Greek Mathematics I

Fall 2023 - R. L. Herman

Greek Numerals

- Decimal (Base 10).
- No zero and Positional.
- Attic Numerals (Athens),
- Ionic (Ionia): 24+3 letters

Digit	1-9	10-90	100-900
1	α	ι	ρ
2	β	κ	σ
3	γ	λ	τ
4	δ	μ	υ
5	ϵ	ν	ϕ
6	ϝ	ξ	χ
7	ζ	ο	ψ
8	η	π	ω
9	θ	ϟ, ϙ	ϡ

Arabic	Attic Greek	
1	I	
5	Γ Π	
10	Δ	deca
50	𐅄	
100	H	hecto
500	𐅅	
1 000	X	kilo
5 000	𐅆	
10 000	M	

2857 = XX𐅅HHH𐅄ΓII

761 = 𐅅HH𐅄ΔI

543 = $\mu\phi\gamma = \rho\rho\rho\rho\rho\kappa\kappa\alpha\beta$

$,\alpha = 1000$

$\overset{\kappa\delta}{\mathrm{M}} = 240{,}000$

Thales of Miletus (ca. 640-546 BCE)

- Ionia, Asia Minor.
- Parents were Greek or Phoenician.
- One of the Seven Sages of Greece.
- Founder of the Milesian School of natural philosophy, and the teacher of Anaximander.
- Credited with 5 theorems in geometry:
    1. A circle is bisected by any diameter.
    2. The base angles of an isosceles triangle are equal.
    3. The angles between two intersecting straight lines are equal.
    4. Two triangles are congruent if they have two angles and one side equal.
    5. An angle in a semicircle is a right angle.

According to Proclus (412-485) and others, head of Plato's Academy, commentaries on mathematicians.

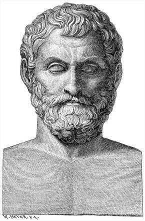


Figure 1: Thales of Miletus taught that 'all things are water.' - Aristotle

Many Other claims:
Predicted solar eclipse (585 BCE).
Measured pyramid heights.

Thales' Theorem

An angle inscribed in a semicircle is a right angle.

- 31st proposition, Book III of Euclid's Elements.
- According to Proclus and Diogenes Laërtius.
- Known earlier to Indian and Babylonian mathematicians.

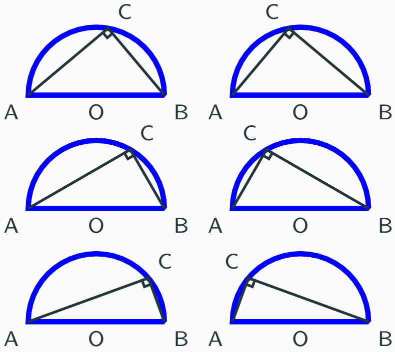


Figure 2: Thales' Theorem demonstrated.

Thales' Theorem: Inscribed Angle = 90^o

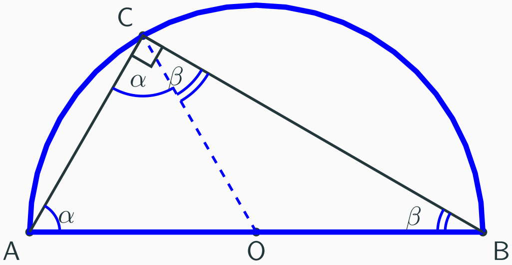


Figure 3: Proof by Picture.

Radii: $\overline{AO} = \overline{OB} = \overline{OC}$.

Isoceles triangles: AOC and OBC.

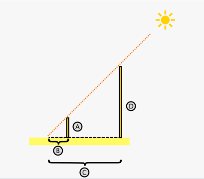
Sum of angles in ABC $= 2\alpha + 2\beta = 180^o$ implies $\alpha + \beta = 90^o$.

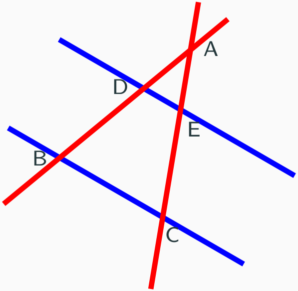
Intercept Theorem

If two (or more) parallel lines (blue) are intersected by two self-intersecting lines (red), then the ratios of the line segments of the first intersecting line is equal to the ratio of similar line segments of the second line.[1]

Prove by using similar triangles:

$$\frac{\overline{DE}}{\overline{BC}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$$





[1] "Hieronymus says that [Thales] measured the height of the pyramids by the shadow they cast, taking the observation at the hour when our shadow is of the same length as ourselves (i.e., as our own height)."

Pythagoras of Samos (570-495 BCE)

- Known from Philolaus and others.
- School in Croton, 530 BCE.
  - vegetarian, communal, secret.
  - All is number.
- Philosophy - love of wisdom.
- Mathematics - that which is learned.

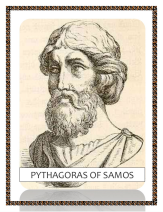


Figure 4: Pythagoras

Figure 5: Locate Samos and Croton.

Numerology - Numbers have meanings.

Even is male; Odd is female.

1. = generator
2. = opinion
3. = harmony
4. = justice
5. = marriage
6. = creation
7. = planets

10 is holiest (tetractys, tetrad, decad).

Figure 6: Tetractys

Triangular numbers:

$$1, 3, 6, 10, \ldots$$

Also the four seasons, planetary motions, music, four elements, fourth triangular number, etc.

Number Theory

- Triangular Numbers:

$$1, 3, 6, 10, \ldots$$

- Perfect Numbers [Sum factors $< n$.]:

$$\begin{aligned} 6 &= 1 + 2 + 3 \\ 10 &\neq 1 + 2 + 5 \\ 28 &= 1 + 2 + 4 + 7 + 14 \end{aligned}$$

- Amicable Numbers:

$$\begin{aligned} 220 &: 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = \\ 284 &: 1 + 2 + 4 + 71 + 142 = \end{aligned}$$

Pythagorean Theorem, $a^2 + b^2 = c^2$

- Known by Babylonians and Egyptians.
- Also, traces in other cultures.
- Many Proofs over the years.
- Attributed to Pythagoras.
- Pythagorean Triples (a, b, c).

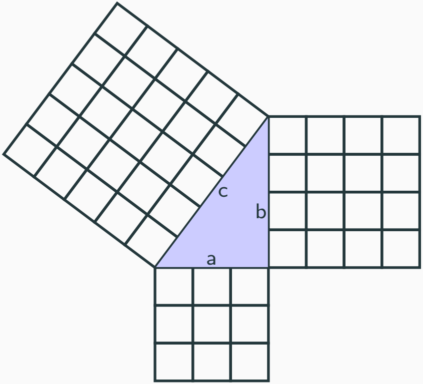


Figure 7: Euclid's Proof.

Ratios

Segments are **commensurable** if there exist a segment EF such that $\overline{AB} = p\overline{EF}$ and $\overline{CD} = q\overline{EF}$, where p and q are integers.
Therefore,

$$\frac{\overline{AB}}{\overline{CD}} = \frac{p}{q}.$$

Sometimes written as $p : q$.
Led to *Music of the Spheres*.

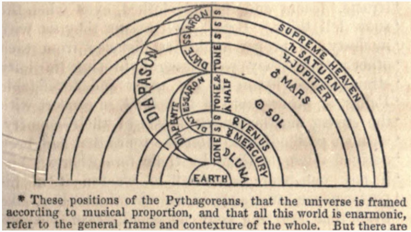


Figure 8: *A General History of the Science and Practice of Music,* Sir John Hawkins, 1853. Also provides story of Pythagoras' death.

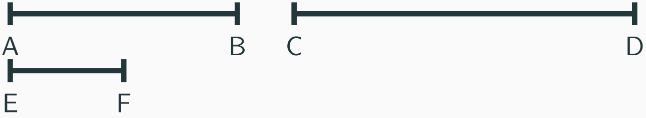


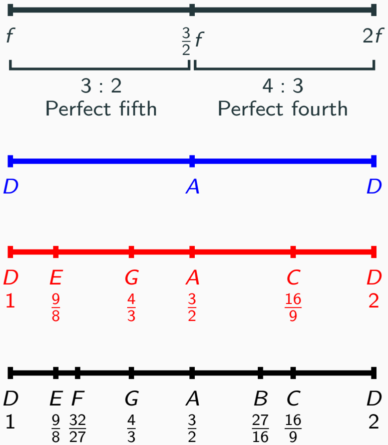
Figure 9: Commensurate Segments.

Pythagorean Scale - Series of Musical Notes

Goal - To produce a music scale.

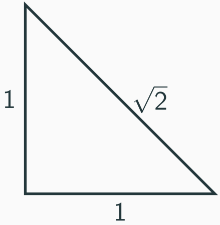
Want sounds that are pleasing when played together. Need simple ratios.

- **Octave**: From f to $2f$ (2^{nd} Harmonic).
  Ex: D goes to D, an octave higher.
- Next Notes?
  Up by **perfect fifth**. $\frac{3}{2}(1) = \frac{3}{2}$, A.
  Down by **perfect fifth**. $\frac{2}{3}(2) = \frac{4}{3}$, G.
- $\frac{3}{2}\left(\frac{3}{2}\right) = \frac{9}{4}$, wrong octave, halve.
- $\frac{2}{3}\left(\frac{4}{3}\right) = \frac{8}{9}$. wrong octave, double.
- Gives E and C.
- Pentatonic scale: D, E, G, A, C, D.
- Western Scale: D, E, F, G, A, B, C, D.
- B: $\frac{3}{2}\left(\frac{9}{8}\right) = \frac{27}{16}$, F: $\frac{2}{3}\left(\frac{16}{9}\right) = \frac{32}{27}$.



Irrational Numbers

- Hippasus of Metapontum (c. 530 - c. 450 BCE).
- Credited proving $\sqrt{2}$ is irrational.
- Drowned - possibly not an accident.
- Plato wrote Theodorus of Cyrene (c. 400 BCE) proved the irrationality of $\sqrt{3}$ to $\sqrt{17}$.
- Greeks knew sum of angles of triangle $= 2(90^o) = 180^o$.
- Construction of figures with compass and straight edge.



Classical Construction Problems

- Squaring the Circle (Quadrature) - Dinostratus (c. 390–320 BCE).
- Doubling the Cube (2×Volume) - Menaechmus (380–320 BCE).
- Trisecting a Angle (using unmarked straightedge and compass.) Hippias (460-400 BCE). Impossibility Proof: 1857, Pierre Wantzel, needs Modern Algebra.

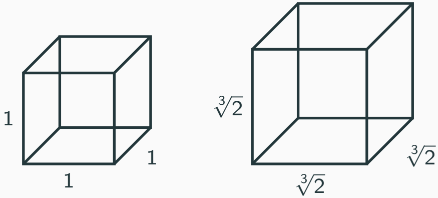


Figure 10: Doubling the Cube.

Hippocrates of Chios (c. 470 - c. 410 BCE)

- Not the Hippocrates of Kos (c. 460 - c. 370 BCE), Father of Medicine, and the Hippocratic Oath.
- Mathematician, geometer, and astronomer.
- Went to Athens.
- Used *reductio ad absurdum* arguments (proof by contradiction).
- Wrote geometry textbook, *Elements*
- Sought Quadrature of Circle.
- Quadrature of Lune.

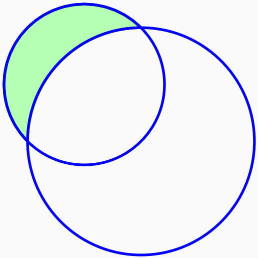


Figure 11: Lune or Crescent.

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.

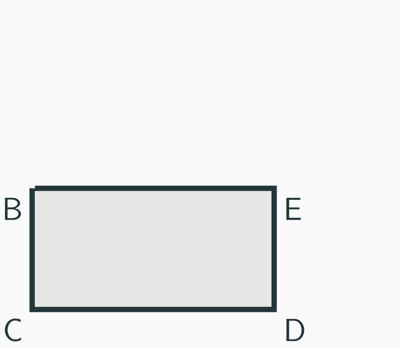


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.

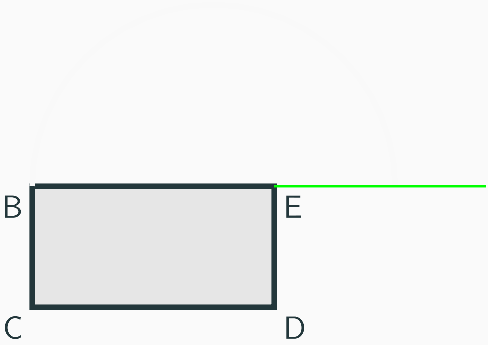
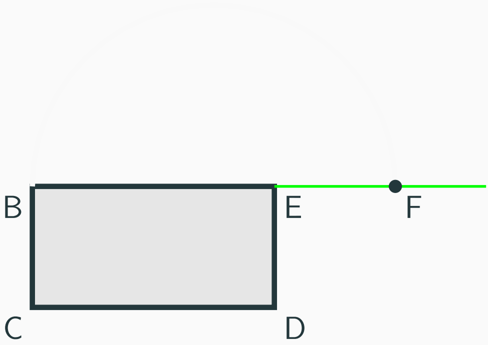


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.



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Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
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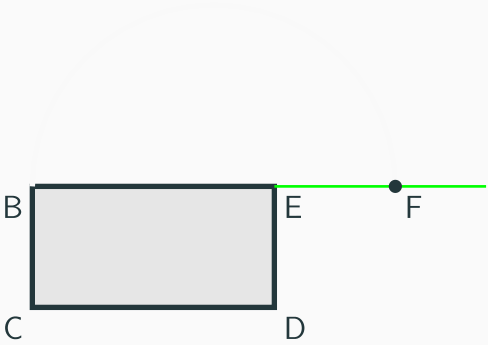
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?



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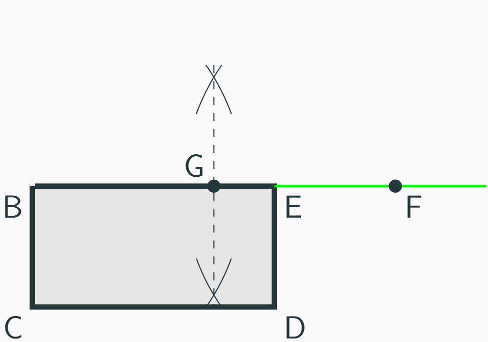
- Bisect segment BF.



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- Start with BCDE.
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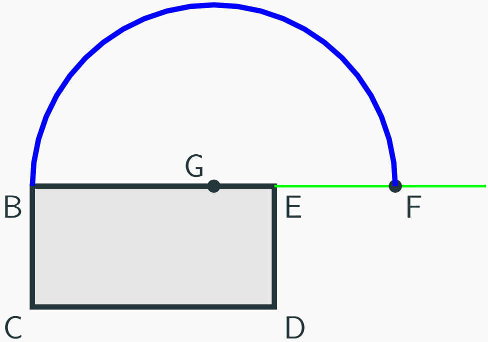
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.



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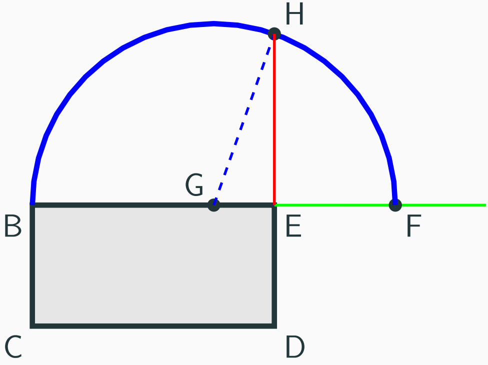
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.



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Quadrature of Rectangle

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- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.
- Construct square EKLH.
- Prove the areas are equal.

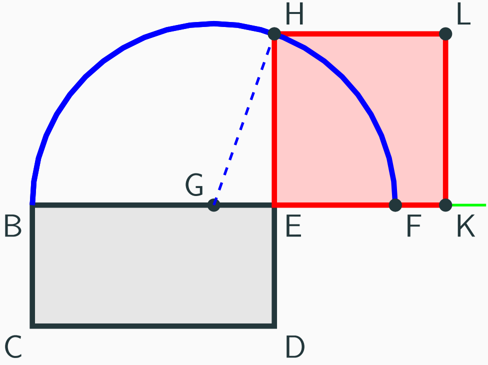


Figure 12: Quadrature of a Rectangle

Proof of Equal Areas

Label lengths a, b, c.

Area of Gray Rectangle BCDE:

$$\begin{aligned} A &= (a+b)\overline{ED} \\ &= (a+b)\overline{EF} \\ &= (a+b)(a-b) \\ &= a^2 - b^2. \end{aligned}$$

Area of Red Square EKLH:

Use Pythagorean Theorem:

$$A = c^2 = a^2 - b^2.$$

Thus, the area of the square is the same as the given rectangle; i.e., we **squared the rectangle**.

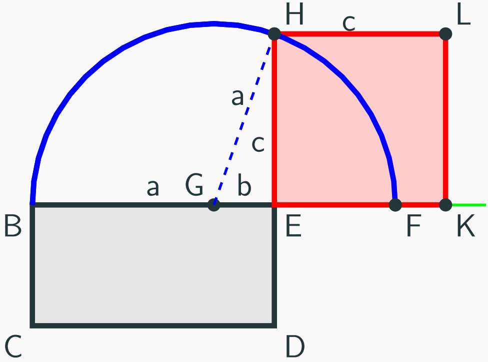
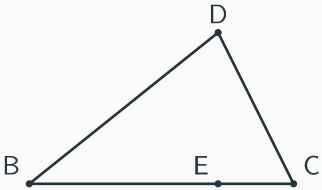


Figure 13: Quadrature of a Rectangle

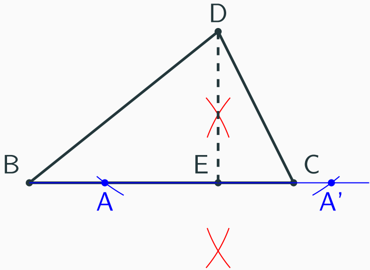
Quadrature of a Triangle

- Start with a triangle.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
    1. Draw blue arcs about D.
    2. Bisect AA' using red arcs.



Quadrature of a Triangle

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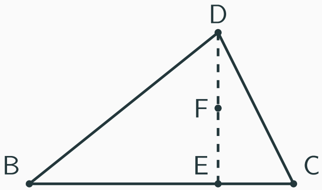
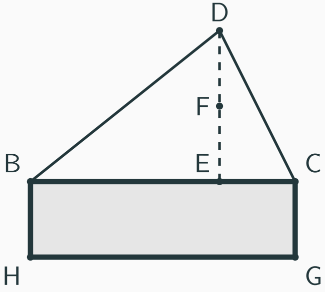
- Bisect perpendicular.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.

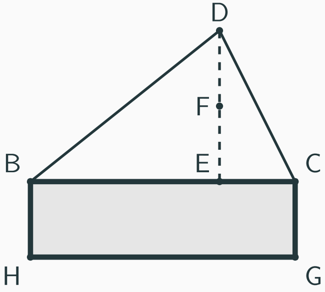


Quadrature of a Triangle

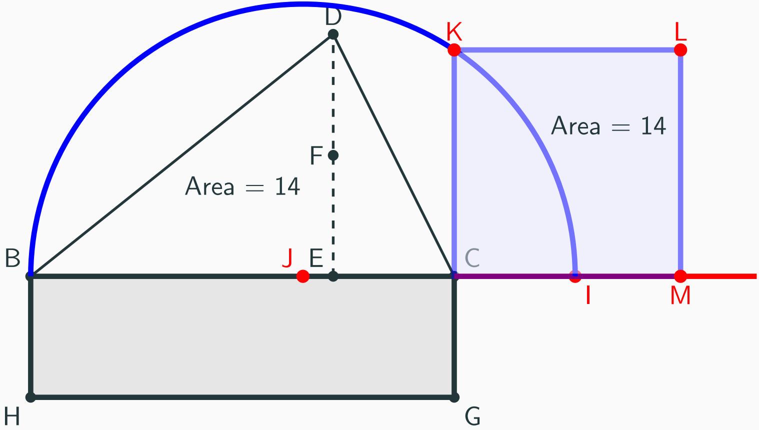
- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.
- Area of Triangle = Area of Rectangle:

$$\begin{aligned} A(BCD) &= \frac{1}{2}\overline{BC}\,\overline{DE} \\ &= \overline{BC}\,\overline{CG} \\ &= A(BCGH). \end{aligned}$$

- Square this rectangle.



Quadrature of a Triangle - Final Construction



Quadrature of a Lune

- Lune is the figure bounded by two circular arcs.
- Hippocrates squared a special lune.
- Based on
  - Pythagorean Theorem.
  - Angle inscribed in semicircle is right.
  - Ratio of Areas of circles
    $$\frac{A_1}{A_2} = \frac{D_1^2}{D_2^2}.$$
  - Triangles are quadrable.
- Hippocrates proof not valid.

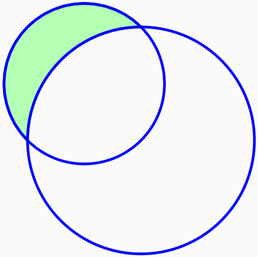


Figure 14: Lune or Crescent.

Hippocrates' Quadrature of a Lune

- $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = 2\overline{AC}^2$
- Semicircle areas

$$\frac{A(AEC)}{A(ACB)} = \frac{\overline{AC}^2}{\overline{AB}^2} = \frac{1}{2}.$$

- Area of Lune = Area of ΔAOC.
- ΔAOC quadrable, so is the lune.

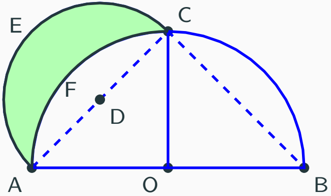


Figure 15: Lune AECF is quadrable.

Can one Square the circle?

Unsolved until Ferdinand Lindemann (1852-1939).

Algebraic Numbers, solutions of polynomial equations with integer coefficients.

Ex: $x^2 - 2 = 0$ has solution $\pm\sqrt{2}$.

Transcendental Numbers, numbers that aren't algebraic.

Timeline of Greek Mathematicians - Where Are WE?