̇ · f m 1 ḏꜣ·t m 8 ỉr·ḫr · k wꜣḥ–tp m 8 sp 8 ḫpr·ḫr · f m 64
Take away thou ⅑ *of it, namely,* 1; *the remainder is :* 8. *Make thou the multiplication :* 8 *times* 8; *becomes it :* 64;

rḫt · f pw m ꜣḥ·t 60[2] śṯꜣ·t 4
the amount of it, this is, in area, 60 setat 4.

ỉr·t my ḫpr
The doing as it occurs:

1	9
9̇ · f	1.
of it	

ẖ[b] ḫnt · f ḏꜣ·t 8
Take away from it; the remainder is 8.

1	8
2	16
4	32
\\8	64

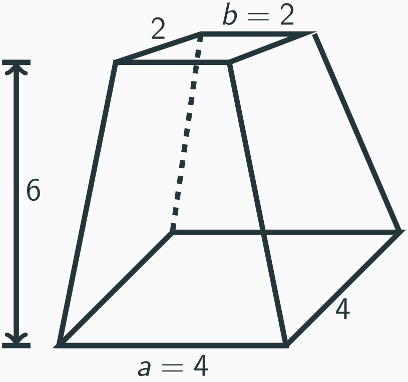
rḫt · f m ꜣḥ·t 60[2] śṯꜣ·t 4
The amount of it in area: 60 setat 4.

[1] The w suggested by the plural strokes has been omitted on the plate. The same omission occurs on the figure in Problem 51, and in Problem 52, line 2.

[2] The scribe has by mistake written here either the number 60 or the special form for 6 used in Problem 48 in writing 6 *setat*. He may have had in his mind the fact that he was actually dealing with 60 *setat* (which, however, would not properly be written in this way), and he had written the abstract number 60 a moment before at the end of the multiplication, or, remembering that 60 *setat* is written with the numeral 6, he did write 6, but used the special sign instead of the ordinary numeral.

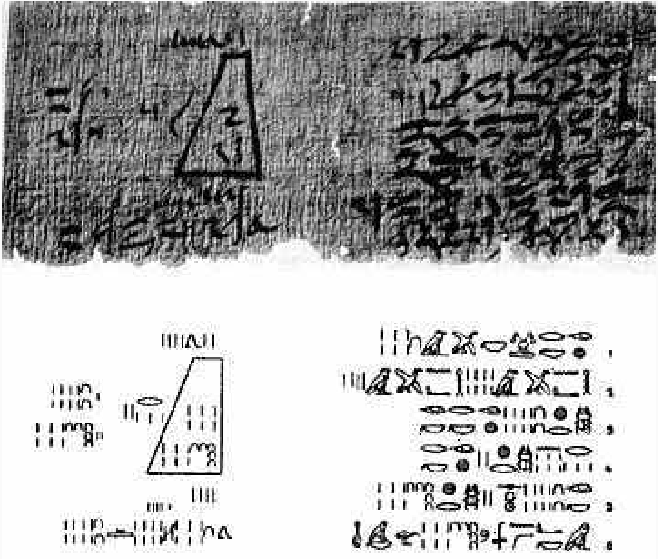
Moscow Papyrus

- From around 1850 BCE.
- Golenishchev bought in 1892 or 1893 in Thebes.
- Housed in Moscow.
- 25 Problems.
- `https://en.wikipedia.org/wiki/Moscow_Mathematical_Papyrus`
- See Problem 14:
  - Frustrum of a Pyramid



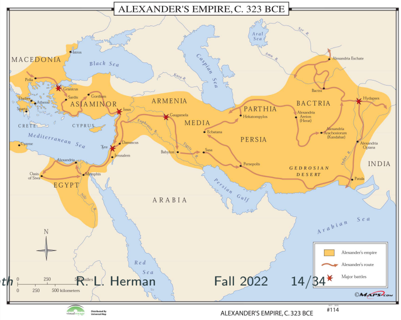
$$V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

Moscow Papyrus - Problem 14 - Frustrum of Pyramid



The Fall of the Egyptian Empire

- Argead dynasty (332–310 BCE)
  - Macedonians (700-310 BCE)
  - Alexander III of Macedon, or Alexander the Great (336–323 BCE) King of Macedonia, Pharaoh of Egypt, King of Persia and of Asia
- Ptolemaic dynasties (310–30 BCE) Cleopatra (69–30 BCE)
- Roman and Byzantine Egypt (30 BCE–641 CE)
- Sasanian (Persian) Egypt (619–629)
- Death of Mohammed (c. 570-632)
- Ruled by Caliphates (641-1517)
- Ottoman Rule (1517-1914)



Mesopotamia (2100 BCE) - Tigris and Euphrates Region

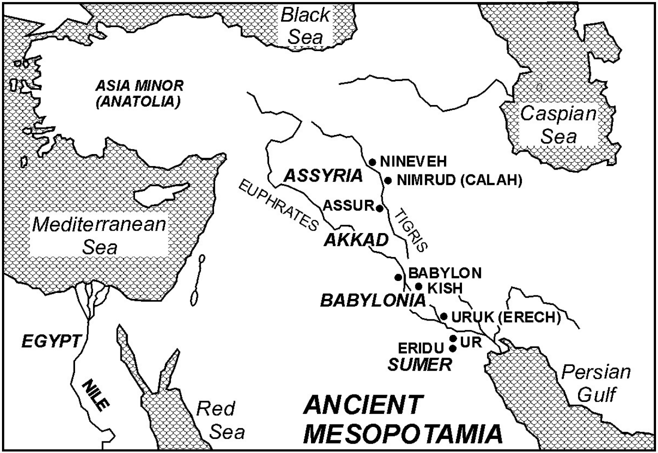


Figure 4: Tigris and Euphrates Rivers

Babylonian and Sumerian Mathematics

- More Advanced.
- Clay Tablets.
- Base 60 Arithmetic.
- Notation: $13_{60} = 1\llcorner 3 = 1/3$.
- Some use commas: $1, 3$.
- Examples:

$$\begin{aligned} 1/3 &= 1(60) + 3 = 63 \\ 1/59 &= 1(60) + 59 = 119 \\ 2/49 &= 2(60) + 49 = 169 \\ 3/31/49 &= 3(60^2) + 31(60) + 49 \end{aligned}$$

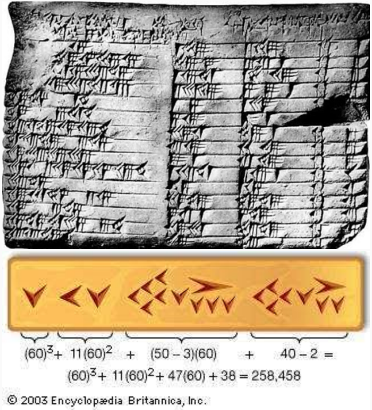


Figure 5: Babylonian tablet - Base 60

Sexagesimal Operations (Base 60)

- Ambiguities:
  - No 0's.
  - No decimal points.
- Special fractions:
  - $8.25_{10} = 8/15 = 8\frac{15}{60}$
  - $8.5_{10} = 8/30 = 8\frac{30}{60}$
  - $8.75_{10} = 8/45 = 8\frac{45}{60}$
- Addition, subtraction, multiplication.

Addition:

$$\begin{array}{r} 14/28/31 \\ +3/35/45 \\ \hline = 18/4/16. \end{array}$$

Multiplication -

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right].$$

No division! - Use reciprocals: See Old Babylonian Multiplication and Reciprocal Tables.

Reciprocal Table

Table of reciprocals $\bar{x}$ of x, where $x\bar{x} = 60^n$, $n = 0, 1, \ldots$.

x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$
2	0/30	8	7/30	16	3/45	30	2
3	0/20	9	6/40	18	3/20	32	1/52/30
4	0/15	10	6	20	3	36	1/40
5	0/12	12	5	24	2/30	40	1/30
6	0/10	15	4	25	2/24	45	1/20

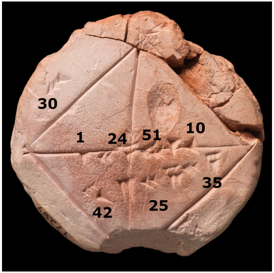
Divide 8 by 2 : $8(0/30) = 8 \times \frac{30}{60} = \frac{240}{60} = 4$, or

$0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 = 1 + 1 + 1 + 1$.

Missing reciprocals: $\frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \ldots$.

Sumerian Tablet - YBC 7289 - imšukkum, or "hand tablet"

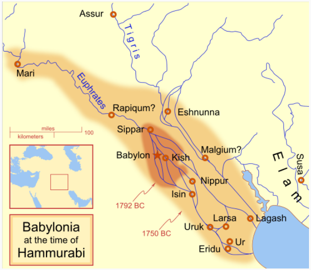
- From southern Iraq, 19th or 18th century BCE.
- Yale Peabody Museum of Natural History, 3D Print.
- Babylonians knew ratio of the side to diagonal in a square, $1 : \sqrt{2}$.



$$1/24/51/10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414213$$

$$42/25/35 = 42 + \frac{25}{60} + \frac{35}{60^2} \approx 42.426$$

Plimpton 322 Clay Tablet (in the news in 2017)

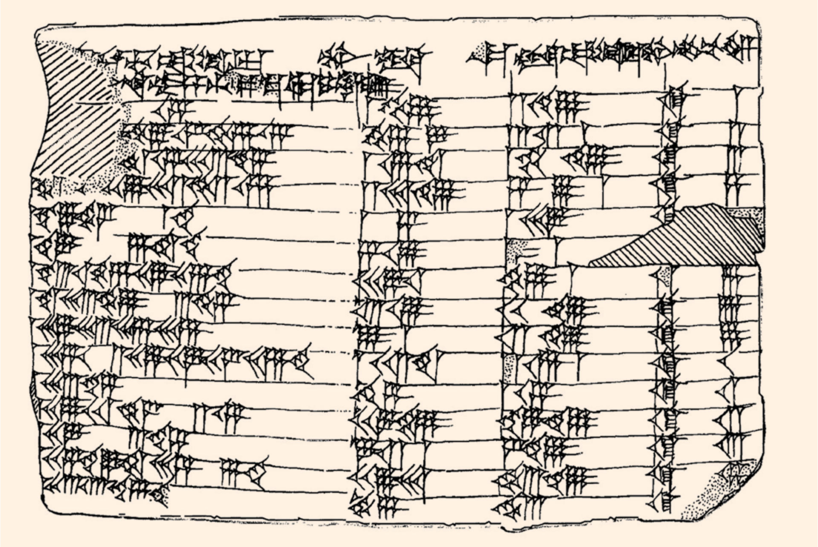


- Larsa (c. 1800 BCE) .
- Removed in 1920s.
- George Plimpton bought it, 1922.
- Left to Columbia University, 1936.
- It's about 13 by 9 by 2 cm.
  (Like a baking dish.)

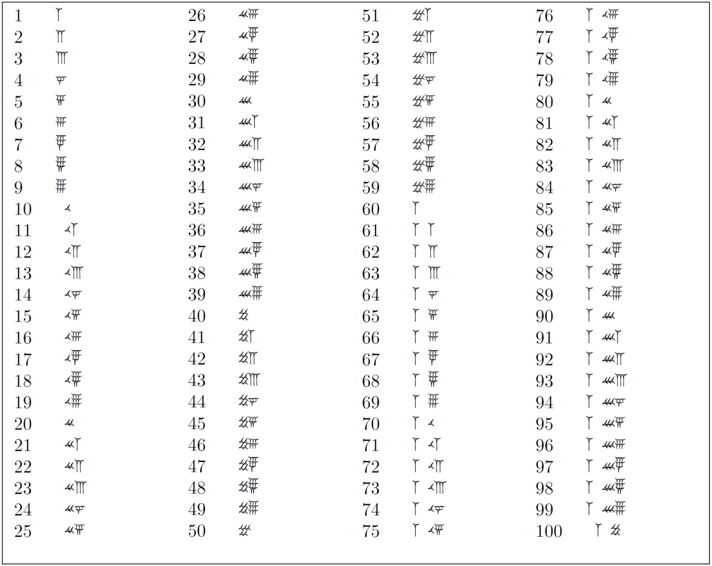
- Four columns, cuneiform numbers.
- 15 rows - Pythagorean triples.
- 2nd column, side of right triangle.
- 3rd column, hypotenuse.
- 4th column, row number.
- What is the 1st Column?

Plimpton 322 Clay Tablet - Homework!

Sketch of the Plimpton 322 Tablet



Babylonian Numerals 1-100 (Base 60)



Akkadian Table of 9’s

2 Akkadian Tablet (-1700)

In the paper “Sherlock Holmes in Babylon,” *Amer. Math. Monthly* 87 (1980), 335-345, C. Buck describes Babylonian mathematics. He begins with a discussion of a clay tablet from 3700 years ago as shown in Table 2. There are four columns. You should convince yourself that this is a table of 9’s.

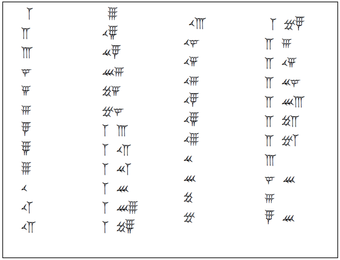


Table 2: Table of 9’s.

As an example, the last entry in the first column is $12 =$. Then, $9 \times 12 = 108 =$. Note that in base 60 we have $108 = 1(60) + 48$.

In the second column is a one ( ) and 48 ( ) separated by a space. Buck introduces a slash notation to write this as 1/48.

It is easy to add in base 60. Buck gives the example $14/28/31 + 3/35/45 = 18/4/16$.

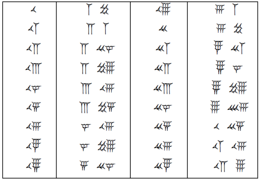
Babylonian Squares

How can a table of squares be useful? In modern notation, we see that

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]. \tag{1}$$

Let's find the product 11×14. Using Table 3, the formula gives

$$\begin{aligned}
11(14) &= \frac{1}{4}\left[(11+14)^2 - (11-14)^2\right] \\
&= \frac{1}{4}(25^2 - 3^2) \\
&= \frac{1}{4}(10/25 - 9) \text{ (base 60)} \\
&= \frac{1}{4}(10/16) \text{ (base 60)} \\
&= \frac{1}{4}(10(60)+16) = \frac{616}{4} = 154.
\end{aligned} \tag{2}$$



10	1/40	19	6/1
11	2/1	20	6/40
12	2/24	21	7/21
13	2/49	22	8/4
14	3/16	23	8/49
15	3/45	24	9/36
16	4/16	25	10/25
17	4/49	26	11/16
18	5/24	27	12/9

Table 3: Table of squares with Babylonian numerals in the left table and slash notation on the right side.

4 Pythagorean Triples

Another interesting tablet from the time is the Plimpton 322 tablet shown in Figure 4. This tablet has a listing of Pythagorean triples. The last column has a list of numbers from 1 to 15. Columns two and three seem to be the hypotenuse, C, and one leg, B, of the right triangle shown in Figure 1. Recall from the Pythagorean Theorem that

$$C^2 = B^2 + D^2.$$

The triple (D, B, C) is called a Pythagorean triple.

We now know that these triples are parametrized by the pair (a, b) as follows:

$$B = a^2 - b^2, \quad C = a^2 + b^2, \quad D = 2ab,$$

since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

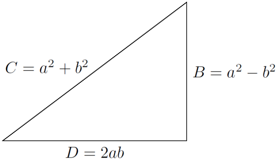


Figure 1: Right triangle with columns two and three as sides B and C, respectively. Pythagorean triples were later found to have a parametrization (a, b).

Transcription - Brackets indicate guesses

[59]/0/15	1/59	2/49	ki	1
[56/56]/58/14/50/6/15	56/7	1/20/25	ki	2
[55/7]/41/15/33/45	1/16/41	1/50/49	ki	3
53/10/29/32/52/16	3/31/49	5/9/1	ki	4
48/54/1/40	1/5	1/37	ki	[5]
47/6/41/40	5/19	8/1	[ki]	[6]
43/11/56/28/26/40	38/11	59/1	ki	7
41/33/45/14/3/45	13/19	20/49	ki	8
38/33/36/36	8/1	12/49	ki	9
35/10/2/28/27/24/26/40	1/22/41	2/16/1	ki	10
33/45	45	1/15	ki	11
29/21/54/2/15	27/59	48/49	ki	12
27/0/3/45	2/41	4/49	ki	13
25/48/51/35/6/40	29/31	53/49	ki	14
23/13/46/40	56	53	ki	[15]

Sketch of the Plimpton 322 Tablet

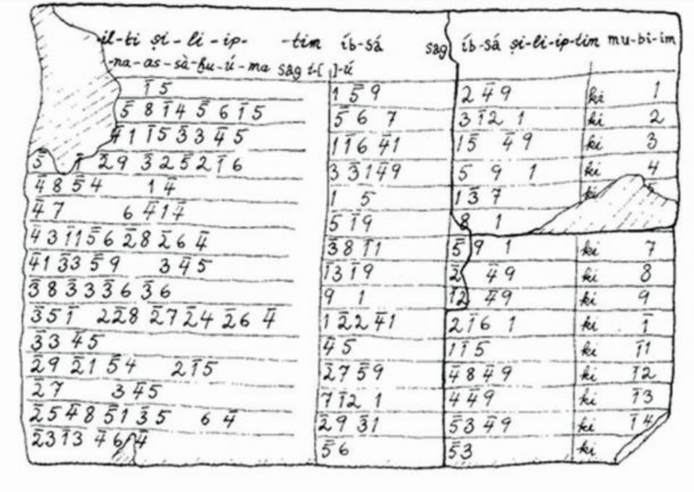


Figure 6: Arabic numerals base 60. The bars designate place holders.

Buck's Corrected Values

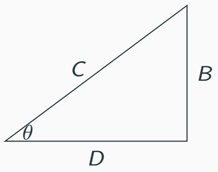
Second column - base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$.

#	A	B	C	a	b
1	59/0/15	119	169	12	5
2	56/56/58/14/50/6/15	3367	4825	64	27
3	55/7/41/15/33/45	4601	6649	75	32
4	53/10/29/32/52/16	12709	18541	125	54
5	48/54/1/40	65	97	9	4
6	47/6/41/40	319	481	20	9
7	43/11/56/28/26/40	2291	3541	54	25
8	41/33/45/14/3/45	799	1249	32	15
9	38/33/36/36	481	769	25	12
10	35/10/2/28/27/24/26/40	4961	8161	81	40
11	33/45	45	75	1	0.5
12	29/21/54/2/15	1679	2929	48	25
13	27/0/3/45	161	289	15	8
14	25/48/51/35/6/40	1771	3229	50	27
15	23/13/46/40	56	106	9	5

First Column Computation

- Buck suggests column A is $\left(\frac{B}{D}\right)^2$.
- Others suggest $\left(\frac{C}{D}\right)^2$ and missing left part of the stone had 1's.
  Noting,

$$\left(\frac{C}{D}\right)^2 = 1 + \left(\frac{B}{D}\right)^2.$$



From row 1: 59/0/15 represents

$$\frac{59}{60} + \frac{0}{60^2} + \frac{15}{60^3} = \frac{14161}{14400}.$$

From row 1: $B = 119,\ C = 169:$

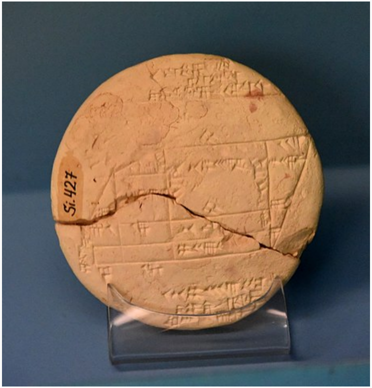
$$B^2 = 119^2 = 14161$$
$$D^2 = 169^2 - 119^2 = 14400$$
$$\left(\frac{B}{D}\right)^2 == \frac{14161}{14400}.$$

Decimal Equivalents for Column One

#	A	Decimal Value	$(B/D)^2$
1	59/0/15	0.983402777777778	0.983402777777778
2	56/56/58/14/50/6/15	0.949158552088692	0.949158552088692
3	55/7/41/15/33/45	0.918802126736111	0.918802126736111
4	53/10/29/32/52/16	0.886247906721536	0.886247906721536
5	48/54/1/40	0.815007716049383	0.815007716049383
6	47/6/41/40	0.785192901234568	0.785192901234568
7	43/11/56/28/26/40	0.719983676268862	0.719983676268862
8	41/33/45/14/3/45	0.692709418402778	0.692709418402778
9	38/33/36/36	0.642669444444444	0.642669444444444
10	35/10/2/28/27/24/26/40	0.586122566110349	0.586122566110349
11	33/45	0.562500000000000	0.562500000000000
12	29/21/54/2/15	0.489416840277778	0.489416840277778
13	27/0/3/45	0.450017361111111	0.450017361111111
14	25/48/51/35/6/40	0.430238820301783	0.430238820301783
15	23/13/46/40	0.387160493827161	0.387160493827161

A	B	C
59 15	1 59	2 49
56 56 58 14 50 6 15	56 7	1 20 25
55 7 41 15 33 45	1 16 41	1 50 49
53 10 29 32 52 16	3 31 49	5 9 1
48 54 1 40	1 5	1 37
47 6 41 40	5 19	8 1
43 11 56 28 26 40	38 11	59 1
41 33 45 14 3 45	13 19	20 49
38 33 36 36	8 1	12 49
35 10 2 28 27 24 26 40	1 22 41	2 16 1
33 45	45	1 15
29 21 54 2 15	27 59	48 49
27 3 45	2 41	4 49
25 48 51 35 6 40	29 31	53 49
23 13 46 40	56	1 46

- 1900-1600 BC.
- Field plan.
- Used Pythagorean triples to make accurate right angles for measuring boundaries.
- Proposes that Plimpton 322 is the world's oldest and most accurate trigonometric table. (8/2017) Youtube
- Robson does not view it that way.



D.F. Mansfield. Plimpton 322: A Study of Rectangles. Found Sci, published online August 3, 2021; Paper.

Babylonian Geometry

- Simple shapes.
- Interested in areas.
- Fields, subdivisions.
- Inclinations, slopes.
- seked, ukuklu, run/rise.

See N. Wildberger's YouTube and reference Neugebaur and Sachs, ed., 1945, *Mathematical Cuneiform Texts*.
Neugebaur and Sachs Introduced Plimpton 322.
Wildberger and Mansfield - Babylonian trigonometry based on ratios.