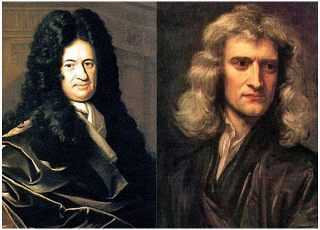
# Emergence of Calculus

Fall 2023 - R. L. Herman



# The Method of Exhaustion and the Infinite

- Zeno's Paradox of the Arrow

  "If a body moves from A to B, then before it reaches B it passes through the mid-point, say $B_1$ of AB. Now to move to $B_1$ it must first reach the mid-point $B_2$ of $AB_1$. Continue this argument to see that A must move through an infinite number of distances and so cannot move. " (450 BCE)
- Eudoxus - Method of Exhaustion.
- Archimedes - area of a segment of a parabola is $\frac{4}{3}$ the area of a triangle with the same base and vertex.
- Luca Valerio (1552-1618) published in 1608 *De quadratura parabolae.*
- Kepler (1571-1630): area as sum of lines. Inpsired Cavalieri's indivisibles.

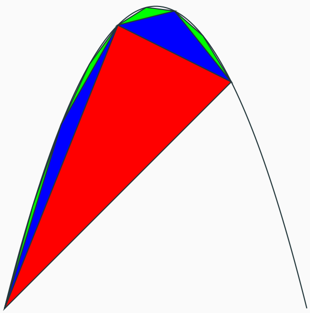


**Figure 1:** Archimedes: First known summation of series.

Area of blue $= \frac{1}{4}$ that of red, $a$. Then,

$$A = a + \frac{1}{4}a + \frac{1}{4^2}a + \ldots.$$

# Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Roberval (1602-1675)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

# Sixteenth Century Science

- Copernicus (1473-1543)
  Commentary - 1514
  *Dē revolutionibus orbium coelestium,* 1542 on death bed.
- Tycho Brahe (1546-1601)
- Galileo Galilei (1564-1642)
  1609 - Telescope
  Jupiter's Moons, Moon, Saturn, Phases of Venus.
  1633 trial and judgement for heresy, under house arrest.
- Johannes Kepler (1571-1630)
  1609 - Study on Mars orbit.
  Laws of Planetary Motion

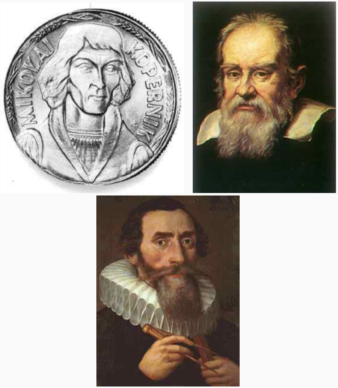



**Figure 2:** Copernicus, Galileo, Kepler

# Seventeenth Century - French, German, English Mathematics

- 1590 Vièta, *The Analytic Art*
- Bonaventura Cavalieri (1598-1647)
- Evangelista Torricelli (1608-1647)
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)


- Descartes

  philosopher, mathematician
  *Discours de la méthode*, Marriage of algebra/geometry - analytic geometry

- Pascal

  Wrote math before 16
  Probability theory
  Theology

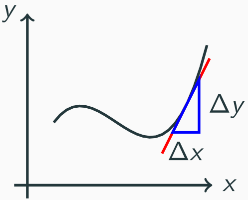
- Fermat

  Created analytic geometry
  Contributions to Calculus
  Number theory
  Scribbled in Diophantus' *Arithmetica*

# Tangents

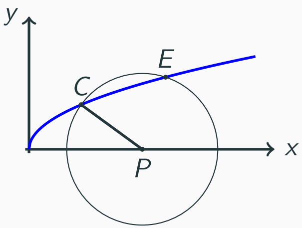
- Pierre de Fermat, René Descartes
- Both studied Apollonius' problem: construct a circle tangent to any three objects.
- Tangent line approximates curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat:
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter: Descartes to Mersenne
  $dy = f(x + dx) - f(x) =?dx.$

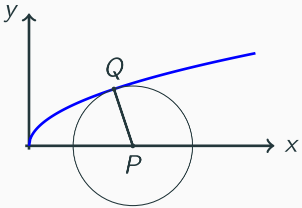
**Figure 3:** Fermat and Descartes



# Descartes vs Fermat - Analytic Geometry, Tangents, Optics

- Decartes published *La Géométrie* - 1637
- Depicted $ax + by = c$ as a line.
- Introduced $x$, $y$.
- Fermat introduced analytic geometry earlier.
- Fermat interested in optimization.
- Fermat: lawyer in Toulouse, Math a hobby.
- Descartes denounced him and challenged him to find tangent to folium, $x^3 + y^3 = 3axy$.
- Descartes' Method of Tangents: Find circles tangent to curves.
- Fermat challenged Descartes to explain refraction. Fermat published in 1662.



# Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - *Geometria indivisibilibus continuorum nova quadam ratione promota,* 1635.

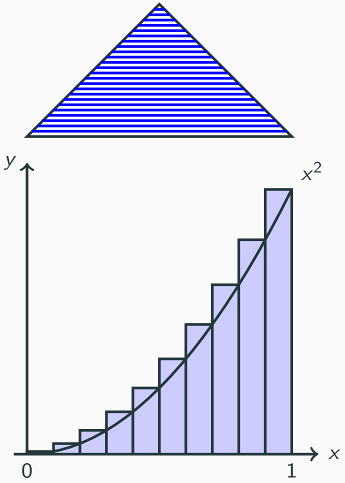
    Fill area with lines.
    But, an infinite number of lines sum to infinity.

- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\, dx.$$

$N$ segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

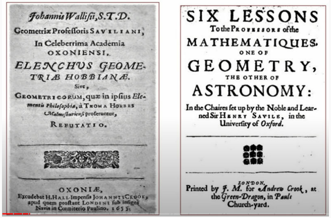
$$A \approx \sum_{k=1}^{N} \frac{1}{N} \left(\frac{k}{N}\right)^2.$$




Cavalieri's Method of Indivisibles

# John Wallis (1616-1703) - A Side Note

- 1649, Savilian professor of geometry at the University of Oxford.
- *Arithmetica Infinitorum*, "The Arithmetic of Infinitesimals", 1655
- Extended Cavalieri's law of quadrature.
- Algebraic vs Geometric approach.
- Influenced Newton.
- *Mathesis Universalis* Developed notation: introduced $\infty$, negative and fractional exponential notation.
- Royal Society of London in 1662.
- *Tractatus de Sectionibus Conicis,* 1659; described as curves using algebra.
- Published colleagues work on quadratures.



Johannis Wallisii, S.T.D.
Geometriæ Professoris SAVILIANI,
In Celeberrima Academia
OXONIENSI,
ELENCHUS GEOMETRIÆ HOBBIANÆ.
Sive,
GEOMETRICORUM, quæ in ipsius Elementis Philosophiæ, à THOMA HOBBES Malmesburiensi proferuntur,
REFUTATIO.

OXONIÆ,
Excudebat H. HALL Impensis JOHANNIS CROOK, apud quem prostant LONDINI sub insigni Navis in Cœmiterio Paulino. 1655.

SIX LESSONS
To the PROFESSORS of the
MATHEMATIQUES,
ONE OF
GEOMETRY,
THE OTHER OF
ASTRONOMY:
In the Chaires set up by the Noble and Learned Sir HENRY SAVILE, in the University of Oxford.

LONDON,
Printed by J. M. for Andrew Crook, at the Green-Dragon, in Pauls Church-yard.

**Figure 4:** Wallace-Hobbes Rivalry

- Thomas Hobbes (1588-1679), called book a "scab of symbols," quarter century controversy.
- Controversies with Huygens, Descartes, Fermat, Pascal.

# Savilian Chairs of Geometry and Astronomy

- The Savilian Chairs of Geometry and Astronomy, University of Oxford, 1619.
- By Henry Savile (1549 - 1622)
- Click to see list.
- 1570 Lectures on Ptolemy
- " he felt that mathematics at that time was not flourishing. Students did not understand the importance of the subject, Savile wrote, there were no teachers to explain the difficult points, the texts written by the leading mathematicians of the day were not studied, and no overall approach to the teaching of mathematics had been formulated."
- Read more at MacTutor.

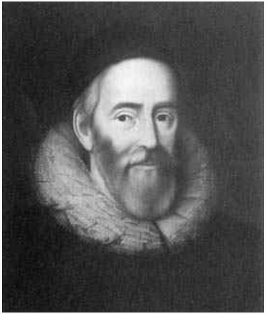


**Figure 5:** Henry Savile

# Back to Wallis' Areas Under Curves

Find the sum

$$
\begin{aligned}
A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\
&= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\
&= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\
&\sim \frac{2N^3}{6N^3} = \frac{1}{3}.
\end{aligned}
$$

Wallis showed

$$\boxed{\int_0^a x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^a = \frac{a^{n+1}}{n+1}}$$

for $k = 1, 2, \ldots, 9$.

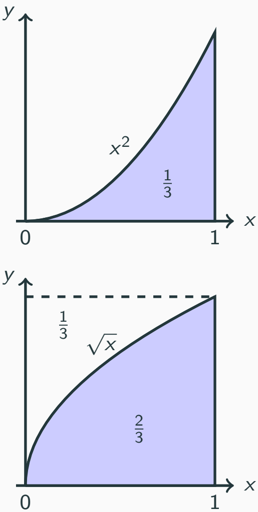
Note:

$$
\begin{aligned}
\sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\
&= N\frac{N+1}{2}.
\end{aligned}
$$

**Figure 6:** John Wallis

# Integrating Powers, $\int x^k\,dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for $k$ up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  **Ex:** $\int_0^1 \sqrt{x}\,dx$
  Use the symmetry in the figures.
- Areas under $x^k$, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.




**Note:** $1^3 + 2^3 + \cdots + k^3 = \frac{(N+1)^2 N^2}{4} = (1 + 2 + \cdots + k)^2$.

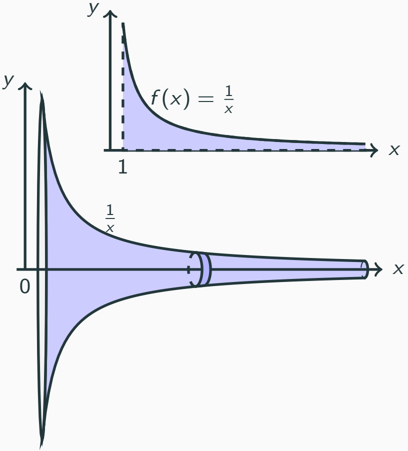
# Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, $x^{-1}$
- Area under $y = \frac{1}{x}$.

$$\int_1^\infty \frac{1}{x}\,dx = \infty.$$

- 1641 Torricelli's trumpet (Gabriel's horn)

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\,dx = \pi. \\ A &= 2\pi \int_1^\infty \frac{dx}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2} \\ &> 2\pi \int_1^\infty \frac{1}{x}\,dx = \infty. \end{aligned}$$

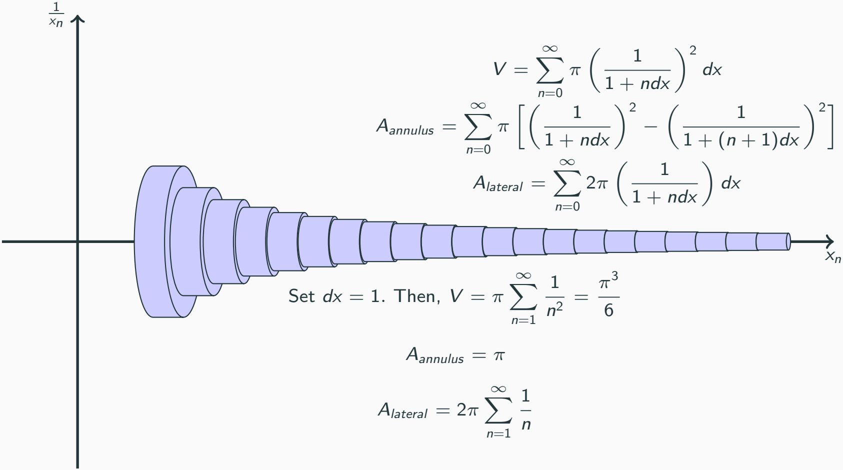



*What? You cannot paint the surface but can fill the trumpet with paint.*



Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

# Gabriel's Wedding Cake - Discrete Case




$$V = \sum_{n=0}^{\infty} \pi \left( \frac{1}{1+ndx} \right)^2 dx$$

$$A_{annulus} = \sum_{n=0}^{\infty} \pi \left[ \left( \frac{1}{1+ndx} \right)^2 - \left( \frac{1}{1+(n+1)dx} \right)^2 \right]$$

$$A_{lateral} = \sum_{n=0}^{\infty} 2\pi \left( \frac{1}{1+ndx} \right) dx$$

Set $dx = 1$. Then, $V = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$

$$A_{annulus} = \pi$$

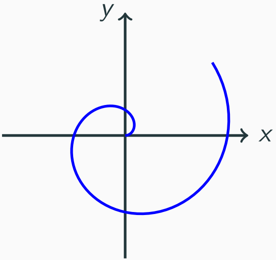
$$A_{lateral} = 2\pi \sum_{n=1}^{\infty} \frac{1}{n}$$

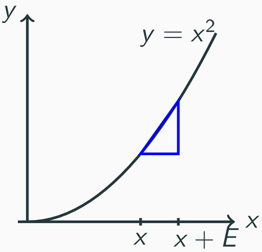
# Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, $E$
- **Example:** $y = x^2$

$$\frac{(x+E)^2 - x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.



# John Wallis' (1655) *Arithmetica Infinitorium*

- Combined Descartes' analytic geometry and Cavalieri's indivisibles.
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals - attacked by Thomas Hobbes (1588-1679).
- 1632 Church banned infinitesimals.
- Formulae for $\pi$ known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found $\pi$ to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\begin{aligned}
\frac{\pi}{4} &= \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\cdot \\
\frac{\pi}{2} &= \left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdots\cdot \\
\frac{4}{\pi} &= 1+\cfrac{1^2}{2+\cfrac{3^2}{2+\cfrac{5^2}{2+\cdots}}}
\end{aligned}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

# Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola*
  Written in 1665,
  1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$ replaced by $x^n$.
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

[Note: Here $\log x = \ln x$.]

Note: Geometric series

$$\begin{aligned} 1 + t + t^2 + \cdots &= \frac{1}{1-t}, |t| < 1. \\ 1 - t + t^2 - \cdots &= \frac{1}{1+t}, |t| < 1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

# Invert Power Series

We have for $y = \log(1 + x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$ . Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of $y$, then ...

# Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned}
0 &= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 \\
1 &= a_1\left(a_0^2 - a_0 + 1\right) \\
0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\
0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right).
\end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!}.$$

# Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

However,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So, we found the series expansion

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

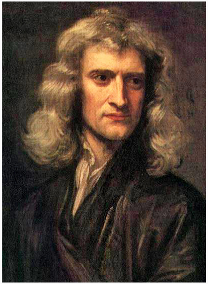


**Figure 7:** Newton

# Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{k=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. Then,

$$
\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!} a^2 + \frac{p(p-1)(p-2)}{3!} a^3 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= \int_0^x \left( 1 + \frac{1}{2} t^2 + \frac{3}{8} t^4 + \cdots + \frac{-\frac{1}{2}(-\frac{3}{2})\cdots(\frac{1}{2}-k)}{k!} (-t^2)^k + \cdots \right) \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\frac{x^{2k+1}}{2k+1} + \cdots
\end{aligned}
$$

Inverting, Newton found $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$ .

# Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx} \int f(x)\, dx = f(x)$.

**Figure 8:** Leibniz

# Infinite Series

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a+ar+ar^2+\cdots+ar^n+\cdots = \frac{a}{1-r}, |r|<1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots \\ = & (1)+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots \\ \geq & \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots=\infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, . . . .

**Figure 9:** Euler

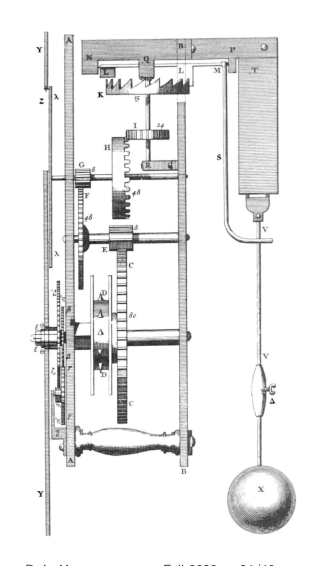
# James Gregory (1638 - 1675)

- Uncle to David Gregory (1659-1708), Professor of Mathematics, University of Edinburgh at 24, filling the chair previously held by James Gregory. Savilian Professor of Astronomy, Oxford, Supported Newton in controversy and first to teach *Principia*.
- First to publish and prove Fundamental Theorem of Calculus, *Geometriae Pars Universalis* (1668) .
- Discovered 7 series before Taylor.
- *Optica Promota,* first practical reflecting (Gregorian) telescope.
- Worked with Angeli at Padua.

**Figure 10:** James and David Gregorie

# Problems of the Day

- Pendulum clock of Galileo - Thought isochronous, time-independent period of swing. Son Vincenzio, worked on it, died 1649.
- Huygens built first pendulum driven clock, 1656.
- Tautochrone - Time taken independent of starting point, Huygens, 1659.
- Brachistochrone - Curve of fastest descent, posed by Johann Bernoulli, 1696.
- Isochrone - Connects points of equal time travel, Leibniz 1687, Jacob Bernoulli 1690.

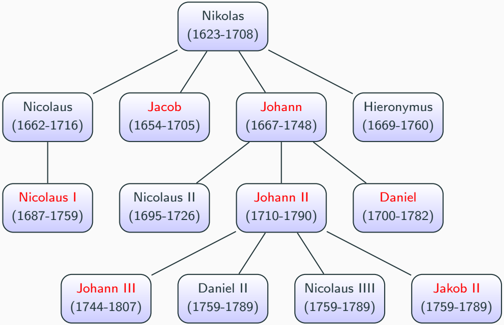


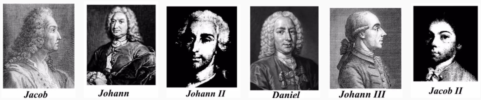
# Calculus Wars

- Newton: 'the method of fluxions'.
- Paper on fluxions in 1666, but not published for decades.
- *Philosophiae naturalis principia mathematica*, published 1687.
- Little explicit calculus.
- Method of fluxions appeared in 1693.
- Leibniz, published first paper on calculus, 1684.
- Said he discovered calculus in 1670s.
- In 1695, Wallis: Leibniz learned about calculus from Newton.
- Nicolas Fatio de Duillier (1664-1753) in 1699 book, Newton's absolute priority.
- Angry responses from Johann Bernoulli and Leibniz.
- John Keill accused Leibniz of plagiarism, 1711.
- Royal Society in England gave report that Leibniz had concealed knowledge of Newton's work, 1712.
- Leibniz accused Newton and followers of stealing his calculus.
- Debate ended when Leibniz died, 1716.

# The Bernoulli Family

- In three generations, there were 8 mathematicians.
- Dominated mathematics and physics, 17-18th centuries - with Newton, Leibniz, Euler, Lagrange, etc.
- Contributions: calculus, geometry, mechanics, probability, ballistics, thermodynamics, hydrodynamics, optics, elasticity, magnetism, astronomy.



# Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = ?$

**Figure 11:** Jacob and Johann

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots \left(\cancel{\frac{1}{N}} - \frac{1}{N+1}\right) \\
&= 1 - \frac{1}{N+1} \underset{N\to\infty}{\Longrightarrow} 1.
\end{aligned}$$

# Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies $p(x) = (x - r)q(x)$, $q(x)$ - polynomial

**Proof:**

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

# Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi,\ n = 0, \pm 1, \pm 2, \ldots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\cdots \\
&= Ax\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x - \frac{x^3}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) + x^5(\cdots) - \cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots.$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \Rightarrow \zeta(2) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

# What are the next coefficients?

We need the $x^5$ terms in the expansion

$$\begin{aligned}\sin x &= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\ &= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots\left(1-\frac{x^2}{m^2\pi^2}\right)\cdots\end{aligned}$$

We multiply $\frac{x^2}{m^2\pi^2}$ times the factors $\frac{x^2}{n^2\pi^2}$, $n\neq m$, and summing:

$$x\sum_{m=1}^{\infty}\frac{x^2}{m^2\pi^2}\sum_{n=1,n\neq m}^{\infty}\frac{x^2}{n^2\pi^2}=\frac{x^5}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1,n\neq m}^{\infty}\frac{1}{n^2}.$$

$$\frac{\pi^4}{5!}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2}\left[\zeta(2)-\frac{1}{m^2}\right]=\frac{1}{2}[\zeta(2)^2-\zeta(4)].$$

So, $\zeta(4)\equiv\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{36}-\frac{\pi^4}{60}=\frac{1}{12}\left(\frac{\pi^4}{3}-\frac{\pi^4}{5}\right)=\frac{\pi^4}{90}$.

# Another Approach to Obtain $\zeta(4)$

Noting that $\frac{d}{dx}(\ln \sin x) = \cot x$ and using the known series expansion for $x \cot x$ in terms of Bernoulli numbers,

$$\begin{aligned}
\sin x &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
\ln \sin x &= \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
x \cot x &= 1 - \sum_{n=1}^{\infty} \frac{2x^2}{n^2\pi^2 - x^2} \\
&= 1 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{x^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{x^2}{n^2\pi^2}\right)^k \\
1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} + \dots &= 1 - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{\pi}\right)^{2k+2} \zeta(2k+2) \\
x \cot x &= 1 + \sum_{k=0}^{\infty} (-1)^k B_{2k} (2x)^{2k}
\end{aligned}$$

# Results for the Riemann Zeta Function, $\zeta(s)$

Therefore, we have

$$\begin{aligned} \zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots &= \frac{\pi^2}{6}, \\ \zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots &= (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \end{aligned}$$

where $B_{2n}$ are Bernoulli numbers,[1] $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots,$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Euler (1748) - Zeta function can be defined for $p$ prime as

$$\begin{aligned} \zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1}\cdots\left(1 - \frac{1}{p^s}\right)^{-1}\cdot. \end{aligned}$$

[1] Jacob Bernoulli, 1713, Seki Takakazu, 1712, published posthumously. $B_0 = 1, B_1 = -\frac{1}{2}$.

# Georg Friedrich Bernhard Riemann[2] (1826-1866)

- Riemann extended Euler's zeta function, $s \in \mathbb{C}$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

$$\begin{aligned} &\zeta(1) = \infty, \text{ harmonic series} \\ &\zeta(2) = \tfrac{\pi^2}{6} \\ &\zeta(3) \quad \text{irrational, Apery (1981)} \end{aligned}$$

- Zeros

$\zeta(-2n) = 0$, $n$ integer $> 0$.

**Riemann Hypothesis:**

$\zeta(\sigma + it) = 0$ when $\sigma = \frac{1}{2}$.

- Connection to primes?

**Figure 12:** Bernhard Riemann

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(2s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \\ \zeta(2n) &= \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \\ \zeta(s) &= 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s) \\ \Gamma(s) &= \int_0^\infty x^{s-1} e^{-x}\, dx \end{aligned}$$

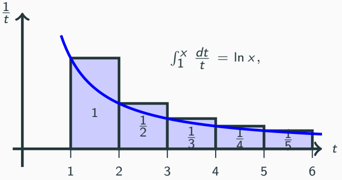
[2] *On the Number of Primes Less Than a Given Magnitude*, 1859

# Connection to Primes and Other Tidbits

$$\begin{aligned}\zeta(s) &= \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\cdots\left(1-\frac{1}{p^s}\right)^{-1}\cdots \\ &= \prod_{p=\text{prime}}\left(1-\frac{1}{p^s}\right)^{-1} \\ &= \prod_{p=\text{prime}}\left[1+\frac{1}{p^s}+\left(\frac{1}{p^s}\right)^2+\cdots\right]\end{aligned}$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant $\gamma \approx 0.577218\ldots.$
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$




$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right) = \gamma.$$

# Euler Eta Function

The Euler, or Dirichlet, eta function is

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad \operatorname{Re}(s) > 0. \tag{1}$$

It is related to the zeta function by

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s).$$

Special values of the eta function are

$$\begin{aligned} \eta(0) &= 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}, \\ \eta(-1) &= 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}, \\ \eta(1) &= 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cdots = \ln 2. \end{aligned} \tag{2}$$

# $1 + 2 + 3 + \cdots = -\frac{1}{12}.$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1. \tag{3}$$

Note that

$$\zeta(-1) = 1 + 2 + 3 + \cdots .$$

But, the Riemann zeta function is not defined for $s = -1$.

So, we can use $\eta(s)$ to analytically continue $\zeta(s) = \dfrac{\eta(s)}{1 - 2^{1-s}}$.

Setting $s = -1$, we obtain

$$1 + 2 + 3 + \cdots = \frac{\eta(-1)}{1 - 2^2} = -\frac{1}{12},$$

assuming that $\eta(-1) = \dfrac{1}{4}$, using Abel summation.

# Leonhard Euler (1707-1783)

Euler (at 14) studied under Johann Bernoulli, graduated in 1723.

Went to St. Petersburg in 1727, Berlin in 1741, and back to St. Petersburg in 1766.

By 1730's - lost vision in right eye and blind by 1771.

866 books and papers - 228 after death. *Opera Omnia* - over 25,000 pgs

First appearance of $e$ - letter to Goldbach in 1731.

Euler published *Introductio in Analysin infinitorum* - 1748

Euler's Formula, $e^{ix} = \cos x + i \sin x$.

Euler's Identity, $e^{i\pi} + 1 = 0$.

Euler's constant, $\gamma$

Euler's Polyhedral Formula, $V + F = E + 2$.

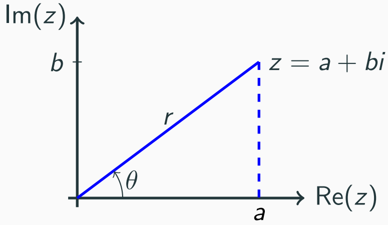
# Amicable Numbers

- Recall Greeks knew 220 and 284; i.e., sum of proper factors of 220 = 284 and vice versa.
- Thabit ibn Qurra (836-901) discovered the next amicable pairs, for example 17296, 18416.
- Pierre Fermat rediscovered this pair in 1636.
- René Descartes discovered Qurra's pair 9,363,584 and 9,437,056 in 1638.
- 1747, Euler published [E100] giving 30 amicable pairs.
- By 1750 - Euler found 61 pairs!

# Euler's Formula - Exponentiate $i\theta$.

- Complex numbers, polar form.
  $z = a + bi$, $a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$




- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

# Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta$.

- $\theta = \pi,\ e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0$.
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\ e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n .$$

- **Example:** $n = 2$

$$\begin{aligned} \cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta. \end{aligned}$$

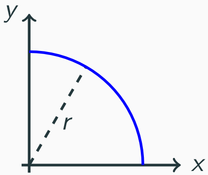
$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta. \end{aligned}$$

# Rectification of a Circle - Recalling Calculus II

- Rectification = Finding arclengths
- The length of the curve $y = y(x)$ :

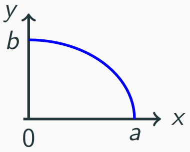
$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

- **Example:** Circle: $x^2 + y^2 = r^2$
  $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$




$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + \frac{x^2}{y^2}}\, dx \\ &= 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r \sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

# Arclength of an Ellipse



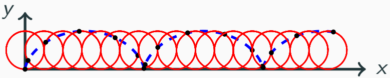

- **Example:** Ellipse
  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$

$$
\begin{aligned}
y &= \frac{b}{a}\sqrt{a^2 - x^2} \\
y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\
1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\
L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx.
\end{aligned}
$$

**Historical Curves**

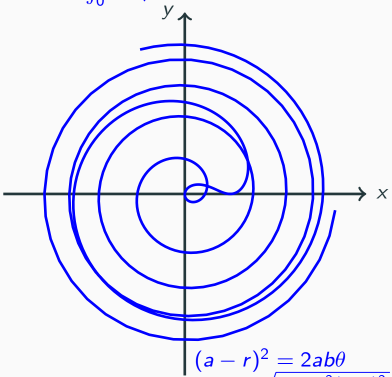
- 1609 - Kepler - Mars' orbit is an ellipse.
- 1659 - Pascal *Dimensions des lignes courbes de toutes les Roulettes.* Roulette curves.
- 1658 Proof by Wren published by Wallis in 1659 - On the rectification of the cycloid.
- 1676 - Newton - infinite series.
- 1742 - Maclaurin - expansion in eccentricities.
- 1691 - Jacob Bernoulli - parabolic spiral.

# Cycloid, Parabolic Spiral, and Lemniscate




$x = r(t - \sin t),\ y = r(1 - \cos t)$

$L = \int_0^{2\pi} r\sqrt{2 - 2\cos t}\, dt = 8r$

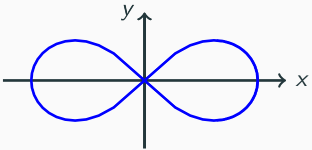


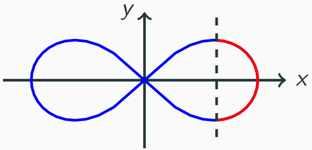

$(a - r)^2 = 2ab\theta$

$s = \int \sqrt{1 + \frac{r^2(a-r)^2}{a^2 b^2}}\, dr$

- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$




Elastica

Sep 1694, Jacob Bernoulli

Oct 1694, Johann Bernoulli

# Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\,dt$, $R$ is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate, 1718.
- Carl Friedrich Gauss (1777-1855) ∼1800 studied inverse $x = sl(u)$ Doubly periodic functions

$$sl(u + 2\bar{\omega}) = sl(u), \quad sl(u + 2i\bar{\omega}) = sl(u)$$

$$\bar{\omega} = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2.62205\ldots.$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

# Addition Theorem for Circle

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2\sin u \cos u \\ &= 2\sin u\sqrt{1-\sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2\int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2\sin u\sqrt{1-\sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

# Elliptic Integral Addition Theorem for Lemniscate

In 1718 Fagnano found formula for doubling arclength of lemniscate.

He solved differential equation

$$\frac{dt}{\sqrt{1-t^4}} = \frac{2dx}{\sqrt{1-x^4}}, \Rightarrow t = \frac{2x\sqrt{1-x^2}}{1+x^4}.$$

So, if the arclength of lemniscate is

$$\int_0^x \frac{dt}{\sqrt{1-t^4}},$$

then double the arclength is

$$\int_0^{\frac{2x\sqrt{1-x^2}}{1+x^4}} \frac{dt}{\sqrt{1-t^4}}.$$

Led Euler to write extensively on elliptic integrals starting in 1752.

# Elliptic Integrals

- Study of Inversions
  Gauss 1790s - $\int \frac{dt}{\sqrt{1-t^3}}$,
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- Legendre - two papers 1786 (40 yrs earlier)
  Legendre classified elliptic integrals into 3 cases,
  Produced 3 volumes, 1811-1816. Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\, d\theta.$$

- Riemann's geometric setting, Riemann surfaces, 1851 - torus.

# Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\ldots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

# Application of $AGM(a, b)$

**Example:** $AGM(1, 2)$. Start with $a_0 = 1, \quad b_0 = 2,$

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

| $a_n$ | $b_n$ |
|---|---|
| 1.0000 | 2.0000 |
| 1.5000 | 1.4142 |
| 1.4571 | 1.4565 |
| 1.4568 | 1.4568 |
| ⋮ | ⋮ |

$$AGM(a, b) = \frac{\pi}{4} \frac{a + b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$