

Cubic Equations

Fall 2022 - R. L. Herman



Solutions of Polynomial Equations

- Linear equations, known solutions.
- Chinese - Gaussian elimination:
Systems of n linear equations and n unknowns.
- Quadratic equations:
Need square roots.
- Cubic equations:
Need square roots and cube roots.
Solved in 16th century.
- Quintic equation: studied in 1820's.
Eventually lead to group theory!



Figure 1: Leonardo da Vinci attempts Delian problem (Doubling cube).

Quadratic Equations

Babylonian Method (Modern Notation)



Find x and y for a given perimeter and area.

$$x + y = p$$

$$xy = q.$$

Eliminate y , $x^2 + q = px$. Then,

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Method - Compute the following

1. $\frac{x+y}{2}$

2. $\left(\frac{x+y}{2}\right)^2$

3. $\left(\frac{x+y}{2}\right)^2 - xy = \frac{(x+y)^2 - 4xy}{4}$

4. $\sqrt{\frac{(x+y)^2 - 4xy}{4}} = \frac{x-y}{2}$

5. By inspection, get x, y from p and q , since

$$\begin{aligned} \frac{x-y}{2} &= \sqrt{\frac{p^2 - 4q}{4}} \\ &= \sqrt{\left(\frac{p}{2}\right)^2 - q}. \end{aligned}$$

Quadratic Equations (cont'd)

- Brahmagupta (628) - Explicit
 $ax^2 + bx = c.$

$$x = \frac{\sqrt{4ac + b^2} - b}{2a},$$

- Euclid - Prop. 28
- al'Khwarizimi

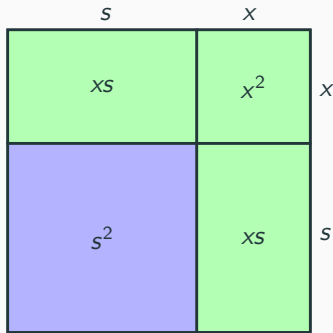
$$x^2 + 2xs = n$$

$$x^2 + 2xs + s^2 = n + s^2$$

$$(x + s)^2 = n + s^2$$

- Quadratic Irrationals

$$\frac{a + \sqrt{b}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$



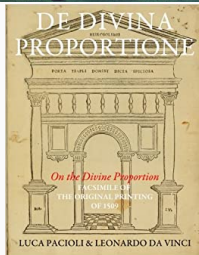
Note from the figure:

Green area = $x^2 + 2xs = n.$

No negative lengths (solutions).

Cubic Equations

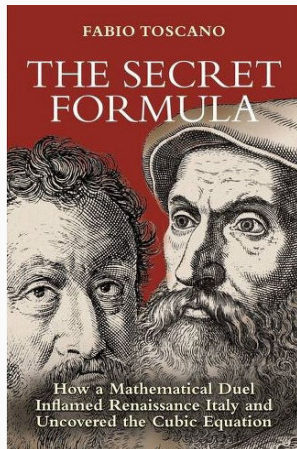
- Babylonians - Table of cubes
- Greeks - Geometric Problems
 - Duplicating cube (Delian Prob.).
 - Intersecting conics.
 - Cutting Sphere with plane.
- Omar Khayyam (1048-1131)
 - First general theory of cubics.
 - Provided several cases.
- Luca Pacioli (1445-1517)
 - Franciscan friar, tutor.
 - *Summa de arithmetica*, 1494.
Father of Accounting.
 - *Divina proportione*, 1509.
 - “Solution to cubic is impossible!”



The Secret Formula

How a Mathematical Duel Inflamed Renaissance Italy and Uncovered the Cubic Equation, by Fabio Toscano, 2020.

- **The Abbaco Master**
Italian Wars, Brescia, 1512, Tartaglia.
 - **The Rule of the Thing**
cosa (thing), *censo* (x^2), *numero*.
 - **The Venetian Challenge**
1535, Fior challenges Tartaglia.
 - **An Invitation to Milan**
Entrance of Gerolamo Cardano.
 - **The Old Professor's Notebook**
da Coi vs. Ferrari, del Ferro's priority:
Solution: things and cube equal to number.
 - **The Final Dual**
Ars Magna, 1545. Ferrari vs. Tartaglia, 1547.
- History of Math*



The Search for Solutions

- Scipio del Ferro (1465-1526)
 - University of Bologna, notebooks
 - Printing press - Guttenberg
 - 1506/1514, solution of **depressed cubic**: $x^3 + ax = b$.
 - Public Challenges led to secrecy.
- Gave to Antonio Maria Fior (Florido).
- Tartaglia (Nicolo Fontana) (1499-1557)
 - 1512, French attack - sabre wound led to stammer.
 - Self-educated
 - 1530 da Coi wrote to him $x^3 + 3x^2 = 5$, $x^3 + 6x^2 + 8x = 1000$.



Figure 2: Tartaglia

The Plot Thickens

- Tartaglia boasted he could solve $x^3 + ax^2 = c$.
- Florido challenged Tartaglia
 - Each posed 30 problems
 - Florido mostly gave problems of form $x^3 + ax^2 = c$.
 - Tartaglia won by solving depressed cubic 1535, but didn't publish.
- Girolamo Cardano (1501-1576)
 - Gambler, astronomer, physician, astrologer, heretic, father of murderer.
 - Begged for solution from Tartaglia. Finally, they met in Milan.
 - Tartaglia eventually gave solution in 1539 as a [Poem](#) if it was kept secret.
 - It was not in Cardano's book.

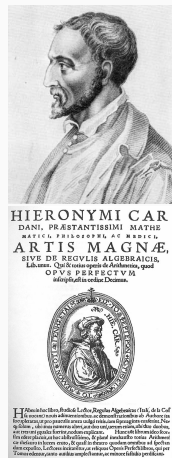
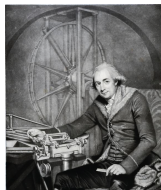


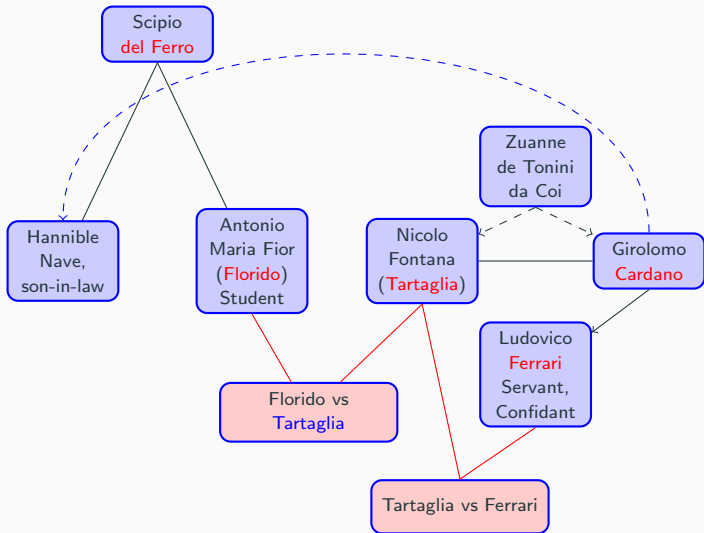
Figure 3: Cardano, *Ars Magna*.

Enter Ludovico Ferrari

- Ludovico Ferrari (1522-1565)
 - Servant at 14
 - Secretary, confidant
 - Worked on problems with Cardano
 - Cubic and biquadratic equations
- da Coi → Cardano → Ferrari
 - 4th degree polynomial
 - Ferrari solution involved solving cubic
 - Publishing was a problem.
- 1543 Trip to Florence, stopped in Bologna on the way.
 - Visited Hannible Nave, del Ferro's son-in-law.
 - Saw del Ferro's notes.
 - Cardano believed he could publish in his *Ars Magna*, 1545.
- Barrage of letters from Tartaglia!



The Players in the Cubic Story



Tartaglia vs Ferrari - 1548

- Public debate in Milan, Ferrari's hometown.
- Cardano was absent.
- Tartaglia lost, blamed crowd.
- Tartaglia worked on arithmetic.
- Ferrari became professor in Bologna, 1565.
Was poisoned 1565, white arsenic, possibly by sister.
- Cardano predicted exact date of his own death in 1576.



Figure 4: Tartaglia and Ferrari

Solution of the Quadratic $x^2 + ax + b = 0$

- Completing the square:
 $(x + \frac{a}{2})^2 + b - \frac{a^2}{4} = 0.$
- Solution: $x + \frac{a}{2} = \pm \sqrt{\frac{a^2}{4} - b}.$
- Graph of parabola
 $y = x^2 + ax + b$
Vertex $(-\frac{a}{2}, b - \frac{a^2}{4})$
- Number of real solutions?
- Substitute $x = u - \frac{a}{2}$:
 $0 = x^2 + ax + b$
 $0 = (u - \frac{a}{2})^2 + a(u - \frac{a}{2}) + b$
 $0 = u^2 + b - \frac{a^2}{4}.$
- Solve for u .

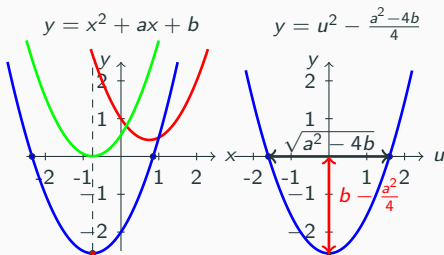


Figure 5: Plots of parabolae. Translating blue parabola on left by $\frac{a}{2}$ results in that on the right.

Plotting a Cubic Function $y = x^3 + ax^2 + bx + c$

- Set $y = 0$, then $x^3 + ax^2 + bx + c = 0$.
- Solutions are black points. Always have a real solution.
- $y' = 3x^2 + 2ax + b$ and $y'' = 6x + a$.
- Inflection point: $y'' = 0$ for $x_0 = -\frac{a}{3}$, $y_0 = c - \frac{ab}{3} + \frac{2a^3}{27}$.
- Slope of tangent at $(x_0, y_0 = q)$ is $p = b - \frac{a^2}{3}$.
- Extrema at $x_{\pm} = -\frac{a}{3} \pm \frac{\sqrt{a^2 - 3b}}{3}$.

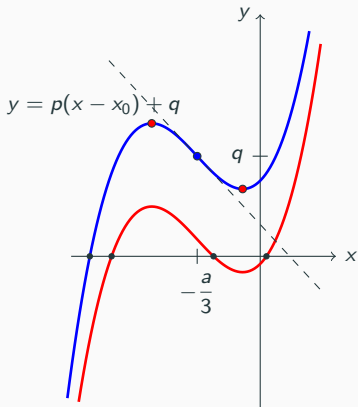


Figure 6: Plots of the cubic function exhibiting either one or three real roots.

Can you complete the cube: $(x + \alpha)^3 = x^3 + 3x^2\alpha + 3x\alpha^2 + \alpha^3$?

Solution of the Cubic $x^3 + ax^2 + bx + c = 0$.

$$\text{Let } x = y - \frac{a}{3}.$$

Then, $y^3 + py + q = 0$, where

$$p = b - \frac{a^2}{3},$$

$$q = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

Let $y = u + v$:

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Let $p + 3uv = 0$, then

$$\begin{aligned}u^3 v^3 &= -\frac{p^3}{27}, \\u^3 + v^3 &= -q.\end{aligned}$$

Now, define $X = u^3$. $Y = v^3$.

We obtain

$$X + Y = -q.$$

$$XY = -\frac{p^3}{27},$$

Does this look familiar?

The solution of Cubic:

$$X, Y = u^3, v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}},$$

$$y = u + v,$$

$$x = y - \frac{a}{3}.$$

Example: $2x^3 - 30x^2 + 162x - 350 = 0$.

Let $x = y - \frac{b}{3a} = y + \frac{30}{6} = y + 5$.

We obtain a **depressed cubic** (del Ferro), $y^3 + 6y - 20 = 0$.

Letting $y = u + v$, $X, Y = u^3, v^3$, we solve

$$X + Y = -q = 20, \quad XY = -\frac{p^3}{27} = -\frac{6^3}{27} = -8.$$

Eliminating Y , $X^2 - 20X - 8 = 0$.

Solving, leads to $X = 10 \pm \sqrt{108}$, $Y = 20 - X = -10 \mp \sqrt{108}$.

So, $u = \sqrt[3]{10 \pm 6\sqrt{3}}$ $v = \sqrt[3]{-10 \pm 6\sqrt{3}}$ and

$$y = u + v = \sqrt[3]{10 \pm 6\sqrt{3}} + \sqrt[3]{-10 \pm 6\sqrt{3}}$$

$$x = \sqrt[3]{10 \pm 6\sqrt{3}} - \sqrt[3]{-10 \pm 6\sqrt{3}} + 5$$

Depressed Cubic $y^3 + 6y - 20 = 0$.

Note: $y = 2$ is a solution. Therefore, $y^3 + 6y - 20 = (y - 2)(y^2 + \alpha y + \beta)$.

Method 1: Expand and match coefficients.

$$y^3 + 6y - 20 = y^3 + (\alpha - 2)y^2 + (\beta - 2\alpha)y - 2\beta$$

$$\alpha - 2 = 0, \quad \beta - 2\alpha = 6, \quad 2\beta = 20 \quad \Rightarrow \quad \alpha = 2, \beta = 10.$$

Method 2: Long division.

$$\begin{array}{r} y^2 + 2y + 10 \\ y - 2 \overline{) y^3 - 20} \\ \underline{-y^3 + 2y^2} \\ 2y^2 + 6y \\ \underline{-2y^2 + 4y} \\ 10y - 20 \\ \underline{-10y + 20} \\ 0 \end{array}$$

Find other roots:

$$y^2 + 2y + 10 = 0$$

$$\begin{aligned} y &= \frac{-2 \pm \sqrt{4 - 40}}{2} \\ &= -1 \pm 3i. \end{aligned}$$

Nested Radicals

We have found that $y = \sqrt[3]{10 \pm 6\sqrt{3}} - \sqrt[3]{-10 \pm 6\sqrt{3}}$. Can one simplify this?
We consider $\sqrt[3]{10 \pm 6\sqrt{3}} = \sqrt{x} \pm \sqrt{y}$ and $\sqrt[3]{-10 \pm 6\sqrt{3}} = -\sqrt{x} \pm \sqrt{y}$. Note:

$$\begin{aligned}(\sqrt{x} \pm \sqrt{y})^3 &= x\sqrt{x} \pm 3x\sqrt{y} + 3y\sqrt{x} \pm y\sqrt{y} \\ &= (x + 3y)\sqrt{x} \pm (3x + y)\sqrt{y} \\ &= 10 \pm 6\sqrt{3}.\end{aligned}\tag{1}$$

$$\begin{aligned}(-\sqrt{x} \pm \sqrt{y})^3 &= -x\sqrt{x} \pm 3x\sqrt{y} - 3y\sqrt{x} \pm y\sqrt{y} \\ &= -(x + 3y)\sqrt{x} \pm (3x + y)\sqrt{y} \\ &= -10 \pm 6\sqrt{3}.\end{aligned}\tag{2}$$

From (1) $y = 3$, $x + 9 = 10$, $3x + 3 = 6$. Then, $x = 1$ and $\sqrt[3]{10 \pm 6\sqrt{3}} = 1 \pm \sqrt{3}$.

Similarly, from (2), $\sqrt[3]{-10 \pm 6\sqrt{3}} = -1 \pm \sqrt{3}$. So,

$$y = \sqrt[3]{10 \pm 6\sqrt{3}} + \sqrt[3]{-10 \pm 6\sqrt{3}} = 2.$$

Complex Solutions $y^2 + 2y + 10 = 0$.

We found $y^3 + 6y - 20 = (y - 2)(y^2 + 2y + 10) = 0$.

we solved the quadratic: $y = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\sqrt{-1}$.

- Cardano, complex numbers
“as subtle as they are useless.”
- Raphael Bombelli (1526-1572)
First to take seriously.
- Ex: $x^3 = 15x + 4$
 $x = \sqrt[3]{2 + 11\sqrt{-11}} + \sqrt[3]{2 - 11\sqrt{-11}}$
- But, $x = 4$ is a solution!
- Complex numbers, $a + bi$, $i = \sqrt{-1}$.
- $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$



Figure 7: Bombelli

Cube Root of Complex Numbers

- Last Example: $x^3 = 15x + 4$
 $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$
- Seek: $\sqrt[3]{2 + 11i} = c + di$.

$$\begin{aligned}\sqrt[3]{2 + 11i} &= c + di \\ 2 + 11i &= (c + di)^3 \\ &= c^3 + 3c^2di + 3c(di)^2 + (di)^3 \\ &= c^3 - 3cd^2 + i(3c^2d - d^3).\end{aligned}$$

Then

$$\begin{aligned}2 &= c^3 - 3cd^2 = c(c^2 - 3d^2), \\ 11 &= 3c^2d - d^3 = d(3c^2 - d^2).\end{aligned}$$

- Bombelli: c, d , positive integers.

Since 2 is prime, $c = 1, 2$.

If $c = 1$, $2 = 1 - 3d^2$. No!

If $c = 2$, then $d = 1$.

$$\begin{aligned}2 &= 8 - 6d^2 \\ 11 &= 12d - d^3\end{aligned}$$

Then,

$$\begin{aligned}x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= (2 + i) + (2 - i) = 4.\end{aligned}$$

François Viète (1540-1603)

- Counselor to Henry III, IV, France
- French Wars of Religion 1562-1598
- Tutored Catherine de Pathenay (1554-1631), noblewoman, mathematician
- 1596 - Adriaan van Roomen
"No French mathematician could solve the 45th degree polynomial."
$$x^{45} - 45x^{43} + 945x^{41} + \dots - 3795x^3 + 45x = A.$$
- Viète solved quickly:
$$2 \sin(45\alpha) = A, \quad x = 2 \sin \alpha.$$
- Trig identity: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$
Let $y = \cos \theta.$ $4y^3 - 3y = c, |c| \leq 1. c = \cos 3\theta.$
Solve for θ given $c.$ Solution, $y = \cos \theta.$
- Use identities to rewrite $2 \sin(45\alpha) = A$ in terms of $2 \sin \alpha.$



Figure 8: Viète, Henry IV, and van Roomen.

Viète's Solution

Define the quantities

$$\begin{aligned}c &= 2 \sin 45\theta, & y &= 2 \sin 15\theta, \\z &= 2 \sin 5\theta, & x &= 2 \sin \theta.\end{aligned}\tag{3}$$

Problem: Find x , given c .

Use the identities:

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha\tag{4}$$

$$\sin 5\alpha = 5 \sin \alpha - 20 \sin^3 \alpha + 16 \sin^5 \alpha\tag{5}$$

Then,

$$c = 2 \sin 45\theta = 6 \sin 15\theta - 8 \sin^3 15\theta = 3y - y^3.\tag{6}$$

$$y = 2 \sin 15\theta = 6 \sin 5\theta - 8 \sin^3 5\theta = 3z - z^3.\tag{7}$$

$$z = 2 \sin 5\theta = 10 \sin \theta - 40 \sin^3 \theta + 32 \sin^5 \theta = 5x - 5x^3 + x^5.\tag{8}$$

Viète's Solution (cont'd)

Since $z = 5x - 5x^3 + x^5$, we write c in terms of x :

$$\begin{aligned}y &= 3z - z^3 \\&= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3 \\&= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x.\end{aligned}\tag{9}$$

$$\begin{aligned}c &= 3y - y^3 \\&= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x] \\&\quad - [-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3 \\&= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} \\&\quad - 14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} \\&\quad + 483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\&\quad - 34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \\&= P_{45}(x).\end{aligned}\tag{10}$$

Math Symbols

- + Oresme (1300)
- – Widman (1400)
- = Recorde (1500)
- × Outred (1500)
- Letters Viète (1500)
- Descartes (1500-1600)
unknowns, constants a, b, c
variables x, y, z
- $\langle \rangle$ Harriot (1600)
- ∞ Wallis (1700)
- imaginary, Descartes
- $x^{3/2}, x^{-1}$, Newton 1600
- $x^2 \rightarrow xx$, Gauss 1800
- π, i, Σ . Euler 1700
- $f(x)$
- $\frac{df}{dx}, \int$ Leibniz 1600

Analytic Geometry

- Fermat (1601-1665)
- Descartes (1596-1650)
- Newton (1642-1727)

Coordinates

- Hipparchus - sky
- Apollonius - conics
- Oresme (1300s) - position, velocity plots
- Fermat-Descartes described curves in coordinate systems
- Degree 1, Linear relations

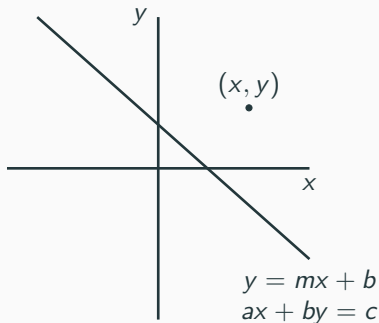


Figure 9: Cartesian system.

Curves of Degree 2 - Quadratics

$$ax^2 + \underbrace{bxy}_{\text{rotation}} + cy^2 + \underbrace{dx + ey + f}_{\text{translation}} = 0$$

- Describes Conics
- $b \neq 0$, rotation
- $d \neq 0$ or $e \neq 0$, translation
- Classification

$$D = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

$D > 0$ ellipse

$D < 0$ hyperbola

$D = 0$ parabola

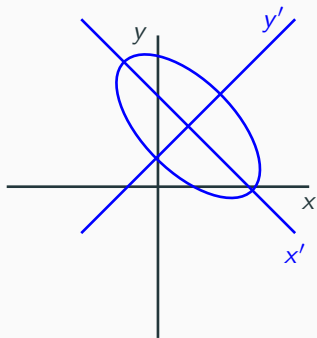


Figure 10: Rotated, translated ellipse.

Curves of Degree 3: $ax^3 + bx^2y + cxy^2 + dy^3 + \dots = 0$, Cubics

- Newton classified cubic curves, 1710, 72 types (missed 6)
- $y = x^3$ and other types.
- Descartes's folium (leaf)
 $x^3 + y^3 = 3axy$
- Parametric Solutions

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

- Rational Points
Ex. $x^3 + y^3 = 1$.
Let $x = \frac{n}{p}$, $y = \frac{m}{p}$. $n^3 + m^3 = p^3$.
- Fermat - only trivial $(0, 1), (1, 0)$.
- Fermat's Last Theorem, 1637

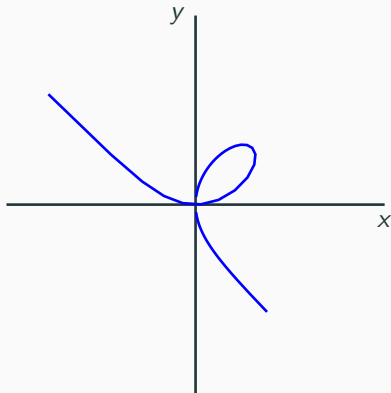


Figure 11: Descartes's Folium.

Fermat's and Bezout's Theorems

- Fermat's Last Theorem, 1637

$$x^n + y^n = z^n$$

Wiles proved in 1995.

- Bezout's Theorem. Let

$$p(x, y) = 0, \quad \text{degree } n.$$

$$q(x, y) = 0, \quad \text{degree } m.$$

Then, p and q intersect in nm points.

- Elimination gives eq of degree nm .
- Need complex numbers, point at infinity.

Next - Projective Geometry.

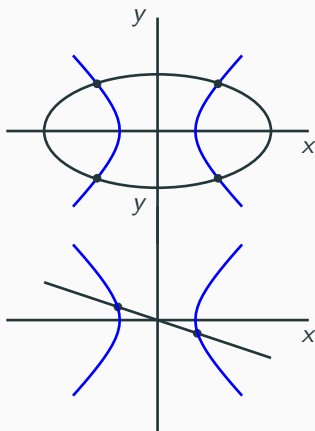


Figure 12: Intersecting Degree 2 curve (blue) with Degree 2 or 1 (black).