Cubic Equations

Fall 2023 - R. L. Herman

Solutions of Polynomial Equations

- Linear equations, known solutions.
- Chinese - Gaussian elimination:
  Systems of n linear equations and n unknowns.
- Quadratic equations:
  Need square roots.
- Cubic equations:
  Need square roots and cube roots.
  Solved in 16th century.
- Quintic equation: studied in 1820's.
  Eventually lead to group theory!

Figure 1: Leonardo da Vinci attempts Delian problem (Doubling cube).

Quadratic Equations

Babylonian Method (Modern Notation)

Find x and y for a given perimeter and area.

$$\begin{aligned} x + y &= p \\ xy &= q. \end{aligned}$$

Eliminate y, $x^2 + q = px$. Then,

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Method - Compute the following

1. $\frac{x+y}{2}$
2. $\left(\frac{x+y}{2}\right)^2$
3. $\left(\frac{x+y}{2}\right)^2 - xy = \frac{(x+y)^2 - 4xy}{4}$
4. $\sqrt{\frac{(x+y)^2 - 4xy}{4}} = \frac{x-y}{2}$
5. By inspection, get x, y from p and q, since

$$\begin{aligned} \frac{x-y}{2} &= \sqrt{\frac{p^2 - 4q}{4}} \\ &= \sqrt{\left(\frac{p}{2}\right)^2 - q}. \end{aligned}$$

Quadratic Equations (cont'd)

- Brahmagupta (628) - Explicit
  $ax^2 + bx = c$.

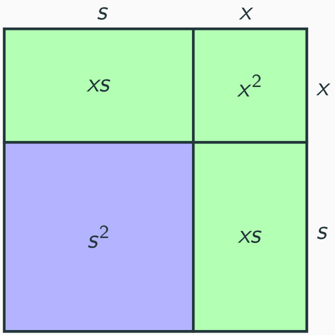
$$x = \frac{\sqrt{4ac + b^2} - b}{2a},$$

- Euclid - Prop. 28
- al'Khwarizimi

$$\begin{aligned} x^2 + 2xs &= n \\ x^2 + 2xs + s^2 &= n + s^2 \\ (x+s)^2 &= n + s^2 \end{aligned}$$

- Quadratic Irrationals

$$\frac{a + \sqrt{b}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$



Note from the figure:
Green area $= x^2 + 2xs = n$.
No negative lengths (solutions).

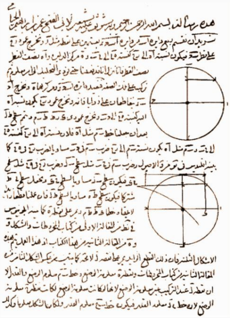
Cubic Equations

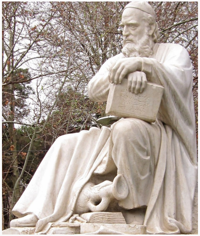
- Babylonians - Table of cubes
- Greeks - Geometric Problems
    - Duplicating cube (Delian Prob.).
    - Intersecting conics.
    - Cutting Sphere with plane.
- Omar Khayyam (1048-1131)
    - First general theory of cubics.
    - Provided 19 types of cubic.
    - Example

$$x^3 + ax^2 = bx + c,$$

Cube and square equals side plus number.

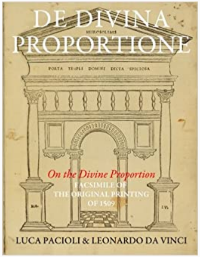
    - Geometric: Intersect two hyperbolae.





Pacioli, da Vinci and della Francesca

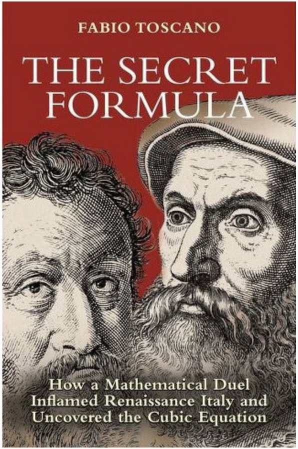
- Luca Pacioli (1445-1517)
  - Franciscan friar, tutor.
  - *Summa de arithmetica*, 1494. Father of Accounting.
  - *Divina proportione*, 1509.
  - “Solution to cubic is impossible!”
  - On table: slate, chalk, compass, dodecahedron. Hanging: Rhombicuboctahedron half-filled with water.
- Leonardo da Vinci (1452-1519)
- Piero della Francesca’s (1420-1492) Painter/mathematician, met Alberti, 1451. Wrote books: algebra, perspective, Archimedean polyhedra. Pacioli used his work.



The Secret Formula

How a Mathematical Duel Inflamed Renaissance Italy and Uncovered the Cubic Equation, by Fabio Toscano, 2020.

- The Abbaco Master
  Italian Wars, Brescia, 1512, Tartaglia.
- The Rule of the Thing
  cosa (thing), *censo* (x^2), *numero*.
- The Venetian Challenge
  1535, Fior challenges Tartaglia.
- An Invitation to Milan
  Entrance of Gerolamo Cardano.
- The Old Professor's Notebook
  da Coi vs. Ferrari, del Ferro's priority:
  Solution: things and cube equal to number.
- The Final Dual
  Ars Magna, 1545. Ferrari vs. Tartaglia, 1547.



The Search for Solutions

- Scipio del Ferro (1465-1526)
    - University of Bologna, notebooks
    - Printing press - Guttenberg
    - 1506/1514, solution of **depressed cubic**: $x^3 + ax = b$.
    - Public Challenges led to secrecy.
- Gave to Antonio Maria Fior (Florido).
- Tartaglia (Nicolo Fontana) (1499-1557)
    - 1512, French attack - sabre wound led to stammer.
    - Self-educated
    - 1530 da Coi wrote to him $x^3+3x^2 = 5$, $x^3+6x^2+8x = 1000$.

Figure 2: Tartaglia

Sidenotes

Pacioli, *Summa de arithmetica* 1494, Not solvable:

$$\begin{aligned} n &= ax + bx^3 \\ n &= ax^2 + bx^3 \\ n &= ax^3 + bx^4 \end{aligned}$$

Tartaglia, Abaco teacher/master but engaged in other activities.

Challenges not uncommon, but expect challenger to know solutions to their questions.

Not happy with da Coi questions.

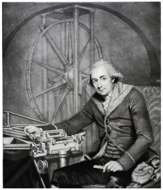
The Plot Thickens

- Tartaglia boasted he could solve $x^3 + ax^2 = c$.
- Florido challenged Tartaglia
  - Each posed 30 problems
  - Florido mostly gave problems of form $x^3 + ax^2 = c$.
  - Tartaglia won by solving depressed cubic 1535, but didn't publish.
- Girolamo Cardano (1501-1576)
  - Gambler, astronomer, physician, astrologer, heretic, father of murderer.
  - Begged for solution from Tartaglia. Finally, they met in Milan.
  - Tartaglia eventually gave solution in 1539 as a Poem if it was kept secret.
  - It was not in Cardano's book.

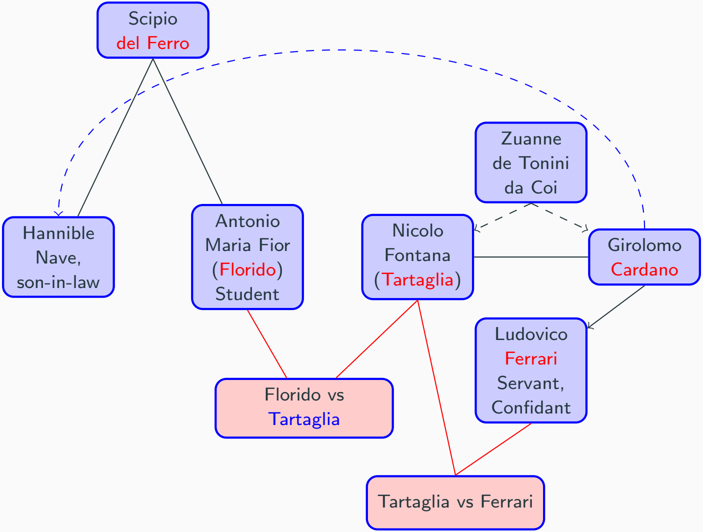
Figure 3: Cardano, *Ars Magna*.

Enter Ludovico Ferrari

- Ludovico Ferrari (1522-1565)
  - Servant at 14
  - Secretary, confidant
  - Worked on problems with Cardano
  - Cubic and biquadratic equations
- da Coi $\rightarrow$ Cardano $\rightarrow$ Ferrari
  - 4th degree polynomial
  - Ferrari solution involved solving cubic
  - Publishing was a problem.
- 1543 Trip to Florence, stopped in Bologna on the way.
  - Visitied Hannible Nave, del Ferro's son-in-law.
  - Saw del Ferro's notes.
  - Cardano believed he could publish in his *Ars Magna*, 1545.
- Barrage of letters from Tartaglia!



The Players in the Cubic Story



Tartaglia vs Ferrari - 1548

- Public debate in Milan, Ferrari's hometown.
- Cardano was absent.
- Tartaglia lost, blamed crowd.
- Tartaglia worked on arithmetic.
- Ferrari became professor in Bologna, 1565.
  Was poisoned 1565, white arsenic, possibly by sister.
- Cardano predicted exact date of his own death in 1576.

Figure 4: Tartaglia and Ferrari

Solution of the Quadratic $x^2 + ax + b = 0$

- Completing the square:
  $\left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4} = 0.$
- Solution: $x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - b}.$
- Graph of parabola
  $y = x^2 + ax + b$
  Vertex $(-\frac{a}{2}, b - \frac{a^2}{4})$
- Number of real solutions?
- Substitute $x = u - \frac{a}{2}$:
  $0 = x^2 + ax + b$
  $0 = (u - \frac{a}{2})^2 + a(u - \frac{a}{2}) + b$
  $0 = u^2 + b - \frac{a^2}{4}.$
- Solve for u.

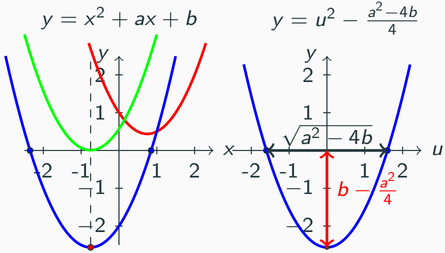


Figure 5: Plots of parabolae. Translating blue parabola on left by $\frac{a}{2}$ results in that on the right.

Plotting a Cubic Function $y = x^3 + ax^2 + bx + c$

- Set $y = 0$, then $x^3 + ax^2 + bx + c = 0$.
- Solutions are black points. Always have a real solution.
- $y' = 3x^2 + 2ax + b$ and $y'' = 6x + a$.
- Inflection point: $y'' = 0$ for $x_0 = -\frac{a}{3}$, $y_0 = c - \frac{ab}{3} + \frac{2a^3}{27}$.
- Slope of tangent at $(x_0, y_0 = q)$ is $p = b - \frac{a^2}{3}$.
- Extrema at $x_\pm = -\frac{a}{3} \pm \frac{\sqrt{a^2-3b}}{3}$.

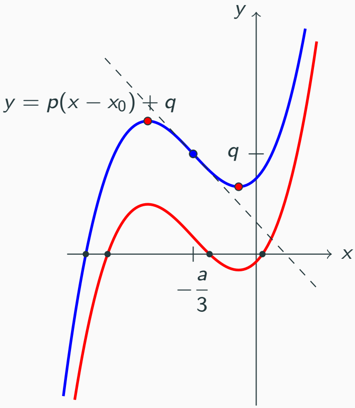


Figure 6: Plots of the cubic function exhibiting either one or three real roots.

Can you complete the cube: $(x + \alpha)^3 = x^3 + 3x^2\alpha + 3x\alpha^2 + \alpha^3$?

Solution of the Cubic $x^3 + ax^2 + bx + c = 0$.

Let $x = y - \frac{a}{3}$.

Then, $y^3 + py + q = 0$, where

$$\begin{aligned} p &= b - \frac{a^2}{3}, \\ q &= c - \frac{ab}{3} + \frac{2a^3}{27}. \end{aligned}$$

Let $y = u + v$:

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Let $p + 3uv = 0$, then

$$\begin{aligned} u^3 v^3 &= -\frac{p^3}{27}, \\ u^3 + v^3 &= -q. \end{aligned}$$

Now, define $X = u^3$. $Y = v^3$.

We obtain

$$\begin{aligned} X + Y &= -q. \\ XY &= -\frac{p^3}{27}, \end{aligned}$$

Does this look familiar?

The solution of Cubic:

$$X, Y = u^3, v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}},$$

$y = u + v$,

$x = y - \frac{a}{3}$.

Example: $2x^3 - 30x^2 + 162x - 350 = 0.$

Let $x = y - \frac{b}{3a} = y + \frac{30}{6} = y + 5.$

We obtain a depressed cubic (del Ferro), $y^3 + 6y - 20 = 0.$

Letting $y = u + v$, $X, Y = u^3, v^3$, we solve

$$X + Y = -q = 20, \quad XY = -\frac{p^3}{27} = -\frac{6^3}{27} = -8.$$

Eliminating Y, $X^2 - 20X - 8 = 0.$

Solving, leads to $X = 10 \pm \sqrt{108}, \qquad Y = 20 - X = -10 \mp \sqrt{108}.$

So, $u = \sqrt[3]{10 \pm 6\sqrt{3}} \qquad v = \sqrt[3]{-10 \pm 6\sqrt{3}}$ and

$$y = u + v = \sqrt[3]{10 \pm 6\sqrt{3}} + \sqrt[3]{-10 \pm 6\sqrt{3}}$$

$$x = \sqrt[3]{10 \pm 6\sqrt{3}} - \sqrt[3]{-10 \pm 6\sqrt{3}} + 5$$

Depressed Cubic $y^3 + 6y - 20 = 0$.

Note: $y = 2$ is a solution. Therefore, $y^3 + 6y - 20 = (y-2)(y^2 + \alpha y + \beta)$.

Method 1: Expand and match coefficients.

$$y^3 + 6y - 20 = y^3 + (\alpha - 2)y^2 + (\beta - 2\alpha)y - 2\beta$$

$$\alpha - 2 = 0, \qquad \beta - 2\alpha = 6, \qquad 2\beta = 20 \quad \Rightarrow \quad \alpha = 2, \beta = 10.$$

Method 2: Long division.

$$\begin{array}{r|l} & y^2 + 2y + 10 \\ \hline y - 2 & y^3 + 6y - 20 \\ & -y^3 + 2y^2 \\ \hline & 2y^2 + 6y \\ & -2y^2 + 4y \\ \hline & 10y - 20 \\ & -10y + 20 \\ \hline & 0 \end{array}$$

Find other roots:

$$y^2 + 2y + 10 = 0$$

$$\begin{aligned} y &= \frac{-2 \pm \sqrt{4 - 40}}{2} \\ &= -1 \pm 3i. \end{aligned}$$

Nested Radicals

We have found that $y = \sqrt[3]{10 \pm 6\sqrt{3}} - \sqrt[3]{-10 \pm 6\sqrt{3}}$. Can one simplify this? We consider $\sqrt[3]{10 \pm 6\sqrt{3}} = \sqrt{x} \pm \sqrt{y}$ and $\sqrt[3]{-10 \pm 6\sqrt{3}} = -\sqrt{x} \pm \sqrt{y}$. Note:

$$\begin{aligned}
(\sqrt{x} \pm \sqrt{y})^3 &= x\sqrt{x} \pm 3x\sqrt{y} + 3y\sqrt{x} \pm y\sqrt{y} \\
&= (x + 3y)\sqrt{x} \pm (3x + y)\sqrt{y} \\
&= 10 \pm 6\sqrt{3}. \qquad (1) \\
(-\sqrt{x} \pm \sqrt{y})^3 &= -x\sqrt{x} \pm 3x\sqrt{y} - 3y\sqrt{x} \pm y\sqrt{y} \\
&= -(x + 3y)\sqrt{x} \pm (3x + y)\sqrt{y} \\
&= -10 \pm 6\sqrt{3}. \qquad (2)
\end{aligned}$$

From (1) $y = 3$, $x + 9 = 10$, $\quad 3x + 3 = 6$. Then, $x = 1$ and $\sqrt[3]{10 \pm 6\sqrt{3}} = 1 \pm \sqrt{3}$.

Similarly, from (2), $\sqrt[3]{-10 \pm 6\sqrt{3}} = -1 \pm \sqrt{3}$. So,

$$y = \sqrt[3]{10 \pm 6\sqrt{3}} + \sqrt[3]{-10 \pm 6\sqrt{3}} = 2.$$

Complex Solutions $y^2 + 2y + 10 = 0.$

We found $y^3 + 6y - 20 = (y-2)(y^2 + 2y + 10) = 0.$

we solved the quadratic: $y = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\sqrt{-1}.$

- Cardano, complex numbers
  "as subtle as they are useless."
- Raphael Bombelli (1526-1572)
  First to take seriously.
- Ex: $x^3 = 15x + 4$
  $x = \sqrt[3]{2 + 11\sqrt{-11}} + \sqrt[3]{2 - 11\sqrt{-11}}$
- But, $x = 4$ is a solution!
- Complex numbers, $a + bi$, $i = \sqrt{-1}.$
- $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$

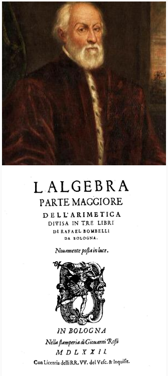


Figure 7: Bombelli

Cube Root of Complex Numbers

- Last Example: $x^3 = 15x + 4$
  $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$
- Seek: $\sqrt[3]{2+11i} = c + di$.

$$\begin{aligned} \sqrt[3]{2+11i} &= c + di \\ 2+11i &= (c+di)^3 \\ &= c^3 + 3c^2di + 3c(di)^2 + (di)^3 \\ &= c^3 - 3cd^2 + i(3c^2d - d^3). \end{aligned}$$

Then

$$\begin{aligned} 2 &= c^3 - 3cd^2 = c(c^2 - 3d^2), \\ 11 &= 3c^2d - d^3 = d(3c^2 - d^2). \end{aligned}$$

- Bombelli: c, d, positive integers.

Since 2 is prime, $c = 1, 2$.

If $c = 1$, $2 = 1 - 3d^2$. No!

If $c = 2$, then $d = 1$.

$$\begin{aligned} 2 &= 8 - 6d^2 \\ 11 &= 12d - d^3 \end{aligned}$$

Then,

$$\begin{aligned} x &= \sqrt[3]{2+11i} + \sqrt[3]{2-11i} \\ &= (2+i) + (2-i) = 4. \end{aligned}$$

Françios Viète (1540-1603)

- Counselor to Henry III, IV, France
- French Wars of Religion 1562-1598
- Tutored Catherine de Pathenay (1554-1631), noblewoman, mathematician
- 1596 - Adriaan van Roomen
  "No French mathematician could solve the 45th degree polynomial."
  $x^{45} - 45x^{43} + 945x^{41} + \cdots - 3795x^3 + 45x = A.$
- Viète solved quickly:
  $2\sin(45\alpha) = A, \quad x = 2\sin\alpha.$
- Trig identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
  Let $y = \cos\theta$. $4y^3 - 3y = c, |c| \le 1$. $c = \cos 3\theta$.
  Solve for θ given c. Solution, $y = \cos\theta$.
- Use identities to rewrite $2\sin(45\alpha) = A$ in terms of $2\sin\alpha$.

Figure 8: Viète, Henry IV, and van Roomen.

Viète's Solution

Define the quantities

$$c = 2\sin 45\theta, \qquad y = 2\sin 15\theta,$$
$$z = 2\sin 5\theta, \qquad x = 2\sin\theta. \tag{3}$$

Problem: Find x, given c.

Use the identities:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \tag{4}$$
$$\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha \tag{5}$$

Then,

$$c = 2\sin 45\theta = 6\sin 15\theta - 8\sin^3 15\theta = 3y - y^3. \tag{6}$$
$$y = 2\sin 15\theta = 6\sin 5\theta - 8\sin^3 5\theta = 3z - z^3. \tag{7}$$
$$z = 2\sin 5\theta = 10\sin\theta - 40\sin^3\theta + 32\sin^5\theta = 5x - 5x^3 + x^5. \tag{8}$$

Viète's Solution (cont'd)

Since $z = 5x - 5x^3 + x^5$., we write c in terms of x :

$$
\begin{aligned}
y &= 3z - z^3 \\
&= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3 \\
&= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x. \qquad (9) \\
c &= 3y - y^3 \\
&= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x] \\
&\quad -[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3 \\
&= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} \\
&\quad -14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} \\
&\quad +483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\
&\quad -34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \\
&= P_{45}(x). \qquad (10)
\end{aligned}
$$

Math Symbols

- $+$ Oresme (1300)
- $-$ Widman (1400)
- $=$ Recorde (1500)
- $\times$ Outred (1500)
- Letters Viète (1500)
- Descartes (1500-1600)
  unknowns, constants a, b, c
  variables x, y, z

- $<>$ Harriot (1600)
- ∞ Wallis (1700)
- imaginary, Descartes
- $x^{3/2}, x^{-1}$, Newton 1600
- $x^2 \to xx$, Gauss 1800
- π, i, Σ. Euler 1700
- $f(x)$
- $\frac{df}{dx}, \int$ Leibniz 1600

Analytic Geometry

- Fermat (1601-1665)
- Descartes (1596-1650)
- Newton (1642-1727)

Coordinates

- Hipparchus - sky
- Apollonius - conics
- Oresme (1300s) - position, velocity plots
- Fermat-Descartes described curves in coordinate systems
- Degree 1, Linear relations

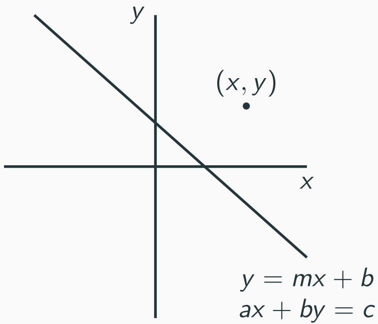


Figure 9: Cartesian system.

Curves of Degree 2 - Quadratics

$$ax^2 + \underbrace{bxy}_{rotation} + cy^2 + \underbrace{dx + ey}_{translation} + f = 0$$

- Describes Conics
- $b \neq 0$, rotation
- $d \neq 0$ or $e \neq 0$, translation
- Classification

$$D = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

$D > 0$ ellipse
$D < 0$ hyperbola
$D = 0$ parabola

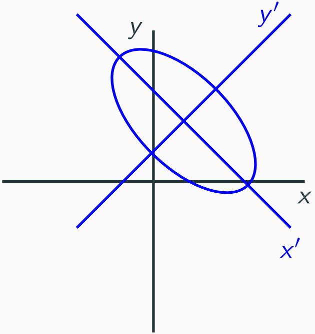


Figure 10: Rotated, translated ellipse.

Curves of Degree 3: $ax^3 + bx^2y + cxy^2 + dy^3 + \cdots = 0$, Cubics

- Newton classified cubic curves, 1710, 72 types (missed 6)
- $y = x^3$ and other types.
- Descarte's folium (leaf) $x^3 + y^3 = 3axy$
- Parametric Solutions

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

- Rational Points
  Ex. $x^3 + y^3 = 1$.
  Let $x = \frac{n}{p}$, $y = \frac{m}{p}$. $n^3 + m^3 = p^3$.
- Fermat - only trivial $(0,1), (1,0)$.
- Fermat's Last Theorem, 1637

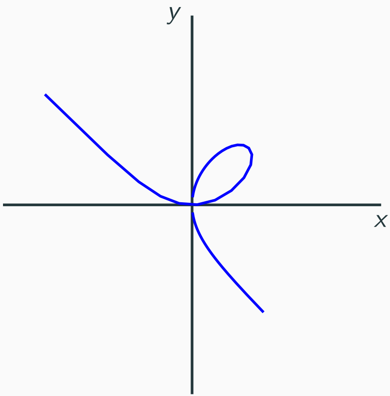


Figure 11: Descarte's Folium.

Fermat's and Bezout's Theorems

- Fermat's Last Theorem, 1637
  $x^n + y^n = z^n$
  Wiles proved in 1995.
- Bezout's Theorem. Let

  $$p(x, y) = 0, \qquad \text{degree } n.$$
  $$q(x, y) = 0, \qquad \text{degree } m.$$

  Then, p and q intersect in nm points.
- Elimination gives eq of degree nm.
- Need complex numbers, point at infinity.
  Next - Projective Geometry.

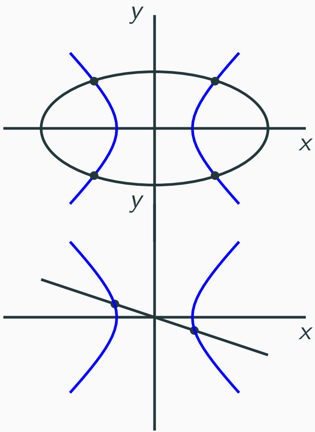


Figure 12: Intersecting Degree 2 curve (blue) with Degree 2 or 1 (black).