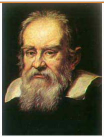
MAT 346 History of Mathematics

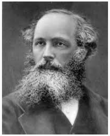
Fall 2023 - Dr. R. L. Herman

Mathematics & Statistics, UNC Wilmington










Course Notes

These notes were compiled in 2019-2023 for a course on the history of mathematics based using many sources and used with the books Dunham, William, *Journey through Genius*, Penguin Books, 1990 and Stillwell, John, *Mathematics and Its History*, 3rd Ed., Spring, 2010. The notes cover various topics in mathematics over 4000 years in several civilizations in the following order:

- Introduction
- Early Mathematics
- Greek Mathematics I
- Euclid's Elements
- Greek Mathematics II
- Asian Mathematics
- Cubic Equation
- Projective Geometry
- Emergence of Calculus
- Complex analysis
- Women in Mathematics
- Non-Euclidean Geometry and Group Theory
- Topology and Knot Theory
- Vibrations and Fourier Analysis
- Mathematics in the 1900s

Introduction to the History of Mathematics

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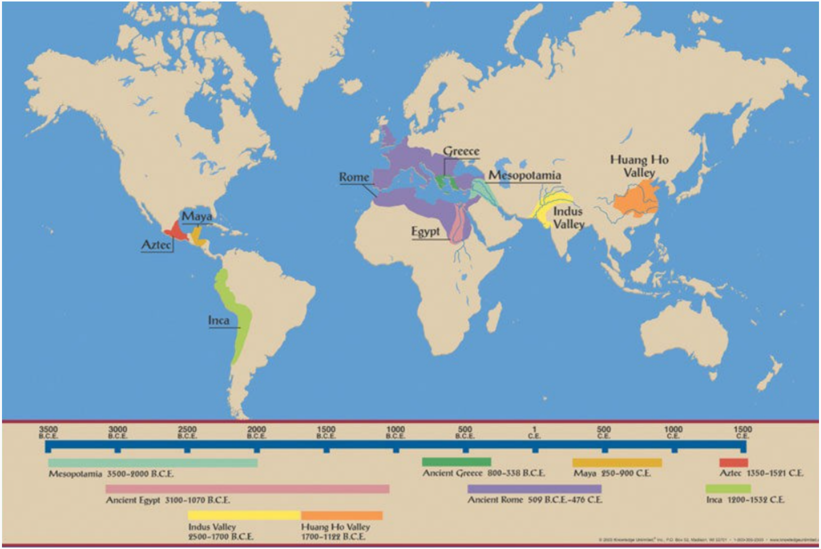
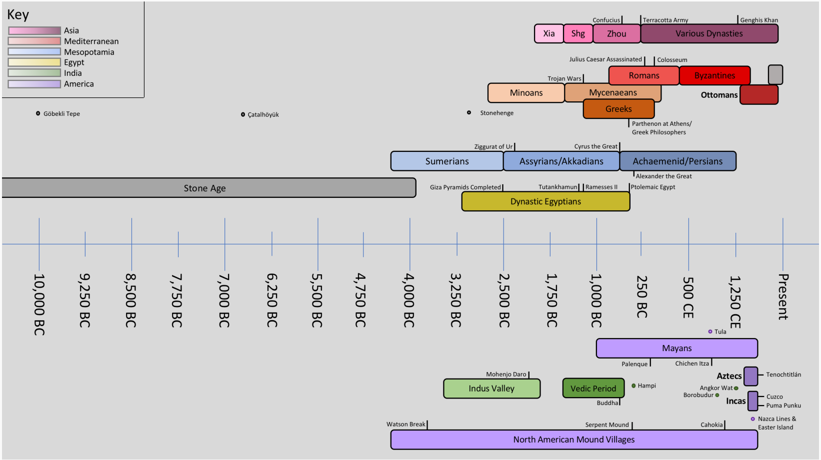
Civilizations - How Did Mathematics Develop?



A Civilization Timeline



Some Early Civilizations

- Egypt (3150-30 BCE)
- Mesopotamia (3100-539 BCE)
- Chinese (1766 BCE-220 CE)
- Indian Mathematics (500-1200)
- Mayan Mathematics (250-900)
- Aztecs and Incans (1345-1560)

Arithmetic, Geometry, No proofs.
Problems were practical or recreational.

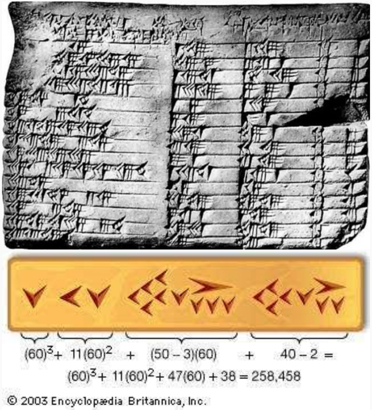


Figure 1: Babylonian Math - Base 60

Greek Civilization

- Deductive Reasoning
  - Definitions, Axioms
  - Propositions via logic
- Geometry, Trigonometry, Astronomy, Numbers, Conics
- Thales (624-546 BCE)
- Pythagoras (6th Century BCE)
- Euclid (4th Century BCE)
  Elements - geometry, numbers
- Archimedes (3rd Century BCE)
- Apollonius (2nd Century BCE)
- Heron (10-70), Diophantus (200-284), Pappas (290-350), Hypatia (370-415)

Figure 2: Euclid

Chinese and Indian

- Chinese Mathematics
  1300 BCE - 1800 CE
  - Pythagorean Thm
  - π estimates
  - Volumes, Applications
  - Pascal's Triangle
  - Chinese Remainder Thm
- Indian Mathematics 1200 BCE, mostly 500-1200 CE
  - Geometry
  - Trigonometry
  - Power series
  - Astronomy
  - π estimates
  - Number system, 0
  - Pell's Equation

Figure 3: Liu-Hong

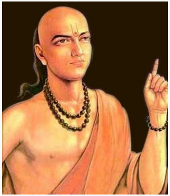


Figure 4: Aryabhatta

Middle Eastern Mathematics - 700-1200 CE

European Dark Ages - 400-1200 CE

- Founding of Islam - 7th Century
- Islamic mathematicians preserve/translate Greek/Asian mathematics into Arabic
- Arabic Numerals by 1000 CE
- Persian mathematicians
    - al-Khwarizmi (780-850) - Algebra (al-Jabr)
    - Omar Khayyam (1048-1131) - geometric solution of cubic

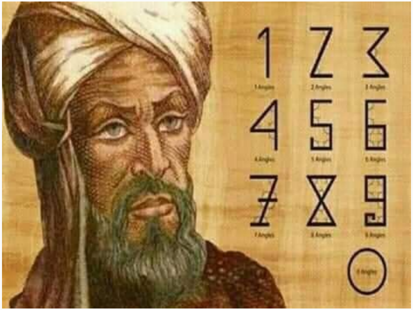


Figure 5: al-Khwarizmi

Around 10th Century - Middle Eastern Mathematics brought to Spain.

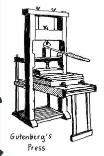
It takes 300 years to accept Hindu-Arabic numerals. - Fibonacci - 1202

The Renaissance

The Renaissance 1400-1600

Leonardo of Pisa (1175-1250)

Black Death (1347-1351)

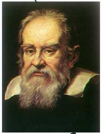


Printing Press (1440)

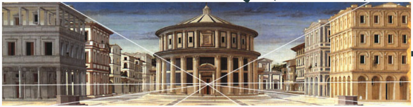
Luca Pacioli (1445-1517)

Giovanni di Medici
Banking (1397)

Galileo (1564-1642)



Perspective in Art (1377-1492)



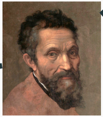
Bruneschelli, Alberti, Piero della Francesca

Leonardo da Vinci (1452-1519)

Reformation (1517)

Cardano (1501-1576)

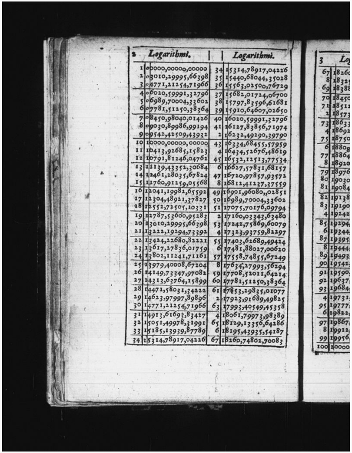
Tartaglia (1500-1557)



Michelangelo (1475-1564)

Beyond Numerals

- Fractions 4000 years ago
- Sexagesimal (base 60) into 17th century
- Decimal (base 10)
  - al-Uqlidisi - (920-980)
  - al-Kashi (1380-1429)
  - Simon Stevin (1548-1620)
- Logarithms
  - John Napier (1550-1617) - used a stange base
  - Henry Briggs (1561-1630)
    - Base 10 Tables
    - 54 square roots of 10 (30 decimal places)
    - Tables - 14 decimal places


2 Logarithmi. Logarithmi.

1	0	0000,00000,00000	34	1	5314,78917,04226
2	0	3010,29995,66398	35	1	5440,68044,35028
3	0	4771,21254,71966	36	1	5563,02500,76729
4	0	6020,59991,32796	37	1	5682,01724,06700
5	0	6989,70004,33602	38	1	5797,83596,61681
6	0	7781,51250,38364	39	1	5910,64607,02650
7	0	8450,98040,01426	40	1	6020,59991,32796
8	0	9030,89986,99194	41	1	6127,83856,71974
9	0	9542,42509,43932	2	1	6232,49290,39790
10	1	0000,00000,00000	43	1	6334,68455,57959
11	1	0413,92685,15823	4	1	6434,52676,48619
12	1	0791,81246,04762	45	1	6532,12513,77534
13	1	1139,43352,30684	6	1	6627,57831,68157
14	1	1461,28035,67824	47	1	6720,97857,93572
15	1	1760,91259,05568	8	1	6812,41237,37559
16	1	2041,19982,65592	49	1	6901,96080,,02851
17	1	2304,48921,37827	50	1	6989,70004,33602
18	1	2552,72505,10331	51	1	7075,70176,09794
19	1	2787,53600,95283	2	1	7160,03343,63480
20	1	3010,29995,66398	53	1	7242,75869,60079
21	1	3222,19294,73392	4	1	7323,93759,82297
22	1	3424,22680,82221	55	1	7403,62689,49424
23	1	3617,27836,01759	6	1	7481,88027,00620
24	1	3802,11241,71161	57	1	7558,74855,67249
25	1	3979,40008,67204	8	1	7634,27993,56294
26	1	4149,73347,97082	59	1	7708,52011,64214
27	1	4313,63764,15899	60	1	7781,51250,38364
28	1	4471,58031,34222	61	1	7853,29835,01077
29	1	4623,97997,89896	2	1	7923,91689,49825
30	1	4771,21254,71966	63	1	7993,40549,45358
31	1	4913,61693,83427	4	1	8061,79973,98389
32	1	5051,49978,31991	65	1	8129,13356,64286
33	1	5185,13939,87789	6	1	8195,43935,54187
34	1	5314,78917,04226	67	1	8260,74802,70083

Figure 6: Brigg's Tables

- Fibonacci (Leonardo of Pisa) (1170-1250) *Liber Abaci*
- Equation Solving contests
- Solutions of cubic and quartic
  - Depressed cubic
    del Ferro (1465-1526)
  - Cubic and quartic equations
    Tartaglia (1500-1557)
    Cardano (1501-1576)

    Ars Magna

    Ferrari (1522-1565)
- Bombelli (1526-1572)
  - Complex numbers
- Viète (1540-1603)
  Adriaan van Roomen Problem

Figure 7: Cardano and Tartaglia
Fight of the Century!

Unification of Geometry and Algebra

- Symbolic Algebra
  - Rhetorical until 15th century
  - Syncopated/abbrev. - 1500
  - Symbolic algebra developed 16-17th century
- Unification
  - Oresme (1320-1382) - Velocity-time graphs, $\sum \frac{1}{n}$
  - Descartes (1596-1650)
    - Rep. curves by equations
    - Coordinate systems - published *The Method*
    - Made use of variables which can vary continuously - lines.
  - Fermat (1607-1665)
    - Rep. equations by curves

Figure 8: Fermat and Descartes

The Rise of Calculus

- Archimedes - 3rd century BCE
- Kepler (1571-1630)
- Cavalieri (1598-1647)
- Fermat (1607-1665)
- Wallis (1616-1673)
- Pascal (1623-1662)
- Barrow (1630-1677)
- Wren (1632-1723)
- Gregory (1638-1675)
- Newton (1642-1726)
  - *Principia* 1687
- Leibniz (1646-1716)
  - *Notation* $\frac{d}{dx}, \int$

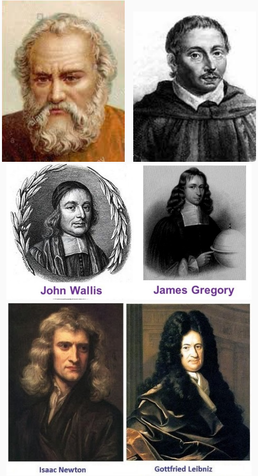
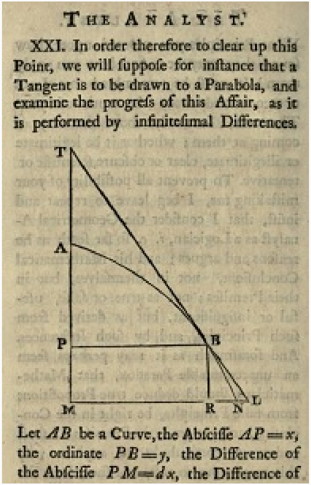


Figure 9: Archimedes, Cavalieri, Wallis, Gregory, Newton, and Leibniz

The Infinitesimal

- Hippasus 500 BCE
  - Pythagorean, $\sqrt{2}$ irrational
- Introduction of Infinitesimals
  - Cavalieri and Torricelli
  - Stevin, Wallis, Harriot
- Religious Critics
  - Era of Copernicus, Galileo
  - Jesuits in Italy, Church bans
  - George Berkeley (1685-1753) *The Analyst*, - *A Discourse Addressed to an Infidel Mathematician*, 1734
  - "Infinitesimals undermine mathematics and rationality"
- Augustin-Louis Cauchy - 1821


THE ANALYST.

XXI. In order therefore to clear up this Point, we will ſuppoſe for inſtance that a Tangent is to be drawn to a Parabola, and examine the progreſs of this Affair, as it is performed by infiniteſimal Differences.

Let AB be a Curve, the Abſciſſe $AP = x$, the ordinate $PB = y$, the Difference of the Abſciſſe $PM = dx$, the Difference of

Figure 10: Berkeley's *The Analyst*

Exploiting Calculus

- Bernoulli Family
- Leonhard Euler (1707-1783)
- Joseph-Louis Lagrange (1736-1813)
- Pierre-Simon Laplace (1749-1827)
- Neptune discovered using math - 1846, Le Verrier (1811-1877)

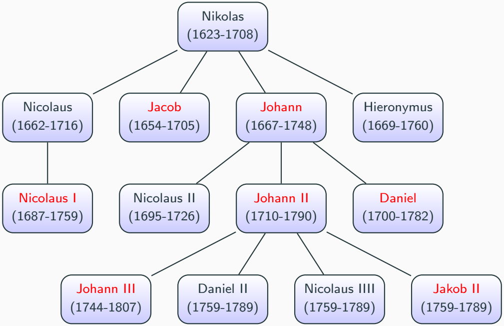


Figure 12: The Bernoulli Family

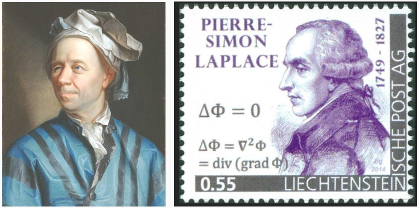
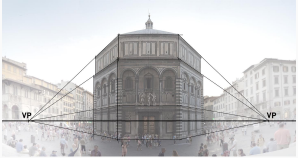
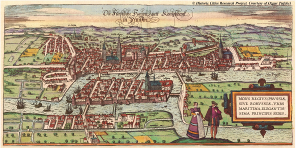


Figure 11: Euler and Laplace

Evolution of Geometry: Projective Geometry and Topology

- Perspective in Art
  - Bruneschelli (1377-1446)
  - Leon Alberti (1404-1472)
  - Girard Desarges (1591-1661)
- Birth of Topology
  - Euler - Königsberg bridge
    Geometry without distance
  - Euler Characteristic
    $\chi = V - E + F$, and
    classification of surfaces
    $\chi = 2 - 2g$, genus
- Birth of Knot Theory
  - Gauss - Intertwining curves
  - Scottish physics and Knots -
    Taits smoke rings.





The Birth of Rigor - 19th Century

Non-Euclidean Geometry

- Parallel Postulate
- Hyperbolic Geometry
    - Nikolai Lobachevsky (1792-1856)
    - Johann Bolyai (1802-1860)
    - Johann Carl Friedrich Gauss (1777-1855)
- Elliptic Geometry
    - Georg Friedrich Bernhard Riemann (1826-1866)
    - Prince of Mathematicians
- By 1870's Euclid in doubt!

Figure 13: Gauss, Lobachevsky, Bolyai

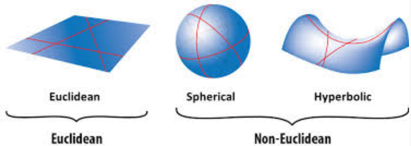


Figure 14: Different Geometries

19th Century Group Theory

The search for the general solution of the quintic.

- Joseph-Louis Lagrange (1736-1813)
- Johann Carl Friedrich Gauss (1777-1855)
- Paola Ruffini (1765-1822) - proof of unsolvability
- Augustin Cauchy (1789-1857)
- Niels Henrik Abel (1802-1829)
- Évariste Galois (1811-1832)
- Arthur Cayley (1821-1895)
- Camille Jordan (1838-1922)

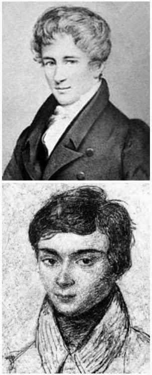
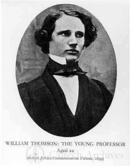


Figure 15: Abel and Galois

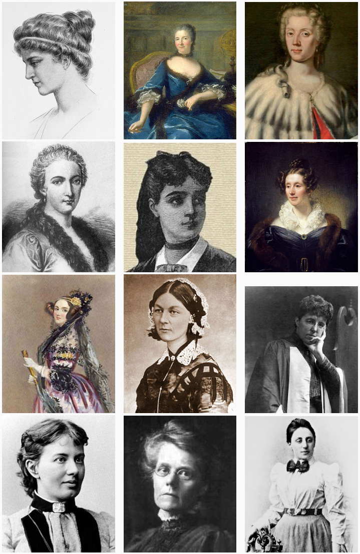
Mathematical Physics - Mathematical Physicists

- Joseph-Louis Lagrange (1736-1813)
- Pierre-Simon Laplace (1749-1827)
- Joseph Fourier (1768–1830)
- Siméon Denis Poisson (1781–1840)
- George Green (1793–1841)
- William Hamilton (1805–1865)
- William Thomson (1824–1907), 1st Baron Kelvin
- James Clerk Maxwell (1831–1879)
- J. Willard Gibbs (1839–1903)
- John William Strutt (1842–1919), 3rd Baron Rayleigh
- Oliver Heaviside (1850–1925)



Famous Women Mathematicians Before 1900

- Hypatia of Alexandria (c. 350-415)
- Émilie du Châtelet (1706-1749)
- Laura Bassi (1711-1788)
- Maria Agnesi (1718-1799)
- Sophie Germain (1776-1831)
- Mary Fairfax Somerville (1780-1872)
- Ada Lovelace (1815-1852)
- Florence Nightingale (1820-1910)
- Charlotte Angas Scott (1848-1931)
- Sofia Kovalevskaya (1850-1891)
- Alicia Boole Stott (1860-1940)
- Amalie 'Emmy' Noether (1882-1935)



19th Century Analysis and Set Theory

- Jean-Baptiste Joseph Fourier (1768-1830)
- Johann Carl Friedrich Gauss (1777-1855)
- Augustin Cauchy (1789-1857)
- Karl Weierstrass (1815-1897)
- George Boole (1815-1864)
- Georg Friedrich Bernhard Riemann (1826-1866)
- Richard Dedekind (1831-1916)
- Georg Ferdinand Ludwig Philipp Cantor (1845-1918)
  - Founder of set theory
  - Defined infinite sets

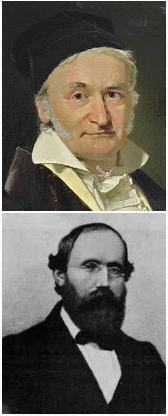


Figure 16: Gauss and Riemann

- Marie-Sophie Germain (1776-1831)
- Johann Carl Friedrich Gauss (1777-1855)
  Disquisitiones Arithmeticae - 1801
- Adrien-Marie Legendre (1752-1833) and
  Peter Gustav Lejeune Dirichlet (1805-1859) prove
  Fermat's Last Theorem for $n = 5$ in 1825
  - Dirichlet, $n = 14$ in 1832.
- Riemann Hypothesis, distribution of primes - 1832.
- Charles Jean de la Vallée-Poussin and Jacques Hadamard - Prime Number Theorem. 1896
- H. Minkowski: Geometry of Numbers, 1896.

Figure 17: Sophie Germain, Adrien-Marie Legendre

The Modern Era

We stop at the turn of the 20th Century: the evolution of mathematics, set theory, physics revolutions, Bourbaki, Hilbert's 23 Problems.

Explore mathematics prizes: Fields Medal, Abel Prize, Wolf Prize, Millenium Prize.

Other Sites

- Chronology of 20th Century Mathematicians
- Greatest Mathematicians born between 1860 and 1975
- Pictures of Famous 20th Century Mathematicians
- The Story of Math Website

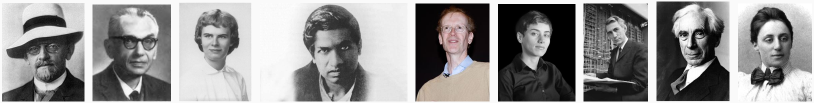


Figure 18: Hilbert, Gödel, Uhlenbeck, Ramanujan, Wiles, Mirzakhani, Shannon, Russell, Noether

Early Mathematics - Egypt and Mesopotamia

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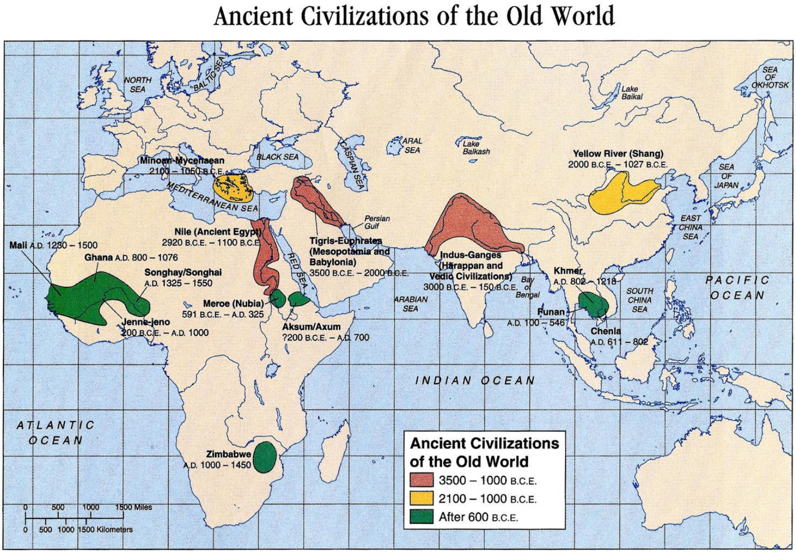
Maps of Ancient Civilizations



Early Civilizations

- Ancient African ¿ 20,000 yrs
- Egypt (3150-30 BCE)
- Mesopotamia (3100-539 BCE)
- Indus (3300-1700 BCE)
- Greek (640 BCE-415 CE)
- Chinese (1766 BCE-220 CE)
- Indian Mathematics (500-1200)
- Islamic Mathematics (700-1200)
- Mayan Mathematics (250-900)
- Aztec Empire (c.1345-1521)
- Inca Civilization (1400-1560)

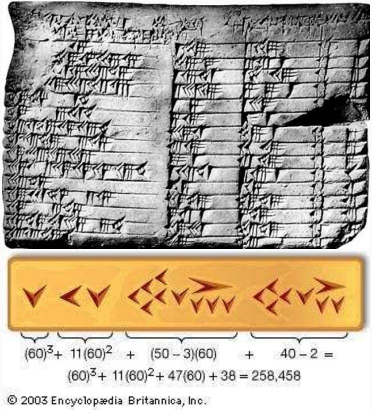
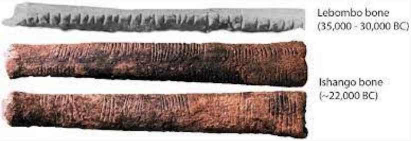


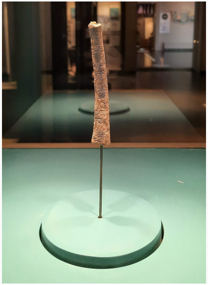
Figure 1: Babylonian tablet - Base 60

Ancient African Mathematics

- Lebombo bone, 43,000-44,200 yrs old.
  Oldest known mathematical artifact,
  29 notches on a baboon's fibula.
  Found in Border Cave, Lebombo Mountains, Swaziland.
- Ishango bone, 20,000 BCE.
  Also baboon bone.
  Ishango, Democratic Republic of Congo.
  Numerical patterns with differing interpretations.



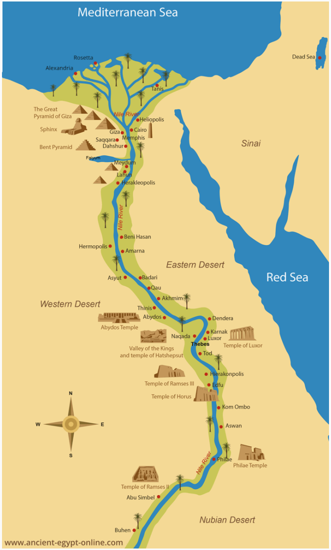
See Blog and Article



See Wikipedia.

Ancient Egypt

- Early Dynastic Period (3150–2686 BCE), writing
- Old Kingdom (2686–2181 BCE) (Great Pyramid of Giza)
- 1st Intermediate Period (2181–2055 BCE)
- Middle Kingdom (2055–1650 BCE), **Reisner Papyri** and **Moscow Papyrus**
- 2nd Intermediate Period (1650–1550 BCE), **Rhind Papyrus**
- New Kingdom (1550–1069 BCE)
- 3rd Intermediate Period (1069–664 BCE)
- Late Period (664–332 BCE)



The Papyri

- Papyri - scrolls.
  - Rhind Papyrus, 1650 BCE.
  - Moscow Papyrus, 1850 BCE.
  - Reisner Papyri, 1950 BCE.
- Reisner Papyri
  - Dr. G.A. Reisner .
  - 1901–04 - southern Egypt.
  - 4 scrolls.
  - Mostly accounts.
- Egyptian Arithmetic.
  - Base-10.
  - heiroglyphic and hieratic numerals.
  - integers, fractions.
  - surveying, building.
  - areas, volumes.

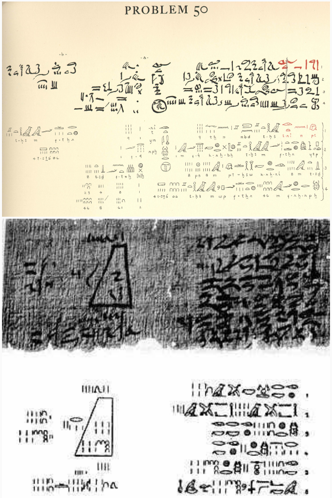
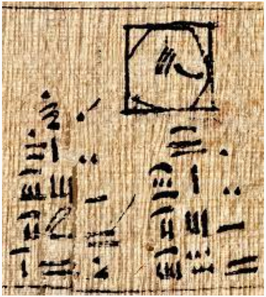


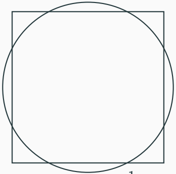
Figure 2: Papyri

The Rhind Papyrus

- Found in Thebes.
- Purchased 1858, by A. Henry Rhind.
- Size: $18' \times 13''$.
- Red and black ink.
- Geometry.
  - Areas, Volumes.
  - Ratios of sides of right triangles.
- Measures - grain.
  - 1 hekat $\approx 29,224$ in$^3 \geq \frac{1}{2}$ peck.
  - 1 ro $= \frac{1}{320}$ hekat.
- Areas of Circles - 48, 50.

$$A = \left(\frac{8}{9}D\right)^2 = \frac{256}{81}r^2 \approx 3.16049r^2.$$





$A_{circle} = A_{square} - \frac{1}{9}A_{square}$

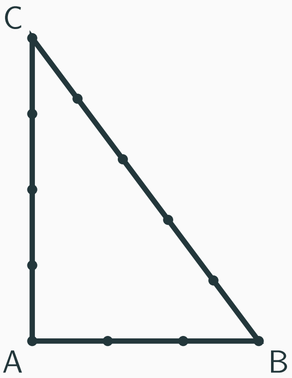
Figure 3: Problem 48

Pythagorean Triples

- Pythagorean Theorem.
- Triples (a, b, c),

$$a^2 + b^2 = c^2.$$

  Examples:
  - 3-4-5.
  - 5-12-13.
- Used to Measure Perimeters.
- Knotted Ropes.
  - Loop with 12 knots.
- Other Units:
  - Finger - 1.9 cm.
  - Palm = 4 fingers - 7.5 cm.
  - Cubit = 7 palms - 52.3 cm.



Multiplication - 17×13.

1	17
2	34
4	68
8	136
13	221

Division - $108/12$.

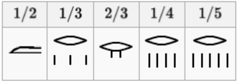
1	12
2	24
4	48
8	96
9	108

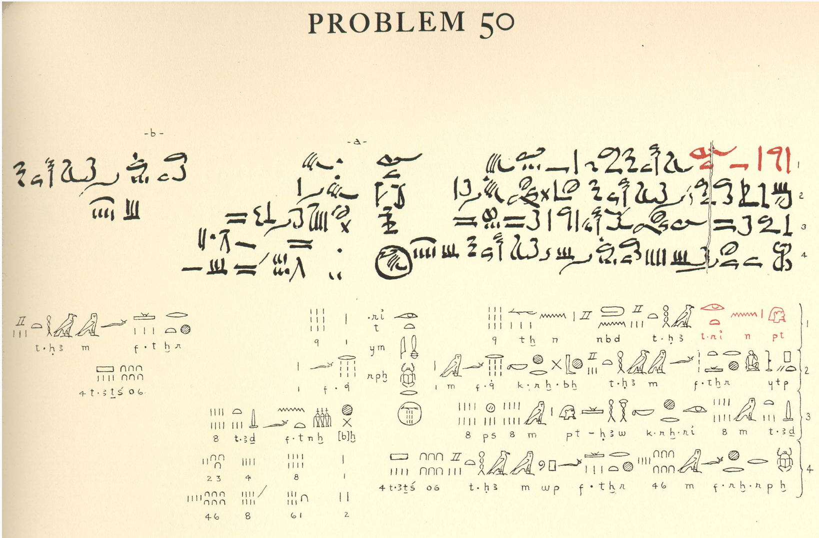
Unit fractions:

$$\frac{4}{7} = \frac{1}{2} + \frac{1}{14}.$$

Counting and Unit Fractions

Hieroglyphics for some fractions[5]




PROBLEM 50

Rhind Papyrus - Problem 50

Problem 50

tp n ỉr·t ꜣḥ·t dbn n ẖt·w[1] 9 pty rḫt · f m ꜣḥ·t
Example of making a field round of khet 9. *What is the amount of it in area?*

ḫb · ḫr · k 9̇ · f m 1 ḏꜣ·t m 8 ỉr·ḫr · k wꜣḥ–tp m 8 sp 8 ḫpr·ḫr · f m 64
Take away thou 1/9 *of it, namely,* 1; *the remainder is* : 8. *Make thou the multiplication* : 8 *times* 8; *becomes it* : 64;

rḫt · f pw m ꜣḥ·t 60[2] śṯꜣ·t 4
the amount of it, this is, in area, 60 setat 4.

ỉr·t my ḫpr
The doing as it occurs:

1	9
9̇ · f	1.
of it	

ḫ[b] ḫnt · f ḏꜣ·t 8
Take away from it; the remainder is 8.

1	8
2	16
4	32
\8	64

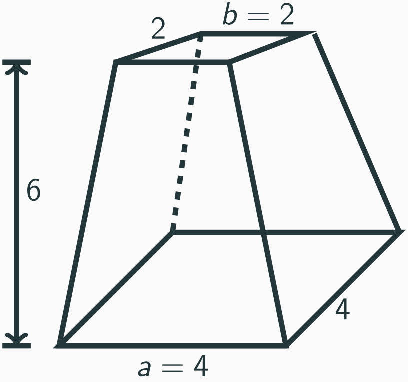
rḫt · f m ꜣḥ·t 60[2] śṯꜣ·t 4
The amount of it in area: 60 setat 4.

[1] The w suggested by the plural strokes has been omitted on the plate. The same omission occurs on the figure in Problem 51, and in Problem 52, line 2.

[2] The scribe has by mistake written here either the number 60 or the special form for 6 used in Problem 48 in writing 6 *setat*. He may have had in his mind the fact that he was actually dealing with 60 *setat* (which, however, would not properly be written in this way), and he had written the abstract number 60 a moment before at the end of the multiplication, or, remembering that 60 *setat* is written with the numeral 6, he did write 6, but used the special sign instead of the ordinary numeral.

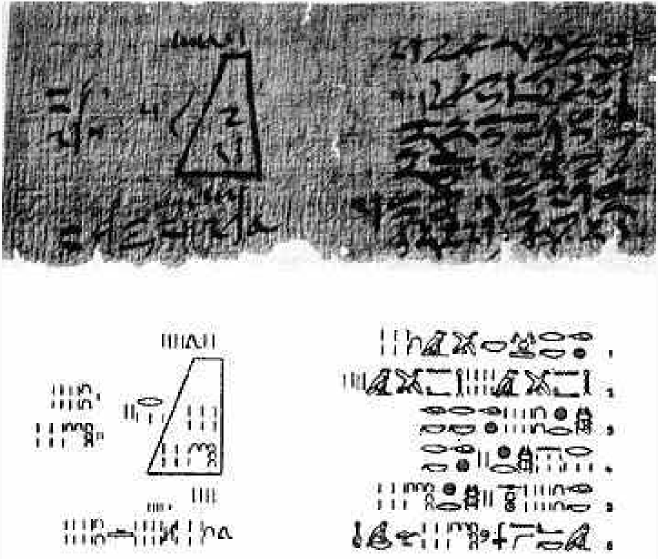
Moscow Papyrus

- From around 1850 BCE.
- Golenishchev bought in 1892 or 1893 in Thebes.
- Housed in Moscow.
- 25 Problems.
- `https://en.wikipedia.org/wiki/Moscow_Mathematical_Papyrus`
- See Problem 14:
  - Frustrum of a Pyramid



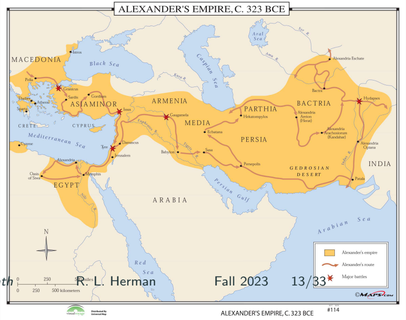
$$V = \frac{h}{3}\left(a^2 + ab + b^2\right)$$

Moscow Papyrus - Problem 14 - Frustrum of Pyramid



The Fall of the Egyptian Empire

- Argead dynasty (332–310 BCE)
    - Macedonians (700-310 BCE)
    - Alexander III of Macedon, or Alexander the Great (336–323 BCE) King of Macedonia, Pharaoh of Egypt, King of Persia and of Asia
- Ptolemaic dynasties (310–30 BCE) Cleopatra (69–30 BCE)
- Roman and Byzantine Egypt (30 BCE–641 CE)
- Sasanian (Persian) Egypt (619–629)
- Death of Mohammed (c. 570-632)
- Ruled by Caliphates (641-1517)
- Ottoman Rule (1517-1914)



Mesopotamia (2100 BCE) - Tigris and Euphrates Region

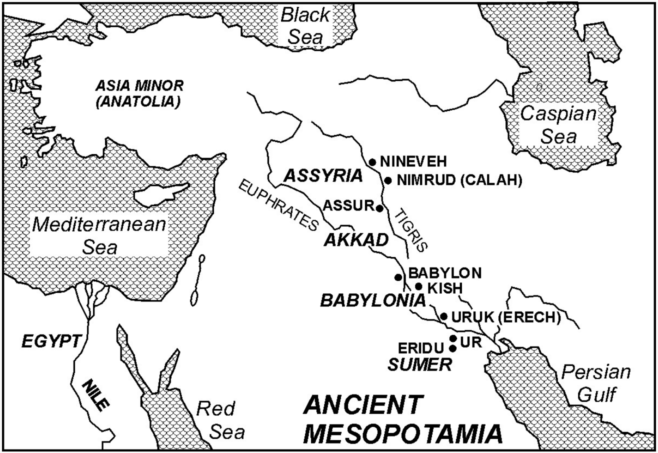


Figure 4: Tigris and Euphrates Rivers

Babylonian and Sumerian Mathematics

- More Advanced.
- Clay Tablets.
- Base 60 Arithmetic.
- Notation: $13_{60} = 1\text{␣}3 = 1/3$.
- Some use commas: $1, 3$.
- Examples:

$$
\begin{aligned}
1/3 &= 1(60) + 3 = 63 \\
1/59 &= 1(60) + 59 = 119 \\
2/49 &= 2(60) + 49 = 169 \\
3/31/49 &= 3(60^2) + 31(60) + 49
\end{aligned}
$$

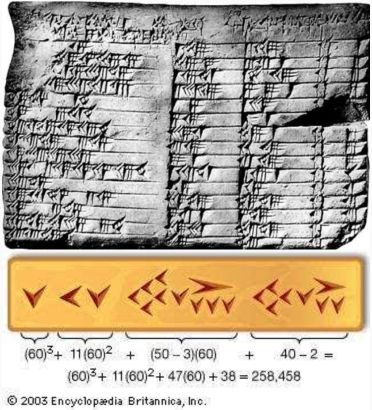


Figure 5: Babylonian tablet - Base 60

Sexagesimal Operations (Base 60)

- Ambiguities:
  - No 0's.
  - No decimal points.
- Special fractions:
  - $8.25_{10} = 8/15 = 8\frac{15}{60}$
  - $8.5_{10} = 8/30 = 8\frac{30}{60}$
  - $8.75_{10} = 8/45 = 8\frac{45}{60}$
- Addition, subtraction, multiplication.

Addition:

$$\begin{array}{r} 14/28/31 \\ +3/35/45 \\ \hline = 18/4/16. \end{array}$$

Multiplication -
$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right].$

No division! - Use reciprocals:
See Old Babylonian Multiplication and Reciprocal Tables.

Reciprocal Table

Table of reciprocals $\bar{x}$ of x, where $x\bar{x} = 60^n$, $n = 0, 1, \ldots$.

x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$	x	$\bar{x}$
2	0/30	8	7/30	16	3/45	30	2
3	0/20	9	6/40	18	3/20	32	1/52/30
4	0/15	10	6	20	3	36	1/40
5	0/12	12	5	24	2/30	40	1/30
6	0/10	15	4	25	2/24	45	1/20

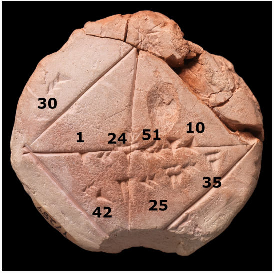
Divide 8 by 2 : $8(0/30) = 8 \times \frac{30}{60} = \frac{240}{60} = 4$, or

$0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 + 0/30 = 1 + 1 + 1 + 1.$

Missing reciprocals: $\frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \frac{1}{17}, \frac{1}{19}, \ldots$.

Sumerian Tablet - YBC 7289 - imšukkum, or “hand tablet”

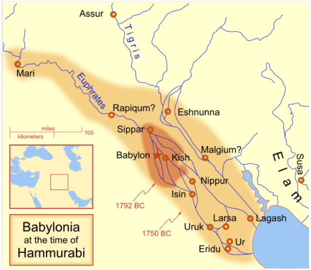
- From southern Iraq, 19th or 18th century BCE.
- Yale Peabody Museum of Natural History, 3D Print.
- Babylonians knew ratio of the side to diagonal in a square, $1 : \sqrt{2}$.



$$1/24/51/10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414213$$

$$42/25/35 = 42 + \frac{25}{60} + \frac{35}{60^2} \approx 42.426$$

Plimpton 322 Clay Tablet (in the news in 2017)

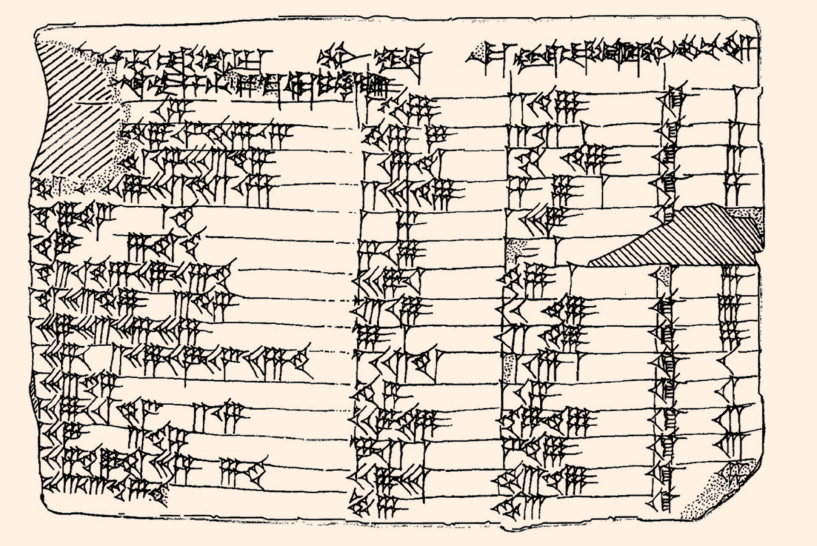


- Larsa (c. 1800 BCE) .
- Removed in 1920s.
- George Plimpton bought it, 1922.
- Left to Columbia University, 1936.
- It's about 13 by 9 by 2 cm.
  (Like a baking dish.)

- Four columns, cuneiform numbers.
- 15 rows - Pythagorean triples.
- 2nd column, side of right triangle.
- 3rd column, hypotenuse.
- 4th column, row number.
- What is the 1st Column?

Plimpton 322 Clay Tablet - Homework!

Sketch of the Plimpton 322 Tablet



Babylonian Numerals 1-100 (Base 60)

1	𒁹	26	𒎙𒐋	51	𒐐𒁹	76	𒁹 𒌋𒐋
2	𒈫	27	𒎙𒑂	52	𒐐𒈫	77	𒁹 𒌋𒑂
3	𒐈	28	𒎙𒑄	53	𒐐𒐈	78	𒁹 𒌋𒑄
4	𒐉	29	𒎙𒑆	54	𒐐𒐉	79	𒁹 𒌋𒑆
5	𒐊	30	𒌍	55	𒐐𒐊	80	𒁹 𒎙
6	𒐋	31	𒌍𒁹	56	𒐐𒐋	81	𒁹 𒎙𒁹
7	𒑂	32	𒌍𒈫	57	𒐐𒑂	82	𒁹 𒎙𒈫
8	𒑄	33	𒌍𒐈	58	𒐐𒑄	83	𒁹 𒎙𒐈
9	𒑆	34	𒌍𒐉	59	𒐐𒑆	84	𒁹 𒎙𒐉
10	𒌋	35	𒌍𒐊	60	𒁹	85	𒁹 𒎙𒐊
11	𒌋𒁹	36	𒌍𒐋	61	𒁹 𒁹	86	𒁹 𒎙𒐋
12	𒌋𒈫	37	𒌍𒑂	62	𒁹 𒈫	87	𒁹 𒎙𒑂
13	𒌋𒐈	38	𒌍𒑄	63	𒁹 𒐈	88	𒁹 𒎙𒑄
14	𒌋𒐉	39	𒌍𒑆	64	𒁹 𒐉	89	𒁹 𒎙𒑆
15	𒌋𒐊	40	𒐏	65	𒁹 𒐊	90	𒁹 𒌍
16	𒌋𒐋	41	𒐏𒁹	66	𒁹 𒐋	91	𒁹 𒌍𒁹
17	𒌋𒑂	42	𒐏𒈫	67	𒁹 𒑂	92	𒁹 𒌍𒈫
18	𒌋𒑄	43	𒐏𒐈	68	𒁹 𒑄	93	𒁹 𒌍𒐈
19	𒌋𒑆	44	𒐏𒐉	69	𒁹 𒑆	94	𒁹 𒌍𒐉
20	𒎙	45	𒐏𒐊	70	𒁹 𒌋	95	𒁹 𒌍𒐊
21	𒎙𒁹	46	𒐏𒐋	71	𒁹 𒌋𒁹	96	𒁹 𒌍𒐋
22	𒎙𒈫	47	𒐏𒑂	72	𒁹 𒌋𒈫	97	𒁹 𒌍𒑂
23	𒎙𒐈	48	𒐏𒑄	73	𒁹 𒌋𒐈	98	𒁹 𒌍𒑄
24	𒎙𒐉	49	𒐏𒑆	74	𒁹 𒌋𒐉	99	𒁹 𒌍𒑆
25	𒎙𒐊	50	𒐐	75	𒁹 𒌋𒐊	100	𒁹 𒐏

Akkadian Table of 9's

2 Akkadian Tablet (-1700)

In the paper "Sherlock Holmes in Babylon," *Amer. Math. Monthly* 87 (1980), 335-345, C. Buck describes Babylonian mathematics. He begins with a discussion of a clay tablet from 3700 years ago as shown in Table 2. There are four columns. You should convince yourself that this is a table of 9's.

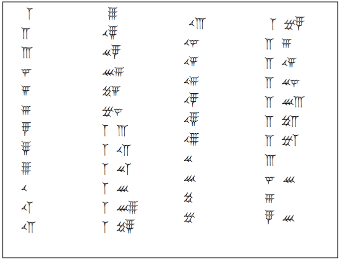


Table 2: Table of 9's.

As an example, the last entry in the first column is 12 = . Then, $9 \times 12 = 108 =$. Note that in base 60 we have $108 = 1(60) + 48$.

In the second column is a one ( ) and 48 ( ) separated by a space. Buck introduces a slash notation to write this as 1/48.

It is easy to add in base 60. Buck gives the example $14/28/31 + 3/35/45 = 18/4/16$.

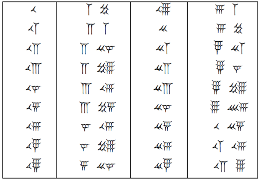
Babylonian Squares

How can a table of squares be useful? In modern notation, we see that

$$ab = \frac{1}{4}\left[(a+b)^2 - (a-b)^2\right]. \tag{1}$$

Let's find the product 11×14. Using Table 3, the formula gives

$$\begin{aligned}
11(14) &= \frac{1}{4}\left[(11+14)^2 - (11-14)^2\right] \\
&= \frac{1}{4}(25^2 - 3^2) \\
&= \frac{1}{4}(10/25 - 9) \text{ (base 60)} \\
&= \frac{1}{4}(10/16) \text{ (base 60)} \\
&= \frac{1}{4}(10(60) + 16) = \frac{616}{4} = 154.
\end{aligned} \tag{2}$$



10	1/40	19	6/1
11	2/1	20	6/40
12	2/24	21	7/21
13	2/49	22	8/4
14	3/16	23	8/49
15	3/45	24	9/36
16	4/16	25	10/25
17	4/49	26	11/16
18	5/24	27	12/9

Table 3: Table of squares with Babylonian numerals in the left table and slash notation on the right side.

4 Pythagorean Triples

Another interesting tablet from the time is the Plimpton 322 tablet shown in Figure 4. This tablet has a listing of Pythagorean triples. The last column has a list of numbers from 1 to 15. Columns two and three seem to be the hypotenuse, C, and one leg, B, of the right triangle shown in Figure 1. Recall from the Pythagorean Theorem that

$$C^2 = B^2 + D^2.$$

The triple (D, B, C) is called a Pythagorean triple.

We now know that these triples are parametrized by the pair (a, b) as follows:

$$B = a^2 - b^2, \quad C = a^2 + b^2, \quad D = 2ab,$$

since

$$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$$

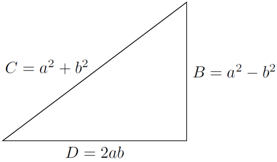


Figure 1: Right triangle with columns two and three as sides B and C, respectively. Pythagorean triples were later found to have a parametrization (a, b).

Transcription - Brackets indicate guesses

[59]/0/15	1/59	2/49	ki	1
[56/56]/58/14/50/6/15	56/7	1/20/25	ki	2
[55/7]/41/15/33/45	1/16/41	1/50/49	ki	3
53/10/29/32/52/16	3/31/49	5/9/1	ki	4
48/54/1/40	1/5	1/37	ki	[5]
47/6/41/40	5/19	8/1	[ki]	[6]
43/11/56/28/26/40	38/11	59/1	ki	7
41/33/45/14/3/45	13/19	20/49	ki	8
38/33/36/36	8/1	12/49	ki	9
35/10/2/28/27/24/26/40	1/22/41	2/16/1	ki	10
33/45	45	1/15	ki	11
29/21/54/2/15	27/59	48/49	ki	12
27/0/3/45	2/41	4/49	ki	13
25/48/51/35/6/40	29/31	53/49	ki	14
23/13/46/40	56	53	ki	[15]

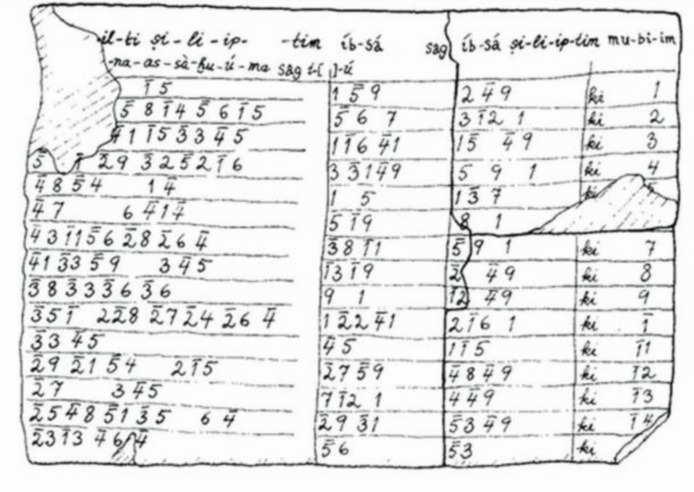


Figure 6: Arabic numerals base 60. The bars designate place holders.

Buck's Corrected Values

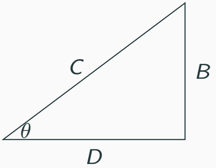
Second column - base 60 values for $(B/D)^2$ with $D^2 = C^2 - B^2$.

#	A	B	C	a	b
1	59/0/15	119	169	12	5
2	56/56/58/14/50/6/15	3367	4825	64	27
3	55/7/41/15/33/45	4601	6649	75	32
4	53/10/29/32/52/16	12709	18541	125	54
5	48/54/1/40	65	97	9	4
6	47/6/41/40	319	481	20	9
7	43/11/56/28/26/40	2291	3541	54	25
8	41/33/45/14/3/45	799	1249	32	15
9	38/33/36/36	481	769	25	12
10	35/10/2/28/27/24/26/40	4961	8161	81	40
11	33/45	45	75	1	0.5
12	29/21/54/2/15	1679	2929	48	25
13	27/0/3/45	161	289	15	8
14	25/48/51/35/6/40	1771	3229	50	27
15	23/13/46/40	56	106	9	5

First Column Computation

- Buck suggests column A is $\left(\frac{B}{D}\right)^2$.
- Others suggest $\left(\frac{C}{D}\right)^2$ and missing left part of the stone had 1's.
  Noting,

$$\left(\frac{C}{D}\right)^2 = 1 + \left(\frac{B}{D}\right)^2.$$



From row 1: 59/0/15 represents

$$\frac{59}{60} + \frac{0}{60^2} + \frac{15}{60^3} = \frac{14161}{14400}.$$

From row 1: $B = 119$, $C = 169$:

$$B^2 = 119^2 = 14161$$

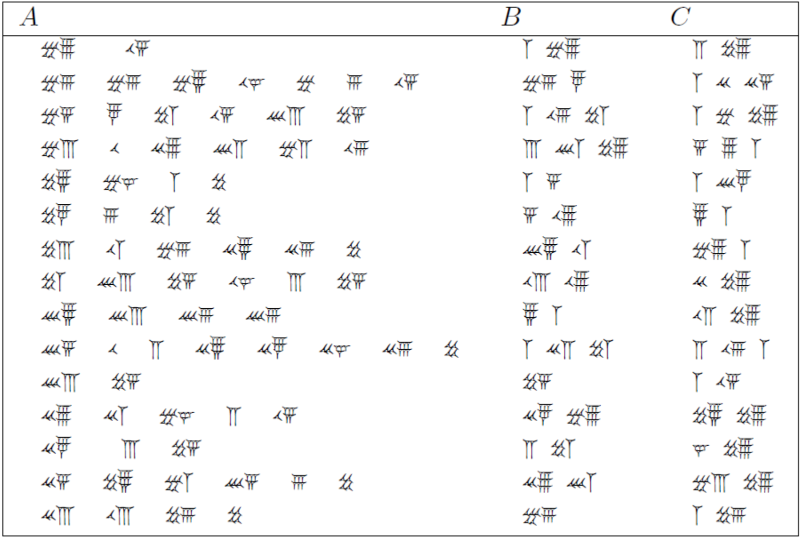
$$D^2 = 169^2 - 119^2 = 14400$$

$$\left(\frac{B}{D}\right)^2 == \frac{14161}{14400}.$$

Decimal Equivalents for Column One

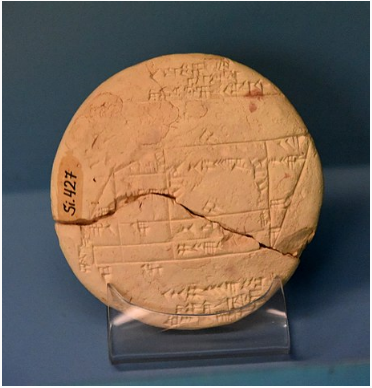
#	A	Decimal Value	$(B/D)^2$
1	59/0/15	0.983402777777778	0.983402777777778
2	56/56/58/14/50/6/15	0.949158552088692	0.949158552088692
3	55/7/41/15/33/45	0.918802126736111	0.918802126736111
4	53/10/29/32/52/16	0.886247906721536	0.886247906721536
5	48/54/1/40	0.815007716049383	0.815007716049383
6	47/6/41/40	0.785192901234568	0.785192901234568
7	43/11/56/28/26/40	0.719983676268862	0.719983676268862
8	41/33/45/14/3/45	0.692709418402778	0.692709418402778
9	38/33/36/36	0.642669444444444	0.642669444444444
10	35/10/2/28/27/24/26/40	0.586122566110349	0.586122566110349
11	33/45	0.562500000000000	0.562500000000000
12	29/21/54/2/15	0.489416840277778	0.489416840277778
13	27/0/3/45	0.450017361111111	0.450017361111111
14	25/48/51/35/6/40	0.430238820301783	0.430238820301783
15	23/13/46/40	0.387160493827161	0.387160493827161

Buck's Corrected Values - Babylonian Numerals



Recent Research - Si.427 - In the news - Aug 4, 2021

- 1900-1600 BC.
- Field plan.
- Used Pythagorean triples to make accurate right angles for measuring boundaries.
- Proposes that Plimpton 322 is the world's oldest and most accurate trigonometric table. (8/2017) Youtube
- Robson does not view it that way.

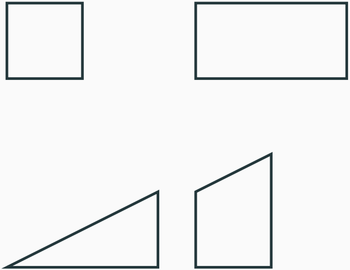


D.F. Mansfield. Plimpton 322: A Study of Rectangles. Found Sci, published online August 3, 2021; Paper.

Babylonian Geometry

- Simple shapes.
- Interested in areas.
- Fields, subdivisions.
- Inclinations, slopes.
- seked, ukuklu, run/rise.

See N. Wildberger's YouTube and reference Neugebaur and Sachs, ed., 1945, *Mathematical Cuneiform Texts*.
Neugebaur and Sachs Introduced Plimpton 322.
Wildberger and Mansfield - Babylonian trigonometry based on ratios.



Greek Mathematics I

Fall 2023 - R. L. Herman

Greek Numerals

- Decimal (Base 10).
- No zero and Positional.
- Attic Numerals (Athens),
- Ionic (Ionia): 24+3 letters

Digit	1-9	10-90	100-900
1	α	ι	ρ
2	β	κ	σ
3	γ	λ	τ
4	δ	μ	υ
5	ϵ	ν	ϕ
6	ϝ	ξ	χ
7	ζ	ο	ψ
8	η	π	ω
9	θ	ϟ, ϙ	ϡ

Arabic	Attic Greek	
1	I	
5	Γ Π	
10	Δ	deca
50	𐅄	
100	H	hecto
500	𐅅	
1 000	X	kilo
5 000	𐅆	
10 000	M	

2857 = XX𐅅HHH𐅄ΓII

761 = 𐅅HH𐅄ΔI

543 = $\mu\phi\gamma = \rho\rho\rho\rho\rho\kappa\kappa\alpha\beta$

$,\alpha = 1000$

$\overset{\kappa\delta}{\mathrm{M}} = 240{,}000$

Thales of Miletus (ca. 640-546 BCE)

- Ionia, Asia Minor.
- Parents were Greek or Phoenician.
- One of the Seven Sages of Greece.
- Founder of the Milesian School of natural philosophy, and the teacher of Anaximander.
- Credited with 5 theorems in geometry:
    1. A circle is bisected by any diameter.
    2. The base angles of an isosceles triangle are equal.
    3. The angles between two intersecting straight lines are equal.
    4. Two triangles are congruent if they have two angles and one side equal.
    5. An angle in a semicircle is a right angle.

According to Proclus (412-485) and others, head of Plato's Academy, commentaries on mathematicians.

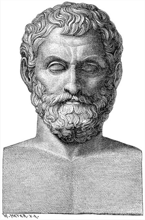


Figure 1: Thales of Miletus taught that 'all things are water.' - Aristotle

Many Other claims:
Predicted solar eclipse (585 BCE).
Measured pyramid heights.

Thales' Theorem

An angle inscribed in a semicircle is a right angle.

- 31st proposition, Book III of Euclid's Elements.
- According to Proclus and Diogenes Laërtius.
- Known earlier to Indian and Babylonian mathematicians.

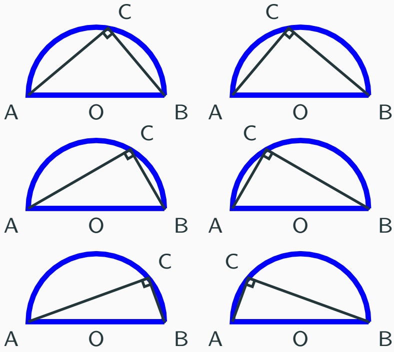


Figure 2: Thales' Theorem demonstrated.

Thales' Theorem: Inscribed Angle = 90^o

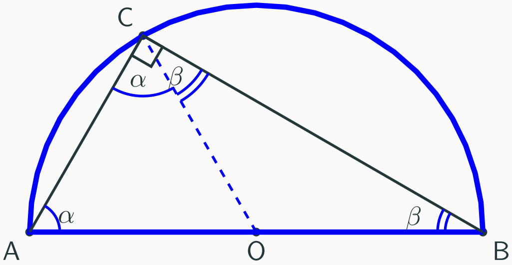


Figure 3: Proof by Picture.

Radii: $\overline{AO} = \overline{OB} = \overline{OC}$.

Isoceles triangles: AOC and OBC.

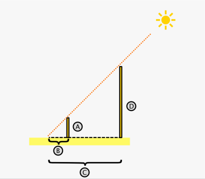
Sum of angles in ABC $= 2\alpha + 2\beta = 180^o$ implies $\alpha + \beta = 90^o$.

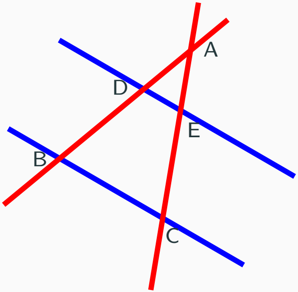
Intercept Theorem

If two (or more) parallel lines (blue) are intersected by two self-intersecting lines (red), then the ratios of the line segments of the first intersecting line is equal to the ratio of similar line segments of the second line.[1]

Prove by using similar triangles:

$$\frac{\overline{DE}}{\overline{BC}} = \frac{\overline{AE}}{\overline{AC}} = \frac{\overline{AD}}{\overline{AB}}$$





[1] "Hieronymus says that [Thales] measured the height of the pyramids by the shadow they cast, taking the observation at the hour when our shadow is of the same length as ourselves (i.e., as our own height)."

Pythagoras of Samos (570-495 BCE)

- Known from Philolaus and others.
- School in Croton, 530 BCE.
  - vegetarian, communal, secret.
  - All is number.
- Philosophy - love of wisdom.
- Mathematics - that which is learned.

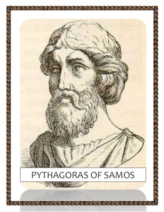


Figure 4: Pythagoras

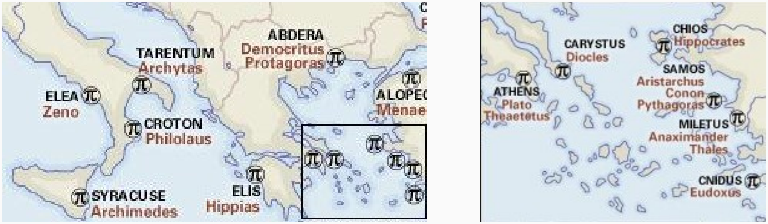


Figure 5: Locate Samos and Croton.

Numerology - Numbers have meanings.

Even is male; Odd is female.

1. = generator
2. = opinion
3. = harmony
4. = justice
5. = marriage
6. = creation
7. = planets

10 is holiest (tetractys, tetrad, decad).

Figure 6: Tetractys

Triangular numbers:

$$1, 3, 6, 10, \ldots$$

Also the four seasons, planetary motions, music, four elements, fourth triangular number, etc.

Number Theory

- Triangular Numbers:

$$1, 3, 6, 10, \ldots$$

- Perfect Numbers [Sum factors $< n$.]:

$$\begin{aligned} 6 &= 1+2+3 \\ 10 &\neq 1+2+5 \\ 28 &= 1+2+4+7+14 \end{aligned}$$

- Amicable Numbers:

$$\begin{aligned} 220 &: 1+2+4+5+10+11+20+22+44+55+110 = \\ 284 &: 1+2+4+71+142 = \end{aligned}$$

Pythagorean Theorem, $a^2 + b^2 = c^2$

- Known by Babylonians and Egyptians.
- Also, traces in other cultures.
- Many Proofs over the years.
- Attributed to Pythagoras.
- Pythagorean Triples (a, b, c).

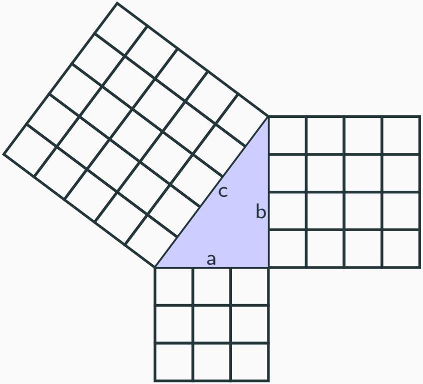


Figure 7: Euclid's Proof.

Segments are **commensurable** if there exist a segment EF such that $\overline{AB} = p\overline{EF}$ and $\overline{CD} = q\overline{EF}$, where p and q are integers.
Therefore,

$$\frac{\overline{AB}}{\overline{CD}} = \frac{p}{q}.$$

Sometimes written as $p : q$.
Led to *Music of the Spheres*.

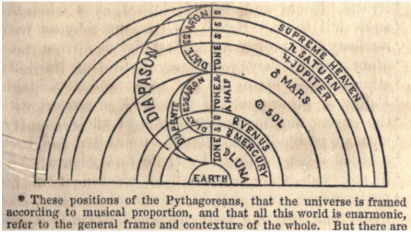


Figure 8: *A General History of the Science and Practice of Music,* Sir John Hawkins, 1853. Also provides story of Pythagoras' death.

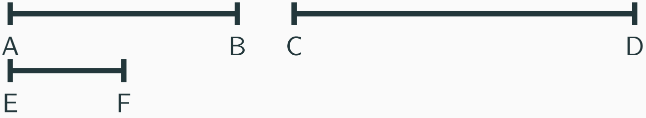


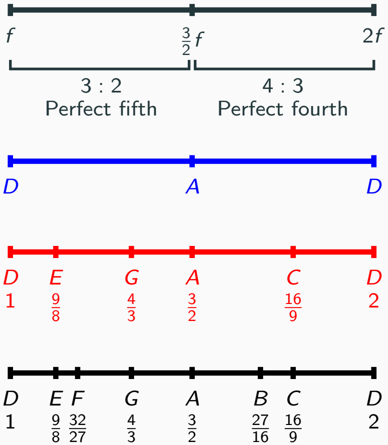
Figure 9: Commensurate Segments.

Pythagorean Scale - Series of Musical Notes

Goal - To produce a music scale.

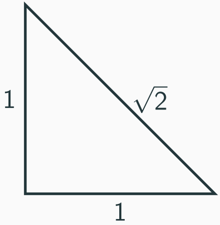
Want sounds that are pleasing when played together. Need simple ratios.

- **Octave**: From f to $2f$ (2^{nd} Harmonic).
  Ex: D goes to D, an octave higher.
- Next Notes?
  Up by **perfect fifth**. $\frac{3}{2}(1) = \frac{3}{2}$, A.
  Down by **perfect fifth**. $\frac{2}{3}(2) = \frac{4}{3}$, G.
- $\frac{3}{2}(\frac{3}{2}) = \frac{9}{4}$, wrong octave, halve.
- $\frac{2}{3}(\frac{4}{3}) = \frac{8}{9}$. wrong octave, double.
- Gives E and C.
- Pentatonic scale: D, E, G, A, C, D.
- Western Scale: D, E, F, G, A, B, C, D.
- B: $\frac{3}{2}(\frac{9}{8}) = \frac{27}{16}$, F: $\frac{2}{3}(\frac{16}{9}) = \frac{32}{27}$.



Irrational Numbers

- Hippasus of Metapontum (c. 530 - c. 450 BCE).
- Credited proving $\sqrt{2}$ is irrational.
- Drowned - possibly not an accident.
- Plato wrote Theodorus of Cyrene (c. 400 BCE) proved the irrationality of $\sqrt{3}$ to $\sqrt{17}$.
- Greeks knew sum of angles of triangle $= 2(90^o) = 180^o$.
- Construction of figures with compass and straight edge.



Classical Construction Problems

- Squaring the Circle (Quadrature) - Dinostratus (c. 390–320 BCE).
- Doubling the Cube (2×Volume) - Menaechmus (380–320 BCE).
- Trisecting a Angle (using unmarked straightedge and compass.) Hippias (460-400 BCE). Impossibility Proof: 1857, Pierre Wantzel, needs Modern Algebra.

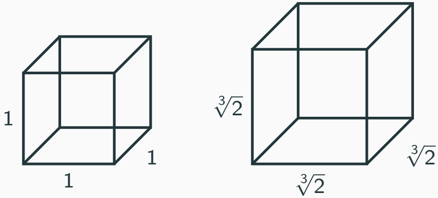


Figure 10: Doubling the Cube.

Hippocrates of Chios (c. 470 - c. 410 BCE)

- Not the Hippocrates of Kos (c. 460 - c. 370 BCE), Father of Medicine, and the Hippocratic Oath.
- Mathematician, geometer, and astronomer.
- Went to Athens.
- Used *reductio ad absurdum* arguments (proof by contradiction).
- Wrote geometry textbook, *Elements*
- Sought Quadrature of Circle.
- Quadrature of Lune.

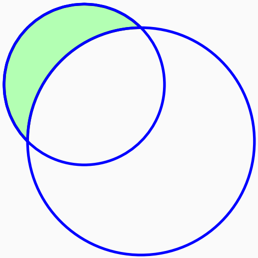


Figure 11: Lune or Crescent.

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.

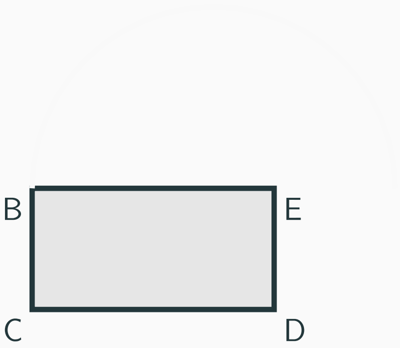


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.

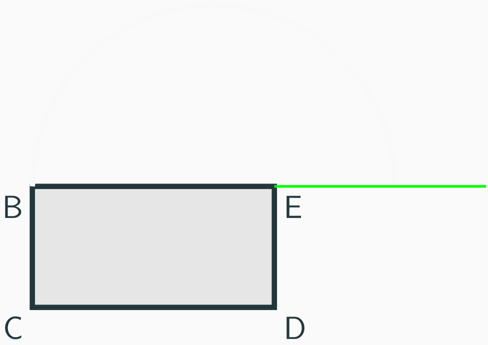


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.

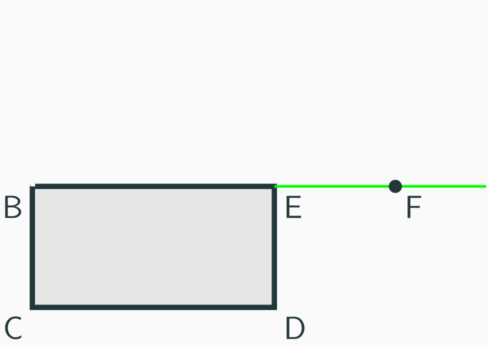


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?

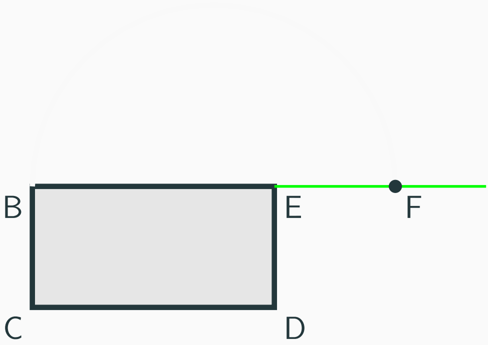


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

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- How do you bisect BF?
- Bisect segment BF.

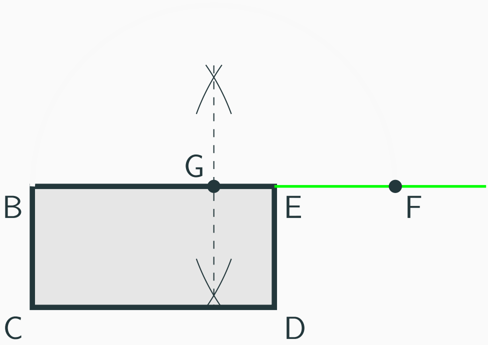


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.

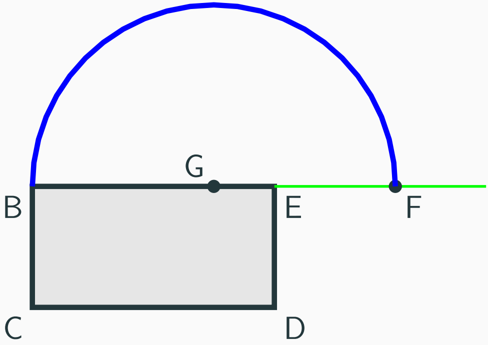


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.

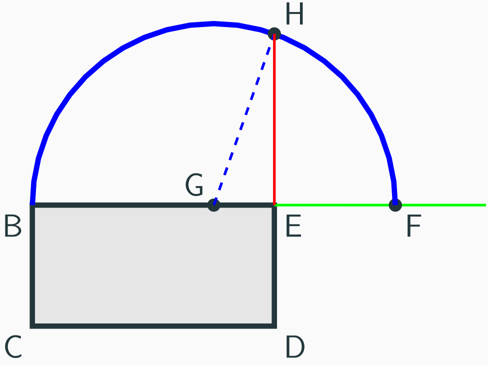


Figure 12: Quadrature of a Rectangle

Quadrature of Rectangle

Quadrature - construction of a square of equal area to a given plane figure.

- Start with BCDE.
- Extend segment BE.
- Get F such that $\overline{EF} = \overline{ED}$.
- How do you bisect BF?
- Bisect segment BF.
- Draw semicircle about G.
- Get point H.
- Construct square EKLH.
- Prove the areas are equal.

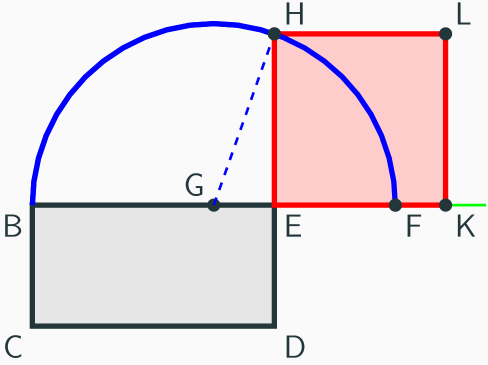


Figure 12: Quadrature of a Rectangle

Proof of Equal Areas

Label lengths a, b, c.

Area of Gray Rectangle BCDE:

$$\begin{aligned} A &= (a+b)\overline{ED} \\ &= (a+b)\overline{EF} \\ &= (a+b)(a-b) \\ &= a^2 - b^2. \end{aligned}$$

Area of Red Square EKLH:

Use Pythagorean Theorem:

$$A = c^2 = a^2 - b^2.$$

Thus, the area of the square is the same as the given rectangle; i.e., we **squared the rectangle**.

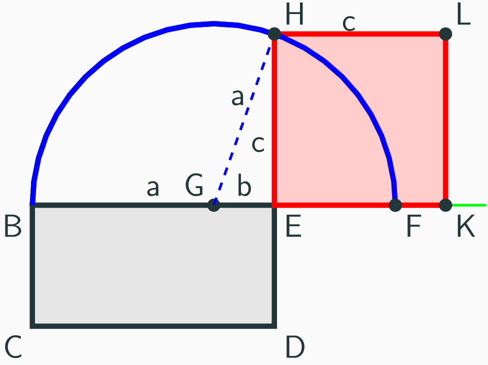
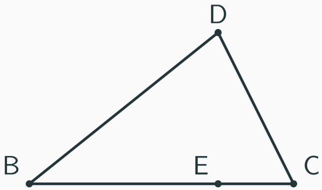


Figure 13: Quadrature of a Rectangle

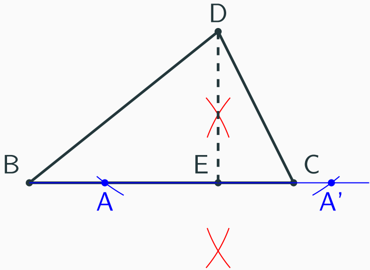
Quadrature of a Triangle

- Start with a triangle.



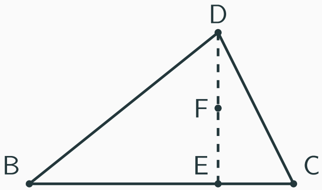
Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
    1. Draw blue arcs about D.
    2. Bisect AA' using red arcs.



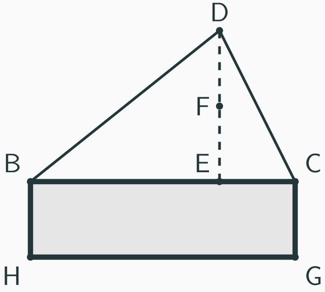
Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
    1. Draw blue arcs about D.
    2. Bisect AA' using red arcs.
- Bisect perpendicular.



Quadrature of a Triangle

- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.

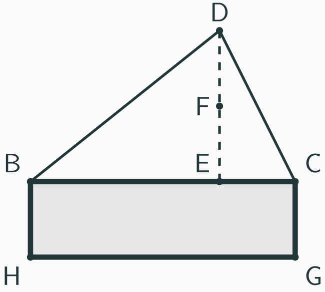


Quadrature of a Triangle

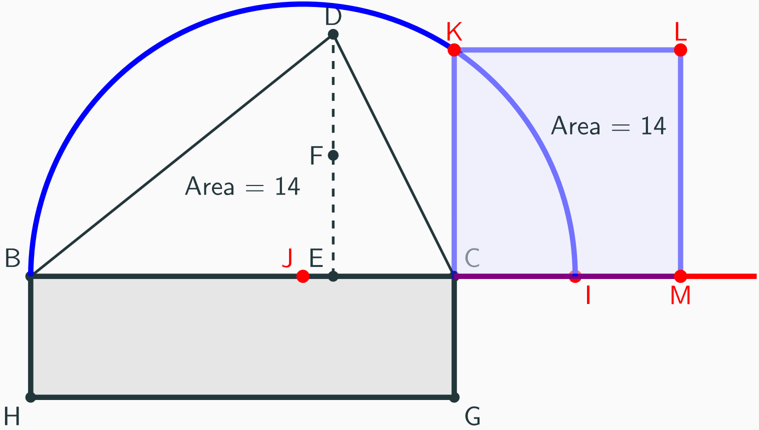
- Start with a triangle.
- Construct perpendicular DE.
  1. Draw blue arcs about D.
  2. Bisect AA' using red arcs.
- Bisect perpendicular.
- Construct a rectangle with height CG = EF.
- Area of Triangle = Area of Rectangle:

$$\begin{aligned} A(BCD) &= \frac{1}{2}\overline{BC}\,\overline{DE} \\ &= \overline{BC}\,\overline{CG} \\ &= A(BCGH). \end{aligned}$$

- Square this rectangle.



Quadrature of a Triangle - Final Construction



Quadrature of a Lune

- Lune is the figure bounded by two circular arcs.
- Hippocrates squared a special lune.
- Based on
    - Pythagorean Theorem.
    - Angle inscribed in semicircle is right.
    - Ratio of Areas of circles

$$\frac{A_1}{A_2} = \frac{D_1^2}{D_2^2}.$$

    - Triangles are quadrable.
- Hippocrates proof not valid.

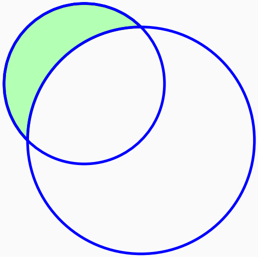


Figure 14: Lune or Crescent.

Hippocrates' Quadrature of a Lune

- $\overline{AB}^2 = \overline{AC}^2 + \overline{CB}^2 = 2\overline{AC}^2$
- Semicircle areas

$$\frac{A(AEC)}{A(ACB)} = \frac{\overline{AC}^2}{\overline{AB}^2} = \frac{1}{2}.$$

- Area of Lune = Area of ΔAOC.
- ΔAOC quadrable, so is the lune.

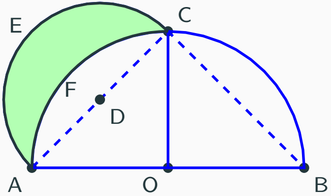


Figure 15: Lune AECF is quadrable.

Can one Square the circle?

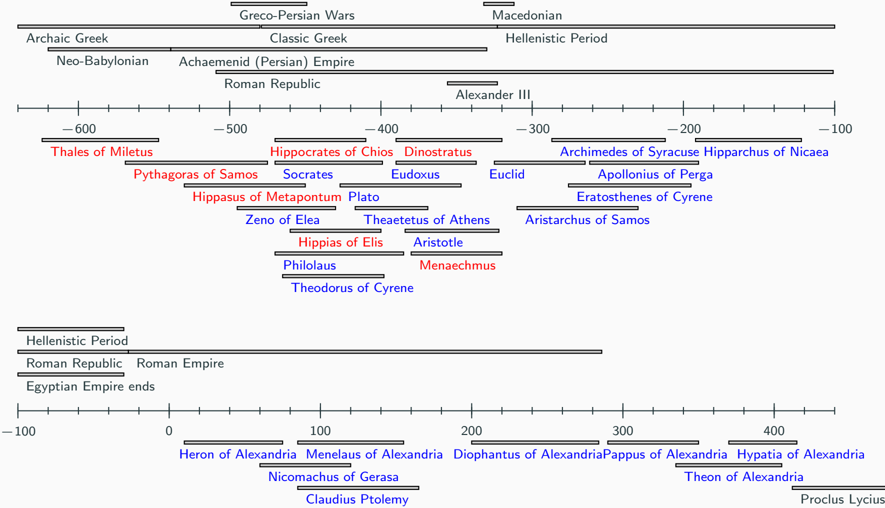
Unsolved until Ferdinand Lindemann (1852-1939).

Algebraic Numbers, solutions of polynomial equations with integer coefficients.

Ex: $x^2 - 2 = 0$ has solution $\pm\sqrt{2}$.

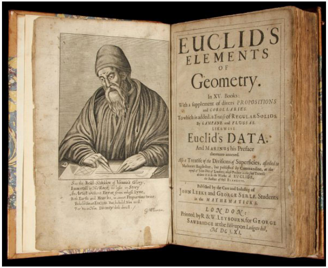
Transcendental Numbers, numbers that aren't algebraic.

Timeline of Greek Mathematicians - Where Are WE?



Euclid's Elements

Fall 2023 - R. L. Herman



Euclid of Alexandria (c. 325 - c. 265 BCE)

- Founder of Geometry.
- Active in Alexandria, Egypt during reign of Ptolemy I (323–283 BC).
- *Elements of Geometry*
  - Most famous mathematical work of classical antiquity.
  - World's oldest continuously used mathematical textbook.
  - Geometry, proportion, and number theory.
  - 13 Books.
  - 465 Propositions.
  - 23 Definitions. (point, line, straight line, . . .)
  - 5 Postulates.
  - 5 Axioms.

Figure 1: Euclid.

The Thirteen Books

Book 1 Fundamental propositions of plane geometry.

- Congruent triangles.
- Theorems on parallel lines.
- Sum of the angles of a triangle.
- The Pythagorean theorem.

Book 2 Geometric algebra.

Book 3 Properties of circles.

- Theorems on tangents and inscribed angles.

Book 4 Inscribed and circumscribed regular polygons around circles.

Book 5 Arithmetic theory of proportion.

Book 6 Theory of proportion in plane geometry.

Book 7 Elementary number theory.

- prime numbers, greatest common denominators, etc.

Book 8 Geometric series.

Book 9 Applications and theorems on the infinitude of prime numbers, and the sum of a geometric series.

The Thirteen Books

Book 10 Incommensurable (irrational) magnitudes using the "Method of Exhaustion." [Eudoxus (390-337 BCE), Antiphon (480-411 BCE).]

Book 11 Propositions of three-dimensional geometry.

Book 12 Relative volumes of cones, pyramids, cylinders, and spheres using the Method of Exhaustion.

Book 13 The five Platonic solids.

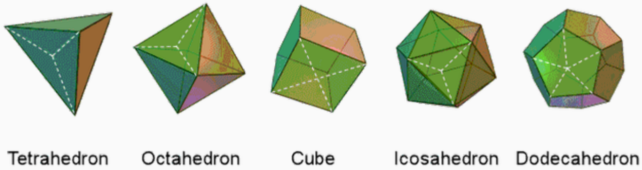


Figure 2: Platonic Solids.

Contributors to *The Elements*

The Elements - a compilation based on books by earlier Greek mathematicians. See Sir Thomas Heath

Proclus (412–485 AD), wrote in his commentary on the Elements: "Euclid, who put together the Elements, collecting many of Eudoxus' theorems, perfecting many of Theaetetus', and also bringing to irrefragable demonstration the things which were only somewhat loosely proved by his predecessors".

Pythagoras - probably the source for most of books I and II

Hippocrates of Chios source for book III,

Eudoxus of Cnidus source for book V

Books IV, VI, XI, and XII probably from Pythagorean or Athenian mathematicians.

Definitions i

Def 1. A point is that which has no part.

Def 2. A line is breadthless length.

Def 3. The ends of a line are points.

Def 4. A straight line is a line which lies evenly with the points on itself.

Def 5. A surface is that which has length and breadth only.

Def 6. The edges of a surface are lines.

Def 7. A plane surface is a surface which lies evenly with the straight lines on itself.

Def 8. A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Def 9. And when the lines containing the angle are straight, the angle is called rectilinear.

Definitions ii

Def 10. When a straight line standing on a straight line makes the adjacent angles equal to one another, each of the equal angles is right, and the straight line standing on the other is called a perpendicular to that on which it stands.

Def 11. An obtuse angle is an angle greater than a right angle.

Def 12. An acute angle is an angle less than a right angle.

Def 13. A boundary is that which is an extremity of anything.

Def 14. A figure is that which is contained by any boundary or boundaries.

Def 15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another.

Def 16. And the point is called the center of the circle.

Def 17. A diameter of the circle is any straight line drawn through the center and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle.

Def 18. A semicircle is the figure contained by the diameter and the circumference cut off by it. And the center of the semicircle is the same as that of the circle.

Def 19. Rectilinear figures are those which are contained by straight lines, trilateral figures being those contained by three, quadrilateral those contained by four, and multilateral those contained by more than four straight lines.

Def 20. Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle that which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.

Definitions iv

Def 21. Further, of trilateral figures, a right-angled triangle is that which has a right angle, an obtuse-angled triangle that which has an obtuse angle, and an acute-angled triangle that which has its three angles acute.

Def 22. Of quadrilateral figures, a square is that which is both equilateral and right-angled; an oblong that which is right-angled but not equilateral; a rhombus that which is equilateral but not right-angled; and a rhomboid that which has its opposite sides and angles equal to one another but is neither equilateral nor right-angled. And let quadrilaterals other than these be called trapezia.

Def 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

Postulates

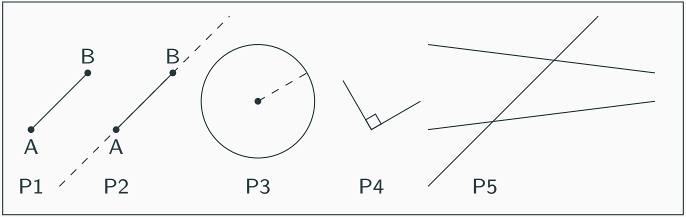
Postulate 1. To draw a straight line from any point to any point.

Postulate 2. To produce a finite straight line continuously in a straight line.

Postulate 3. To describe a circle with any center and radius.

Postulate 4. That all right angles equal one another.

Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



Common Notions

Notion 1. Things which equal the same thing also equal one another.

Notion 2. If equals are added to equals, then the wholes are equal.

Notion 3. If equals are subtracted from equals, then the remainders are equal.

Notion 4. Things which coincide with one another equal one another.

Notion 5. The whole is greater than the part.

Proposition 1

To construct an equilateral triangle on a given finite straight line.

- Start with segment AB.

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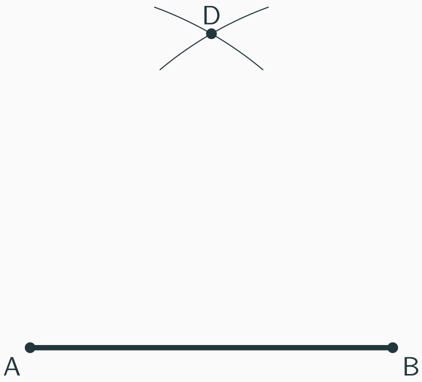
To construct an equilateral triangle on a given finite straight line.

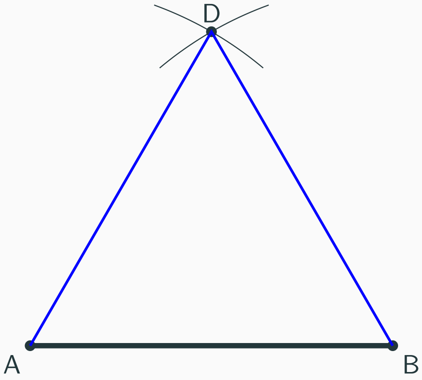


- Start with segment AB.
- Draw circular arcs about A, B of radius AB.

Proposition 1

To construct an equilateral triangle on a given finite straight line.

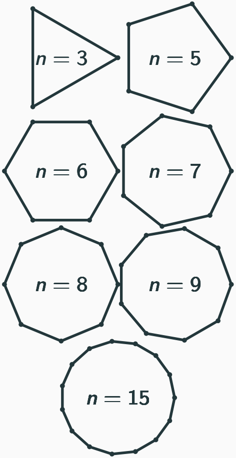
- Start with segment AB.
- Draw circular arcs about A, B of radius AB.
- Draw line segments AD, BD.



Regular Polygons

Construct using straight edge and compass.

- Triangles (Euclid I.1)
- Squares (Euclid I.46)
- Pentagons (Euclid IV.11)
- Hexagons (Euclid IV.15)
- Septagon (heptagon) (no)
- Octagon (Euclid III.30)
- Nonagon (no)
- 15-gon (Euclid IV.16) pentadecagon
- Double the number of sides of a given regular polygon, 8, 10, 12, 16, 20, 24, etc. (Euclid III.30)



Constructible regular n-gons

Is it possible to construct all regular polygons with compass and straightedge? If not, which n-gons (that is polygons with n edges) are constructible and which are not?

- Young C. F. Gauss, 1796: the regular 17-gon (Heptadecagon) is constructible.
- Theory of Gaussian periods in his *Disquisitiones Arithmeticae*. 1801.
- Gave sufficient condition for the constructibility.
- Proof of necessity - Pierre Wantzel in 1837.
- Gauss–Wantzel theorem:
  A regular n-gon can be constructed with compass and straightedge if and only if n is the product of a power of 2 and any number of distinct Fermat primes, p_ℓ:[only 3, 5, 17, 257, 65537.]

$$n = 2^m p_1 p_2 \cdots p_k, \qquad p_\ell = 2^{2^\ell} + 1.$$

Book 13 - Platonic Solids - Regular Polyhedra

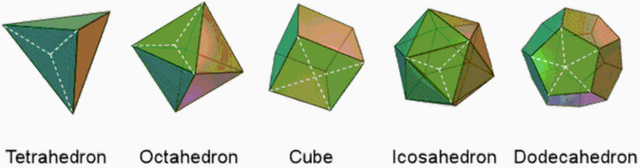


Figure 3: Platonic Solids: Fire, Air, Earth, Water, Universe.

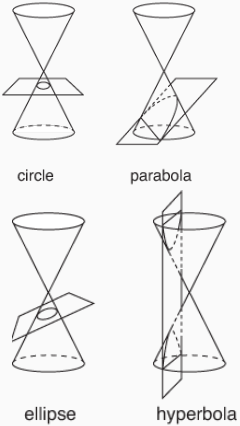
Polyhedron	Faces	Edges	Vertices
Tetrahedron	$4\ \Delta's$	6	4
Cube	6 □'s	12	8
Octahedron	$8\ \Delta's$	4	6
Dodecahedron	12 Pentagons	30	20
Icosahedron	$20\ \Delta's$	30	12

Note: Johannes Kepler (1571-1630) systematized and extended what was known about polyhedra. See *Harmonice Mundi*, 1619. Proposed relationships between six known planets and the Platonic solids.

Conic Sections

Possibly discovered by **Menaechmus**[1] (380–320 BCE) to duplicate cube: Intersect parabola $y = \frac{1}{2}x^2$ and hyperbola $xy = 1$.

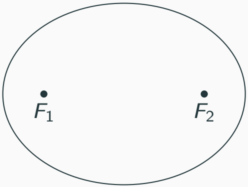
- **Euclid** - four lost books on conics.
- **Archimedes** of Syracuse (287-212 BCE) studied conics, area bounded by a parabola and a chord in *Quadrature of the Parabola*.
- **Apollonius** of Perga (262-190 BCE), eight-volume *Conics*.
  Terms: parabola, ellipse, hyperbola
- **Pappus** of Alexandria (290 – 350) - focus directrix.
- Applied by Kepler (1609), Newton (1687).



[1] To Alexander, *"O king, for travelling through the country there are private roads and royal roads, but in geometry there is one road for all."*

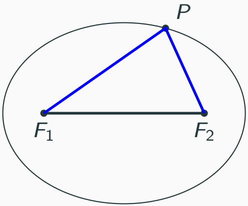
Ellipse

- Focal points: F_1, F_2.



Ellipse

- Focal points: F_1, F_2.
- $\overline{F_1P} + \overline{F_2P} = 2a$.



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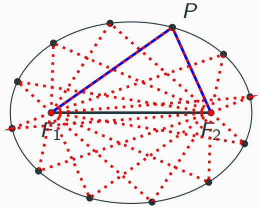
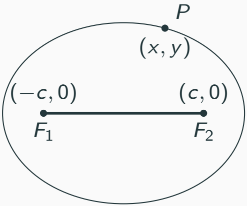


Ellipse

- Focal points: F_1, F_2.
- $\overline{F_1P} + \overline{F_2P} = 2a$.
- Algebra leads to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

a, $b =$ semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.



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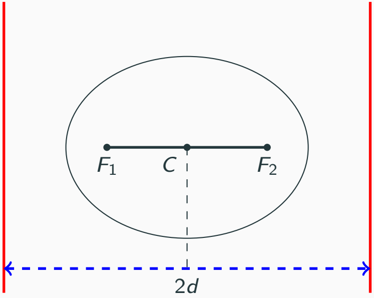
- Directrix $d = \frac{a^2}{c}$,



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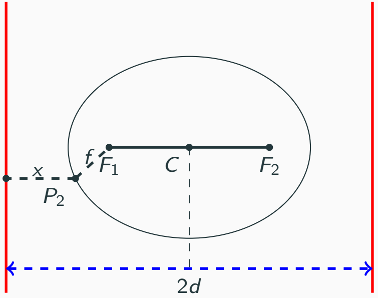
  a, $b =$ semimajor/semiminor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.



Ellipse

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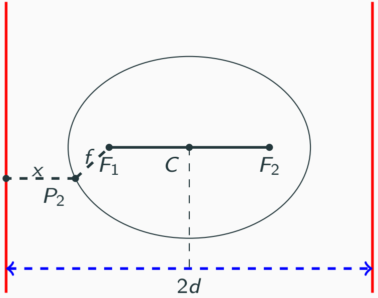
  a, b = semimajor/seminimor axes with $c = \sqrt{a^2 - b^2}$, $a > b$.
- Directrix $d = \frac{a^2}{c}$,
- Eccentricity $\epsilon = \frac{f}{x} = \frac{c}{a}$.
- Eccentricty of Conics:

  $\epsilon = 0$, circle.

  $0 < \epsilon < 1$, ellipse.

  $\epsilon = 1$, parabola.

  $\epsilon > 1$, hyperbola.



Dandelin Sphere - Germinal Pierre Dandelin (1794-1847)

- Inscribed spheres tangent to cone and intersecting plane.
- Intersection is a conic.
- Tangent pts to sphere are focal points.
- Used to prove theorems of Apollonius.
  - Conic section is the set points such that the sum of the distances to two fixed points is constant.
  - The distance from the focus is proportional to the distance from a fixed line (directrix).
  - The constant of proportionality is the eccentricity

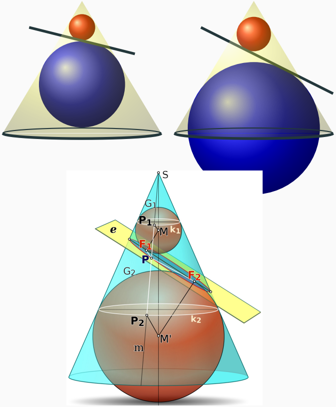
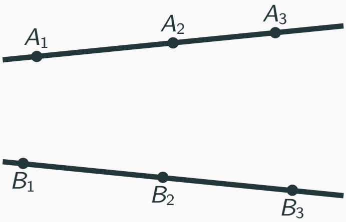


Figure 4: Dandelin Spheres (Paper in 1822)

Other Geometry Gems

- Pappas' Hexagon Theorem (290-350)
  - Connect 6 pts on two lines.
  - A_1-B_2, B_2-A_1, etc.



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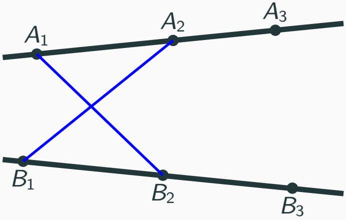

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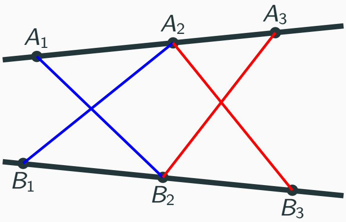

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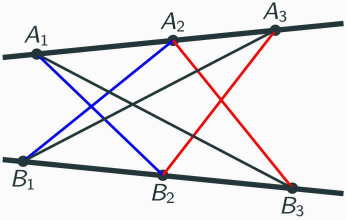

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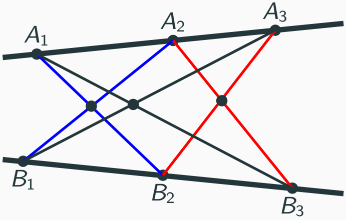

Other Geometry Gems

- Pappas' Hexagon Theorem (290-350)
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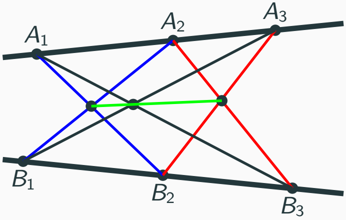
- The three points are collinear.
- The beginning of projective geometry.



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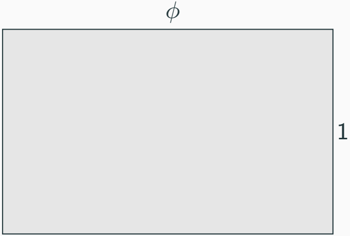
- Golden Ratio (section, mean), ϕ, τ.
  - Euclid, "extreme and mean ratio"



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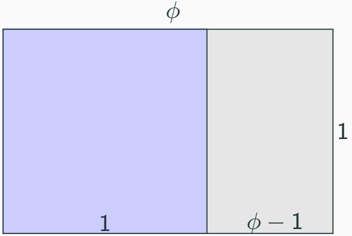
  - Golden Rectangle: $\frac{\phi}{1} = \frac{1}{\phi - 1}$. [Gray region is similar to the large rectangle.]



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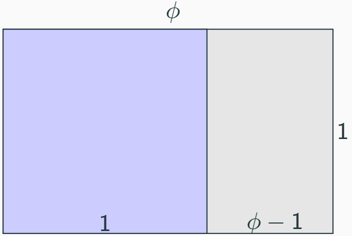
  - Golden Rectangle: $\frac{\phi}{1} = \frac{1}{\phi - 1}$. [Gray region is similar to the large rectangle.]
  - Solution $\phi^2 = \phi + 1$:



$$\phi = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2} \approx 1.61803\ldots.$$

Fibonacci and Lucas Numbers

Fibonacci Numbers:

$$
\begin{aligned}
F_n &= 1, 1, 2, 3, 5, 8, \ldots \\
\sqrt{5} F_n &= \phi^n - (-\phi)^{-n} \qquad (1)
\end{aligned}
$$

Lucas Numbers:

$$
\begin{aligned}
L_n &= 2, 1, 3, 4, 7, \ldots \\
&= \phi^n + (-\phi)^{-n} \qquad (2)
\end{aligned}
$$

Ratio limits

$$
\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \phi
$$

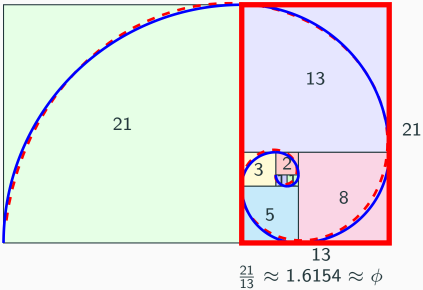


Figure 5: Golden Spiral, or Fibonacci Spiral, is approximately a logarithmic spiral, $r = a\phi^{2\theta/\pi}$.

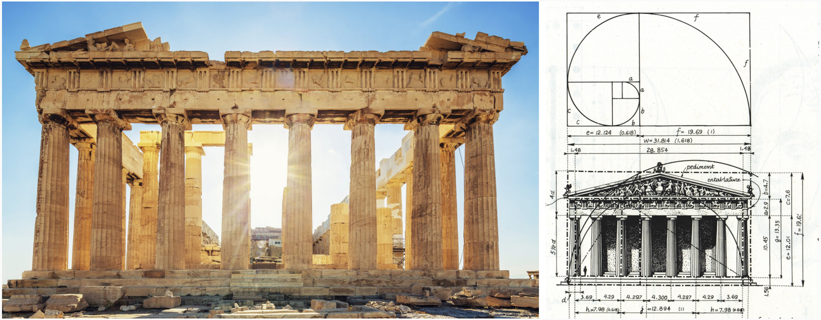


Figure 6: The Parthenon and the Golden Rectangle

Other Examples in Nature

Greek Mathematics II

Fall 2023 - R. L. Herman

Greek Number Theory

- **Pythagorean Theorem**
  $x^2 + y^2 = z^2$, x, y, z integers.
- **Diophantine Equations**
  Solve $3x + 5y = 1$, x, y integers.
- Euclid
  - Proved **# primes infinite**, Book IX, Prop 20.
  - **Perfect Numbers**, Book VII, Def 22, Book IX, Prop 36.
    Euclid proves:
    If $2^n - 1$ is prime, then $(2^n - 1)\,2^{n-1}$ is perfect.
    Mersenne prime: $2^n - 1$.
  - **Polygonal Numbers**

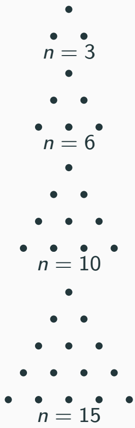


Figure 1: Polygonal Numbers.

Euclidean Algorithm - Book VII, Prop 1

- gcd(a, b): **Greatest common divisor** of a and b.
- Algorithm

$$\begin{aligned} a_1 &= \max(a,b) - \min(a,b) \\ b_1 &= \min(a,b) \\ &\text{repeat} \end{aligned}$$

Terminates when $a_{i+1} = b_{i+1}$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} a_1 &= 210 - 45 = 165 \\ b_1 &= 45 \end{aligned}$$

Continuing the computation:

a_i	b_i
210	45
165	45
120	45
75	45
30	45
15	30
15	15

We find $\gcd(210, 45) = 15$.

Euclidean Algorithm - Another Approach

Find the greatest common divisor of positive integers, a and b.

- If $a < b$, exchange a and b.
- Divide a by b and get the remainder, r. Thus,

$$a = qb + r.$$

  - If $r \neq 0$, replace a by b and b by r. Repeat the division.
  - If $r = 0$, report $gcd(a, b) = b$.

Example: Find $\gcd(210, 45)$.

$$\begin{aligned} 210 &= 4 \cdot 45 + 30 \\ 45 &= 1 \cdot 30 + 15 \\ 30 &= 2 \cdot 15 \end{aligned}$$

Thus, $\gcd(210, 45) = 15$.

Pell's Equation (1611-1685)

- $x^2 - Ny^2 = 1$,
  N is a nonsquare integer, and
  x, y are integer solutions.
- Example of a Diophantine equation.
- Related to $\sqrt{2}$: $x^2 - 2y^2 = 0$,
  $y = 1 \Rightarrow x = \sqrt{2}$.
- $x^2 - 2y^2 = 1$,
  If x, y large, then $\frac{x}{y} \approx \sqrt{2}$.
- Known to Pythagoreans, Diophantus, and
- Archimedes' Cattle Problem can be reduced to solving Pell's Equation.
- Brahmagupta (598-570) first to solve.

From *The New York Times*
January 18, 1931, p. 54

CATTLE PROBLEM SOLVED

Moreover, Final Conditions Set by Archimedes Can Be Worked Out

To the Editor of The New York Times:

Frank G. Nelson, whose interesting letter regarding his solution of the cattle problem of Archimedes appeared in THE TIMES, would feel flattered if he had the translation of this problem which is possessed by me, for, according to Archimedes, he is no mere "novice in numbers," since no such person could be expected to arrive at a correct solution—as has Mr. Nelson—of the first seven equations presented by the problem.

But Mr. Nelson's conclusion that the final conditions set by the problem cannot be solved is erroneous—at least according to a large number of mathematicians who have worked on it. As far back as 1860, Amthor showed that the total of the cattle would be represented by a number containing 206,545 figures, the printing of which would require about two full pages of THE NEW YORK TIMES. Since it has been calculated that it would take the work of a thousand men for a thousand years to determine the complete number, it is obvious that the world will never have a complete solution, which should relieve the mind of any linotype operator who fears that he might be called on to set it. However, the first thirty-one figures have been computed, as have the last twelve, and the solution, for those who are interested, is

7,760,271 081,800

in which the line of dots represents thirty solved and 206,502 unsolved numbers.

The above solution was worked out by the Hillsboro Mathematical Club of Hillsboro, Ill., which was formed by A. H. Bell in 1889 to labor on the problem. Nearly four years were spent by the three club members on the work, and the results were published in the American Mathematical Monthly in 1895. An interesting summary of the mathematical steps involved in the determination of these enormous numbers—there are ten altogether, each containing 206,544 or 206,545 figures—is to be found in Recreations in Mathematics, by H. E. Licks (Van Nostrand, 1917).

Archimedes was evidently fond of problems involving enormous numbers, as his book "Arenarius" discusses the solution of the problem of determining the number of grains of sand in a sphere the size of the earth. This number is, however, of insignificant size in comparison with that representing the solution of the cattle problem. In fact, it has been calculated that if the cattle represented by this number were reduced to the size of the smallest bacterium, they could not be contained in a sphere having the diameter of the Milky Way, across which astronomers calculate that it takes light, traveling at about 186,000 miles a second, 10,000 years to travel.

NORMAN MERRIMAN.
New York, Jan. 12, 1931.

Pell's Equation General Solution

- $x^2 - ny^2 = 1$,
  n is a nonsquare integer and
  x, y are integer solutions.
- Let $z = x + y\sqrt{n}$, $x, y \in \mathbb{Z}$
  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.

Pell's Equation General Solution

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  n is a nonsquare integer and
  x, y are integer solutions.
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  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.

Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

Then

$$\begin{aligned} x_3 &= x_1x_2 + ny_1y_2, \\ y_3 &= x_1y_2 + x_2y_1. \end{aligned}$$

Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

Pell's Equation General Solution

- $x^2 - ny^2 = 1$,
  n is a nonsquare integer and
  x, y are integer solutions.
- Let $z = x + y\sqrt{n}$, $x, y \in \mathbb{Z}$
  and $\bar{z} = x - y\sqrt{n}$.
- $\text{Norm}(z) = z\bar{z} = x^2 - ny^2 = 1$.
- $\text{Norm}(zw) = \text{Norm}(z)\text{Norm}(w)$.
- Example: $x^2 - 3y^2 = 1$
- Guess $(2, 1)$.
  So, $z = 2 + \sqrt{3} = w$.

$$\begin{aligned} zw &= \left(2 + \sqrt{3}\right)^2 \\ &= 7 + 4\sqrt{3}. \end{aligned}$$

Let

$$\begin{aligned} z &= x_1 + y_1\sqrt{n}, \\ w &= x_2 + y_2\sqrt{n}, \\ zw &= x_3 + y_3\sqrt{n}, \end{aligned}$$

Then

$$\begin{aligned} x_3 &= x_1x_2 + ny_1y_2, \\ y_3 &= x_1y_2 + x_2y_1. \end{aligned}$$

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w &= x_2 + y_2\sqrt{n}, \\
zw &= x_3 + y_3\sqrt{n},
\end{aligned}
$$

Then

$$
\begin{aligned}
x_3 &= x_1 x_2 + n y_1 y_2, \\
y_3 &= x_1 y_2 + x_2 y_1.
\end{aligned}
$$

Since $\text{Norm}(zw) = 1$,
(x_3, y_3) is a solution.

Then, $(7, 4)$ is a solution.

Eudoxus of Cnidus (c.390 – c. 337 BCE)

- Studied under Plato.
- Taught Aristotle.
- Astronomer, Mathematician.
- Theory of Proportions:

  Circles: $A \propto r^2$,
  Spheres: $V \propto r^3$,
  Volume of a pyramid .
  Volume of a cone.
- Studied reals, continuous quantities.
- Method of Exhaustion:
  Due to Antiphon (480–411 BCE).
  Area from a sequence of inscribed polygons.

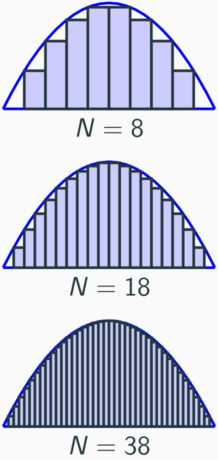
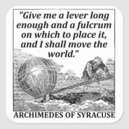
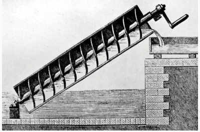


Figure 2: Method of Exhaustion.

Archimedes of Syracuse (287-212 BCE)

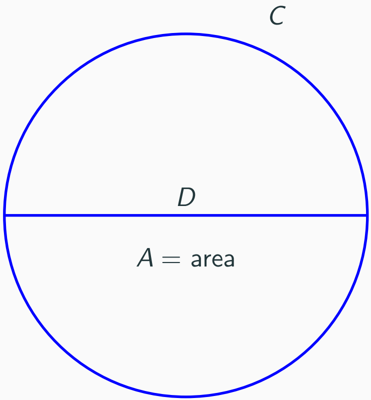
- Went to Alexandria, Egypt
  then, back to Syracuse, Sicily.
- Greatest Mathematician of Antiquity.
- Mathematician, Engineer, Inventor.
  - Archimedean screw, lever, pulley.
- King Heiro II's crown - Eureka.
  Archimedes Principle of Bouyancy.
- According to Plutarch (46-120):
  - Marcellus - Syracuse 212 BCE.
  - Claw of Archimedes.
  - Heat Ray.
  - Prone to intense concentration.
  - Death of Archimedes.





Archimedes' Mathematics

- Mastered Euclid and Eudoxus' (c. 390-337 BCE) Method of Exhaustion.
- *Measurement of a Circle*
  $\frac{C}{D} = \text{const.}, \quad \frac{A}{D^2} = \text{const.}$



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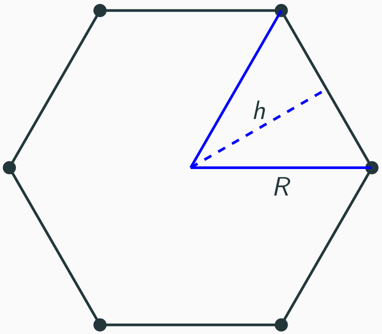
  $A_p = \frac{1}{2}hQ$, $Q =$ Perimeter.



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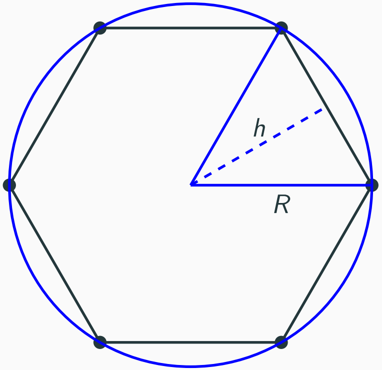

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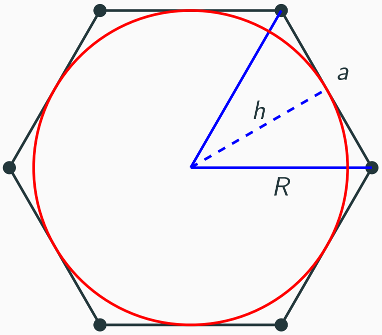
  $A_p = \frac{1}{2}anh <$ area of circle.
- Circumscribed Polygon
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.



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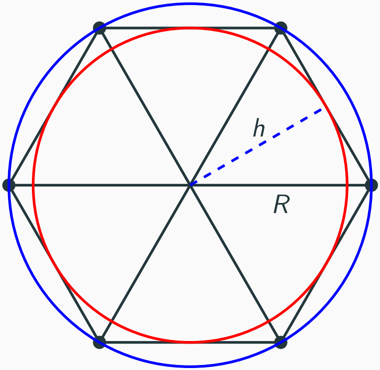
  $a = 2R\sin\frac{180}{n}, h = \sqrt{R^2 - \frac{a^2}{4}}$.
- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$.

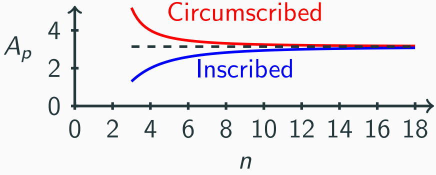


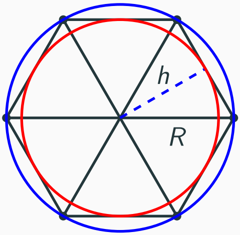
Estimating π

- Approximation of π,
  $\frac{A_p}{R^2} < \pi < \frac{A_p}{h^2}$,
- Recall
  $a = 2R \sin \frac{180}{n}$,
  $h = \sqrt{R^2 - \frac{a^2}{4}} = R \cos \frac{180}{n}$,
  $A_p = \frac{1}{2} anh = nhR \sin \frac{180}{n}$.
- Therefore,

$$\frac{n}{2} \sin \frac{360}{n} < \pi < n \tan \frac{180}{n},$$

- Hexagon ($n = 6$),
  $2.598 < \pi < 3.464$.
- Archimedes - up to 96-gon
  $3.1394 < \pi < 3.1427$.





Archimedes' Inscribed and Circumscribed n-gons

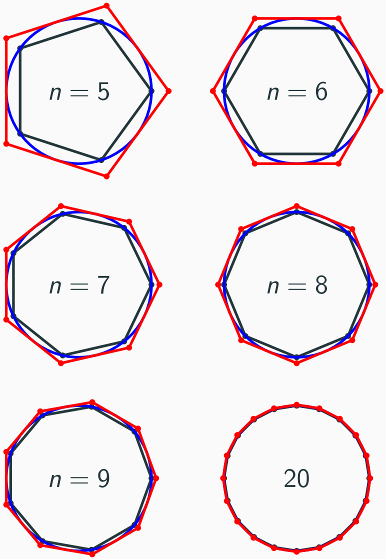
Consider a fixed circle of radius R.

- Inscribed n-gon: $h = R\cos\frac{180}{n}$, $A_i = nhR\sin\frac{180}{n}$.
- Circumscribed n-gon: $r = \frac{R}{\cos\frac{180}{n}}$, $A_c = nHr\sin\frac{180}{n}$.
- Thus,

$$A_i = nR^2\tan\frac{180}{n},$$
$$A_c = \frac{n}{2}R^2\sin\frac{360}{n}.$$

- This gives

$$\frac{n}{2}\sin\frac{360}{n} < \pi < n\tan\frac{180}{n},$$



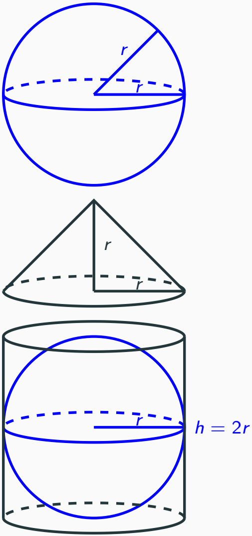
Early Approximations of π: Peripherion $\pi\epsilon\rho\iota\phi\epsilon\rho\epsilon\iota\alpha$

- Bible, $\pi \approx 3$.
- Babylonian $3 + \frac{1}{8}$.
- Egyptians, $\left(\frac{4}{3}\right)^4 = \frac{256}{81} \approx 3.1604938$.
- Sulbasutrakaras ($<$ 800 BCE), 3.08.
- Archimedes (250 BCE) $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
- Aryabhata (499), $\frac{62832}{20000}$.
- Ptolemy (150), 360-gon, 3.14166.
- Chinese (430-501) $\frac{355}{113} \approx 3.14159292$.
- Hindu (1100) $\frac{3927}{1250} \approx 3.1416$.
- Viete', 393,216-gon, π to 9 places.
- van Ceulen (1540-1610) Dutch, 35 places.
- William Shanks (1873) 527 digits.

- Lambert (1728-1777) - irrationality proof.
- William Jones (1706) introduced π.
- Euler popularized notation.
- See Approximations of π .
- Leibniz-Madhava
  $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$
- Euler
  $$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$
- Ramanujan
  $$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!}{k!^4}\frac{1103 + 26390k}{396^{4k}}$$

On the Sphere and the Cylinder, **Archimedes**

- Spirals, Area of Parabolae.
- Volumes and Surface Areas of 3D Objects.
- A(Sphere) = 4 A(Great circle) $A = 4(\pi r^2)$.
- V(Sphere) = 4 V(Cone) $V = 4\left(\frac{1}{3}\pi r^3\right)$.
- Sphere inside Cylinder
  Cylinder Area =
  $2\pi r(2r) + 2(\pi r^2) = 6\pi r^2$
  $= \frac{3}{2}$ Sphere Area.
  Volume $= (\pi r^2)h = 2\pi r^3$
  $= \frac{3}{2}$ Sphere Volume.



Archimedes' Manuscripts

- What we know is from 3 books.
- Codex A Lost in 1564.
- Codex B Lost in 1311.
- Codex C Discovered 1906.
  - 4th century Parchment bound.
  - 10th Century Book, Constantinople housed great texts.
  - 1204 4th Crusade destroyed books.
  - 87 Sheets (43.5 goat skins).
  - 1229 Century taken apart, scraped, cut in half, written over with Christian prayer.
  - Moved to Palestine, 400 yrs.

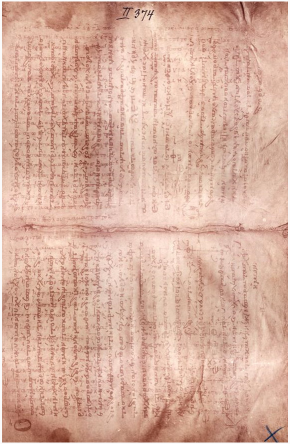


Figure 3: Codex C Page.

The Walters Museum - http://www.archimedespalimpsest.net

Palimpsest, http://www.archimedespalimpsest.org/about/

- 1846, It was in Istanbul, leaf removed to Cambridge.
- 1906, Johan Heiberg took pictures and translated.
- 1922, It went missing.
- 1998, Sold for $2,000,000 - Christies of NY auction.
  Moldy, Charred,
- 7 Manuscripts

  The Equilibrium of Planes, Spiral Lines, The Measurement of the Circle, Sphere and Cylinder, On Floating Bodies, The Method of Mechanical Theorems, and the Stomachion.

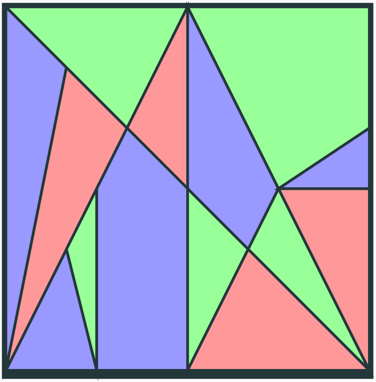


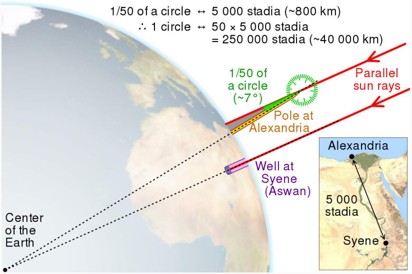
Figure 4: Number of different arrangements of the Stomachion, 17,152.

Eratosthenes of Cyrene (276 - 194 BCE)

- Chief Librarian at Library of Alexandria. (300 BCE-?). Burned 641 by Arabs?
- Sieve of Eratosthenes
- Finding primes:

$$\not{1}, 2, 3, \not{4}, 5, \not{6}, 7, \not{8}, \not{9}, \not{10}, 11, \ldots$$

- Circumference of Earth:
  - Syene - 1st day summer Sun directly over deep well.
  - Alexandria - small shadow.
  - 250,000 stadia.
    1 stade ≈ 526.37 ft
    Equals $24,466$ mi.
    Current - $24,860$ mi.

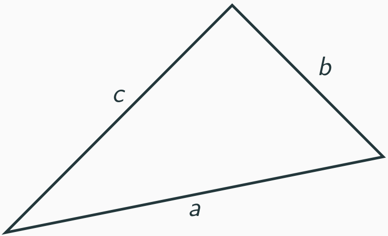


Heron of Alexandria (c. 10-70)

- Inventor.
- Aeolipile,rocket-like reaction engine.
- First-recorded steam engine.
- Hero's wind-powered organ.
- The first vending machine.
- A wind-wheel operating an organ,
- The force pump.
- A syringe-like device.
- Principle: shortest path of light:
- Standalone fountain.
- A programmable cart.

Heron's Formula

$$A = \sqrt{(s-a)(s-b)(s-c)}$$
$$S = \frac{1}{2}(a+b+c)$$



Last of the Ancient Greek Mathematicians

- Ptolemy (100-170)
  - Astronomy.
  - Geocentric model - until ...
  - Copernicus (1500). Heliocentric
- Diophantus (200's)
  - Equations with integer solutions.
  - Series of 13 books, *Arithmetica*, - algebraic equations.
- Hypatia (370-415)
  - Father - Theon.
  - Martyr.
  - Movie - *Agora*.

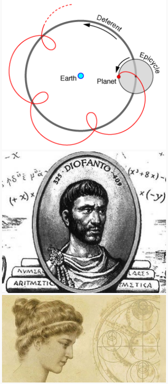
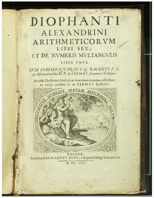


Figure 5: Epicycles, Diophantus, and Hypatia.

Romans - Little contribution to mathematics.

Diophantus' Epitaph



"Here lies Diophantus.

God gave him his boyhood one-sixth of his life;

One twelfth more as youth while whiskers grew rife;

And then yet one-seventh 'ere marriage begun.

In five years there came a bouncing new son;

Alas, the dear child of master and sage,

After attaining half the measure of his father's life, chill fate took him.

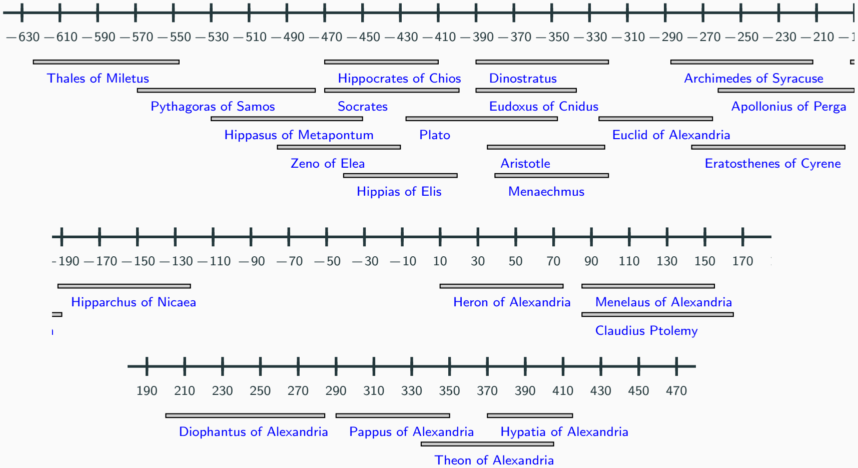
After consoling his fate by the science of numbers for four years, he ended his life."

- *Metrodorus, Greek Anthology.*

One Last Thing - The Antikythera Mechanism

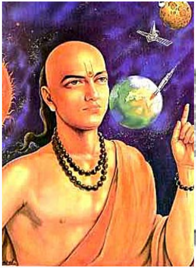
- Found in a wreck in 1900 near Antikythera.
- Recovered statues and other items.
- Small corroded fragments found.
- Fragment A has 27 gears.
- 100 years later ...
- It is an astronomical calculator.
- Predicts moon's position and phase, solar eclipses, motion of planets, and more.
- Possibly from 1st or 2nd century BCE.
- See March 2021 Paper, Freeth, et al.

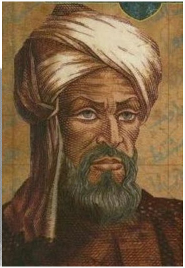
Timeline of Greek Mathematicians



Early Asian Mathematics

Fall 2023 - R. L. Herman

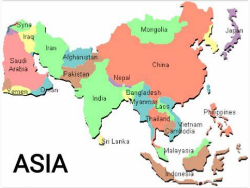




Overview

China

- Unique development
- *Zhoubi Suanjing* - c. 300 BCE
- *Tsinghua Bamboo Slips*, - decimal times table. 305 BCE
- Chinese abacus ($<$190 CE)
- After book burning (212 BCE), Han dynasty (202 BCE–220) produced mathematics works.



India

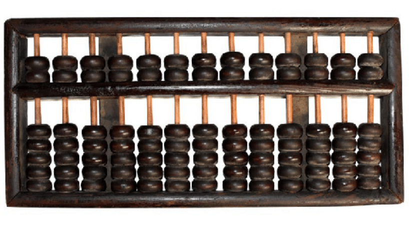
- *Pingala* (3rd–1st cent. BCE) - binary numeral system, binomial theorem, Fibonacci numbers.
- *Siddhantas*, 4th-5th cent. astronomical treatises, trigonometry.

Arabian-Islamic (330-1450)

- Preserved Greek texts
- 7th-14th cent. Development of algebra, etc.
- Hindu-Arabic numerals

Bamboo Slips and Suanpan

- *Tsinghua Bamboo Slips*, - Decimal Times Table. 305 BCE. By People from Warring States Period (476-221 BC), On right - Public Domain Image
- *Suanpan*, Chinese abacus (<190 CE), Below.



Chinese Dynasties

- Xia Dynasty (c. 2070-1600 BCE)
- Shang Dynasty (c. 1600-1046 BCE)
- Zhou Dynasty (c. 1046-256 BCE)
  Periods: Western Zhou, Spring and Autumn (770), Warring States (475)
- Qin Dynasty (221-206 BCE)
- Han Dynasty (206 BCE-220 AD)
- Six Dynasties Period (220-589)
- Sui Dynasty (581-618)
- Tang Dynasty (618-907)
- Five Dynasties and Ten Kingdoms (907-960)
- Song Dynasty (960-1279)
- Yuan Dynasty (1279-1368)
- Ming Dynasty (1368-1644)
- Qing Dynasty (1644-1912)

Figure 1: Liu Hui's circle.

Before Qin Dynasty (< 221 BCE)

- Known from myths and legends.
- Grasped numbers, figures.
- Quipu knots to record events, numbers.
- Used gnomon and compass.
- Artifacts - patterns on bone utensils.
- Plastrons (turtle shells) and oracles bones found in 1800's from Shang dynasty.
- Earliest known Chinese writing.
- Numbers up to 30,000. Special characters for numbers like 30,000, 20,000, 10,000, etc. [No zero.]
- Bone script from Zhou dynasty.

Calculations

- Ancient tools.
  - Babylonians - clay tablets.
  - Egyptians - hieroglyphs, papyri.
  - Indian, Arab - sand boards.
  - Chinese - counting rods.
- Short bamboo rods.
  - Han Dynasty: 1/10" $\times$ 6".
  - Sui Dynasty: 1/5" $\times$ 3".
  - No later than Warring States period.
  - Arranged - decimal place-value.
- Arithmetic operations. $+ - \times \div$
- Nine-nines rhyme - seen on bamboo strips. Spring and Autumn period (770-476).

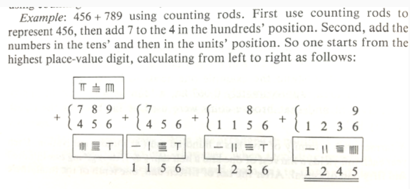
Chinese Numbers

rod	char.	digit	name
𝍩	一	1	i
𝍪	二	2	erh
𝍫	三	3	san
𝍬	四	4	ssu
𝍭	五	5	wu
𝍮	六	6	liu
𝍯	七	7	ch'i
𝍰	八	8	pa
𝍱	九	9	chiu
𝍠 ····	十	10	shih
𝍩 ···· ····	百	100	pai
	千	1000	ch'ien

Arithmetic Operations

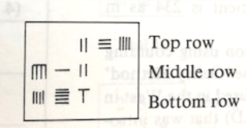
Clips from *Chinese Mathematics A Concise History* by Yan and Shiran.

Addition - $456 + 789$


Example: 456 + 789 using counting rods. First use counting rods to represent 456, then add 7 to the 4 in the hundreds' position. Second, add the numbers in the tens' and then in the units' position. So one starts from the highest place-value digit, calculating from left to right as follows:

Multiplication - 234×456

Start with 2×456



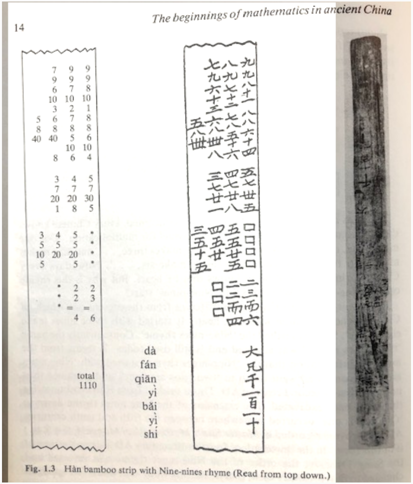
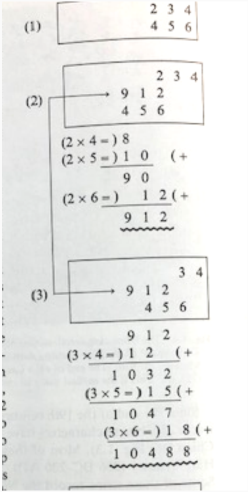
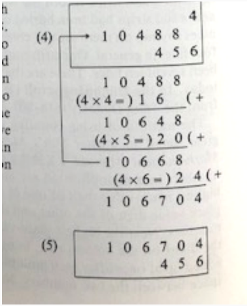


Figure 2: Nine-nines Rhyme.

Multiplication 234×456





The Answer: $234 \times 456 = 106704$.

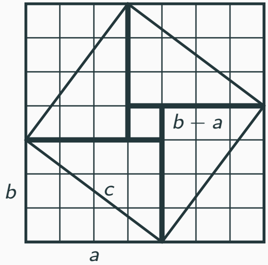
Early Mathematics

- Early works - construction techniques and statistics
- Fractions, measurements, angles, geometry, limit.
- Education - Six Gentlemanly Arts of the Zhou Dynasty (*Zhou Li - The Zhou Rites*).
    - Ritual,
    - Music,
    - Archery,
    - Horsemanship,
    - Calligraphy,
    - Mathematics.

Increased productivity in Han Dynasty (206 BCE-220 CE).

Zhoubi suanjing

- *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*,
- Written 100 BCE-100 CE.
- Uses **Gōugǔ Theorem**.
  [Around 1100 BC, Western Zhou period; Shanggao first described the Theorem.]
- Shadow gauges - vertical stakes for observing sun's shadow in Zhou.
- Fractions
  - 7 extra lunar months in 19 yr.
  - So, $12\frac{7}{19}$ mo/yr gives $29\frac{499}{940}$ da/mo.
  - Moon goes $13\frac{7}{19}^{o}$ per day. Then, 1 year $= 365\frac{1}{4}$ days.



$$c^2 = (b-a)^2 + 4(\tfrac{1}{2}ab) = a^2 + b^2$$

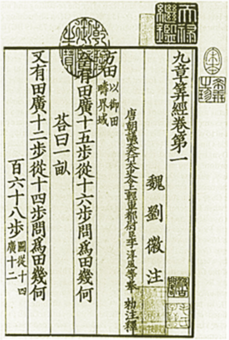
- Gǔ = vertical gauge.
- Gōu = shadow.
- Xián = Hypotenuse.

Jiuzhang Suanshu *The Nine Chapters on the Mathematical Art.*

- Complete by Eastern Han Dynasty (AD 25-220),
- Composed by generations of scholars starting 10th century BCE.
- Designated by Imperial Court as nation's standard math text - During Tang (618-907) and Song (960-1279) Dynasties.
- 246 word problems in 9 chapters on agriculture, business, geometry, engineering, surveying.
- Proof for the Pythagorean theorem.
- Formula for Gaussian elimination.
- Provides values of π. [They had approximated as 3.]
- Used negative numbers.

Chapters of the Mathematical Art

1. Rectangular Fields (Fangtian);
2. Millet and Rice (Sumi);
3. Proportional Distribution (Cuifen);
4. The Lesser Breadth (Shaoguang);
5. Consultations on Works (Shanggong);
6. Equitable Taxation (Junshu);
7. Excess and Deficit (Yingbuzu);
8. The Rectangular Array (Fangcheng); and
9. Base and Altitude (Gougu)


九章算經卷第一
魏 劉 徽 注
方田 以御田疇界域
今有田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
百六十八步

Chapters 1 and 9 summed up the accumulated knowledge of geometry and introduced a proposition - the Pythagorean theorem.

Influenced mathematical thought in China for centuries, introduced into Korea during the Sui Dynasty (581-618) and into Japan during the Tang Dynasty.

Notes and Commentaries

- Liu Hui (c. 225-295), mathematician and Li Chunfeng (602-670), astronomer and mathematician, known for commentaries on *The Nine Chapters on the Mathematical Art*.
- Chapters 2, 3 and 6: Proportion problems. 1st time.
- Chapter 7: The rule of False Double Position for linear problems. 1st time. [In Europe in the 13th century, Fibonacci.]
- World's earliest systematic explanation of fractional arithmetic.
- Chapter 8: Gaussian elimination.
  $> 1,500$ years before Carl Friedrich Gauss.
- Introduces negative numbers (1st),
  Rules for addition/ subtraction of positive and negative numbers.
  Brahmagupta (598-665) came up with the idea of negative numbers in India and Bombelli (1572) in Europe.

Summary of Chinese Mathematics i

- Shang - simple math, Oracle bones
- I Ching influenced Zhou Dynasty - use of hexagrams, binary (Leibniz).
- Decimal system since Shang Dynasty. 1st to use negative numbers.
- Suan shu shu, *A Book on Numbers and Computation*, 202-186 BCE.
  190 Bamboo strips, Found in 1984.
  Roots by False Position and Systems of equations.
- Zhoubi suanjing, *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*, 100 BCE-100 CE.
  Gougu Thm. ["Around 1100 BC, the Western Zhou period, the ancient Chinese mathematician, Shanggao, first described the Gougu Theorem. "]
- Jiuzhang Suanshu, *The Nine Chapters on the Mathematical Art.*
- Sun Zi (400-460) *Sunzi suanjing* (Sun Zi's Mathematical Manual) - Chinese Remainder Thm, Diophatine equations.

Summary of Chinese Mathematics ii

- Zhang Qiujian (430-490) - Manual, Sum arithmetic series, systems of 2 eqns and 3 unknowns.
- Before Han Dynasty - Addition. Subtraction, Multiplication, Division
- After Han Dynasty - Square roots and cube roots.
- Liu Hong (129-210) Calendar, Motion of moon.
- Computing π
    - Liu Xin (d. 23 AD), $\pi \approx 3.1457$.
    - Zhang Heng (78–139), $\pi \approx 3.1724, 3.162$ using $\sqrt{10}$.
    - Liu Hui (3rd century), commented on the Nine Chapters, $\pi = 3.14159$ from 96,192-gon, Exhaustion for circles, Gave method for V_{cyl} (Cavalieri's Principle).
    - Zu Chongzhi (5th century), Mathematical astronomy - Da Ming Li calendar. $\pi = 3.141592$ using 12,288-gon, Remained the most accurate value almost 1000 years. $\pi \approx \frac{355}{113}$ Gave method (Cavalieri's principle) for V_{sphere}.

Summary of Chinese Mathematics iii

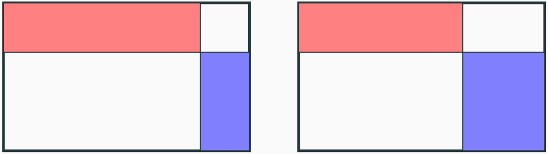
- Liu Zhuo (544-610) - quadratic interpolation.
- Yi Xing (683-727) Tangent table.
- Qin Jiushao (1202–1261) - Treatise of 81 problems
  Up to 10th degree eqns, Chinese Remainder Theorem, Euclidean Algorithm.
- Li Chih (1192-1297) 12 Chapters, 120 problems - Right triangles with circles inscribed/superscribed, geometric problems via algebra.
- Yan Hui (1238–1298), magic squares
- Zhu Shijie (1260 - 1320) - higher degree equations, binomial coefficients, uses zero digit.

One needs to view the methods from their point of view and not ours. In some cases, they provide a simpler "proof by picture."

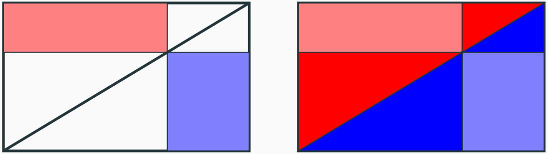
Next: The In-Out Complementary Principle.

In-Out Complementary Method

When do the red and blue rectangles have the same area?



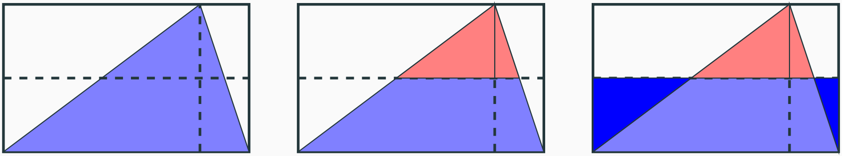
What does this suggest to you?



The case of "Equals subtracted from equals are equal."

Area of a Triangle

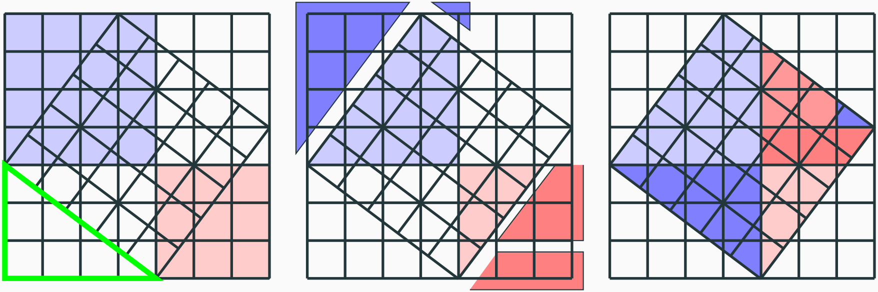
How would you prove that the area of a triangle is $A = \frac{1}{2}bh$?



Move the red "Out" triangles in the middle block to form the blue "In" triangles. The proof should be obvious at this point.

Gōugǔ Theorem

- Gǔ = vertical gauge.
- Gōu = shadow.
- Xián = hypotenuse.



According to J.W. Dauben, *Int. J. of Eng. Sci.* 36 (1998), Liu Hui explains, "The Gou-square is the red square, the Gu -square is the blue square. Putting pieces inside and outside according to their type will complement each other, then the rest (of the pieces) do not move. Composing the Xian-square, taking the square root will be Xian."

Commentary on Ch. 9 of *The Nine Chapters on the Mathematical Art*, by Liu Hui, in 263 AD.

- Surveying a sea island, set up two three zhang poles at one thousand steps apart, let the two poles and the island in a straight line. Step back from the front post 123 steps, with eye on ground level, the tip of the pole is on a straight line with the peak of island. Step back 127 steps from the rear pole, eye on ground level also aligns with the tip of pole and tip of island. What is the height of the island, and what is the distance to the pole ?
- The height of the island is four li and 55 steps, and it is 120 li and 50 steps from the pole.

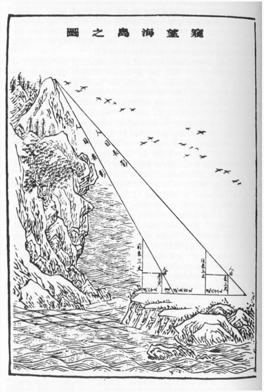


Figure 3: Liu Hui's Sea Island

Pascal's Triangle

Known by early Chinese mathematicians:

- Systems of linear equations
- Chinese Remainder Theorem
- Square roots
- Pythagorean Theorem
- Euclidean algorithm
- Pascal's Triangle
  - Typical term, $a^{n-k}b^k$, $k = 0, 1, \ldots, n$.
  - What is the coefficient?

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= 1a + 1b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4
\end{aligned}$$

Figure 4: Binomial Expansion, $(a+b)^n = \sum_{k=0}^{n} C_{n,k} a^{n-k} b^k$, $n = 0, 1, \ldots$.

Yáng Huī, (ca. 1238–1298) presented Jiǎ Xiàn's (ca. 1010–1070) triangle. Used for extracting roots.

Blaise Pascal (1623–1662)

Pascal's Triangle

$$1 = 1$$

$$1 + 1 = 2$$

Sum each row: $1 + 2 + 1 = 4$

Sum $= 2^n$. $1 + 3 + 3 + 1 = 8$

$$1 + 4 + 6 + 4 + 1 = 16$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

Figure 5: Pascal's Triangle, $C_{n,k} = \binom{n}{k} \equiv \frac{n!}{(n-k)!k!}$

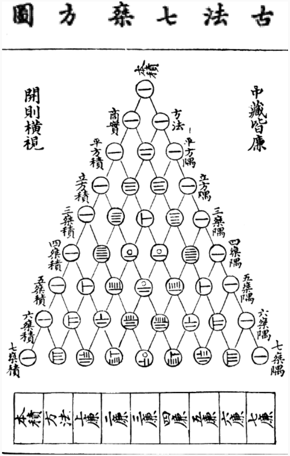


Figure 6: Jia Xian triangle published in 1303 by Zhu Shijie

Euclidean Algorithm Example

Example

One can only go 23 units or 79 units left or right. How many of each would it take to go from 0 to 1?

- $79m + 23n = 1.$
- Use Euclidean Algorithm

$$\begin{aligned} 79 &= 3\cdot 23 + 10 \\ 23 &= 2\cdot 10 + 3 \\ 10 &= 3\cdot 3 + 1 \\ 3 &= 3\cdot 1 \end{aligned}$$

In Reverse:

$$\begin{aligned} 1 &= 10 - 3\cdot 3 \\ &= 10 - 3(23 - 2\cdot 10) \\ &= 7\cdot 10 - 3\cdot 23 \\ &= 7\cdot(79 - 3\cdot 23) - 3\cdot 23 \\ &= 7\cdot 79 - 24\cdot 23 \end{aligned}$$

Thus, $m = 7, n = -24.$

Chinese Remainder Theorem

The Chinese remainder theorem: If one knows the remainders of the Euclidean division of an integer x by several integers, then one can determine uniquely the remainder of the division of x by the product of these integers, assuming the divisors are pairwise coprime. Earliest - Sun-tzu in *Sunzi Suanjing*.

If $p_1, p_2, \ldots, p_n$ are relatively prime, then

$$\begin{aligned} x &= r_1 \pmod{p_1} \\ x &= r_2 \pmod{p_2} \\ &\vdots \\ x &= r_n \pmod{p_n} \end{aligned}$$

always has a solution.

Chinese Remainder Theorem Example

Example

$$\begin{aligned} x &= 2 \pmod 3 \\ x &= 3 \pmod 5 \\ x &= 2 \pmod 7 \end{aligned}$$

First equation means $x = 3n + 2$. Insert into second:

$$\begin{aligned} 3n + 2 &= 3 \pmod 5 \\ 3n &= 1 \pmod 5 \\ 3n &= 6 \pmod 5 \\ n &= 2 \pmod 5 \end{aligned}$$

So,

$$\begin{aligned} x &= 3n + 2 \\ &= 3(5m + 2) \\ &= 15m + 8. \end{aligned}$$

From third equation

$$\begin{aligned} 15m + 8 &= 2 \pmod 7 \\ 15m + 1 &= 2 \pmod 7 \\ 15m &= 1 \pmod 7 \\ 15m &= 15 \pmod 7 \\ m &= 1 \pmod 7 \end{aligned}$$

Therefore, $m = 7k + 1$ and $x = 105k + 23$.

Indian Mathematics (500-1200)

- Major mathematicians
  - Aryabhata (476-550?)
  - Bhaskara I (600-680)
  - Brahmagupta (598-668)
  - Bhaskara II (1114-1185)
  - Madhava (1350-1425)
- Contributions
  - Algebra
  - Geometry
  - Trigonometry
  - Spherical trigonometry
  - Diophantine Equations
  - Mathematical astronomy
  - Place-value decimal system

Brahmagupta:
$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
$s = \frac{1}{2}(a+b+c+d)$ is semiperimeter

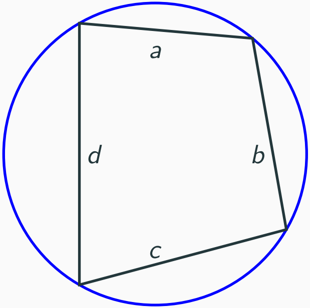


Figure 7: Cyclic Quadrilaterals

Aryabhata (476-550)

- Major work, *Aryabhatiya*, mathematics and astronomy,

  arithmetic, algebra,
  plane trigonometry,
  spherical trigonometry.
  continued fractions,
  quadratic equations,
  sums of power series, and
  table of sines.

- 108 verses, 13 introductory verses
- Relativity of motion
- *Arya-siddhanta*,

  Astronomical computations
  Astronomical instruments

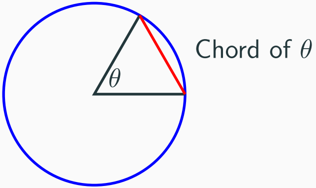
Figure 8: Aryabhata on the grounds of IUCAA, Pune.

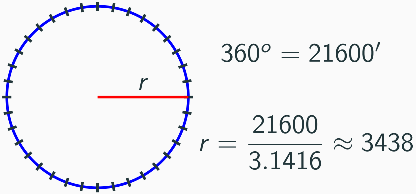
Table of Sines

- Introduction of sine
- Aryabhata's sine table
- Based on half chords
  vs Hipparchus, Menlaus, Ptolemy.
- Also, provided differences
- From Babylonians, base 60
  degrees, minutes, seconds
- Circumference $= 21600'$.
- Aryabhata, $\pi = 3.1416$
- Bhaskara I approximation

$$\sin\theta \approx \frac{4\theta(180-\theta)}{40500-\theta(180-\theta)}.$$

- Mādhava's - more accurate.





Pell's Equation, $x^2 - Ny^2 = 1$, N Nonsquare

Brahmagupta (628): *samasa*, Method of Composition

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2$$

If $x_1^2 - Ny_1^2 = k_1$ and $x_2^2 - Ny_2^2 = k_2$, then

$$\begin{aligned} x &= x_1x_2 + Ny_1y_2 \\ y &= x_1y_2 + x_2y_1 \end{aligned}$$

solves $x^2 - Ny^2 = k_1k_2$.

This gives a composition of triples, (x_1, y_1, k_1) and (x_2, y_2, k_2) to give (x, y, k_1k_2).

Example (Brahmagupta) $x^2 - 92y^2 = 1.$

Note: $10^2 - 92(1)^2 = 8$. Thus, triple $= (10, 1, 8)$.

Pell's Equation (cont'd)

- $10^2 - 92(1)^2 = 8 \to (10, 1, 8)$.
- Compose $(10, 1, 8)$ with itself. $(10 \cdot 10 + 92 \cdot 1 \cdot 1, 10 \cdot 1 + 1 \cdot 10, 8 \cdot 8) = (192, 20, 64)$
- or, $192^2 - 92(20)^2 = 64$
  $24^2 - 92\left(\frac{5}{2}\right) = 1$
- Compose $(24, \frac{5}{2}, 1)$ with itself: $(1151, 120, 1)$.
- Bhaskara II (1150) - cyclic process always works - *chakravala*.
- Proved by Lagrange (1768) $gcd(a, b) = 1,\ a^2 - Nb^2 = k$.
- Compose (a, b, k) with $(m, 1, m^2 - N)$ gives $(am + Nb, a + bm, k(m^2 - N))$.
- Rescale

$$\left(\frac{am + Nb}{k}, \frac{a + bm}{k}, m^2 - N\right)$$

- Fermat (1657), $x^2 - 61y^2 = 1$, $x = 1766319049$, $y = 226153980$.

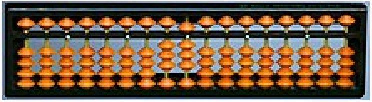
Japanese Mathematics (Wasan)

- Developed in Edo Period (1603-1867) rule of the Tokugawa shogunate.
- Economic growth, strict social order, isolationist foreign policies, a stable population, perpetual peace, and popular enjoyment of arts and culture.
- Foreign trade restrictions
- Mathematics for taxes
- Used Chinese counting rods.
- Imported Chinese texts.
- Sangi rods, adopted from Chinese rods, called saunzi.

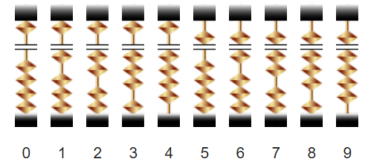
Second period (552-1600) - Influx of Chinese learning - first through Korea.

Figure 9: Seki Takakazu.

The Soroban



- Abacus developed in Japan.
- Derived from ancient Chinese **suanpan**,
- Imported to Japan, 14th century.
- go-dama = 5, ichi-dama = 1.
- Addition, substraction, multiplication, division.



Yoshida Mitsuyoshi (1598–1672), *Jinkōki,* (1627) oldest existing Japanese math text, subject of **soroban arithmetic**, incl. square and cube roots.

Student of Kambei Mori, **first Japanese mathematician** with students Imamura Chishō, and Takahara Kisshu ("Three Arithimeticians").

Seki Takakazu (1642–1708), infinitesimal calculus and Diophantine equations, **"Japanese Newton."**

The Rise of Islam

- Mohammad was born in Mecca (570)
- Began preaching in Mecca, Escaped to Medina (622), later returned (629).
- Under Umar, Son of Al-Khattab, the empire spread. Ruled the Sasanian Empire and more than two-thirds of the Byzantine Empire
- Eventually spread from India to Spain. Main centers at Baghdad and Cordoba, Spain.
- Caliph Al-Ma'mun created Bait Al-hikma (House of Wisdom), Around 800 CE.
- Arabic became the common language.
- Translation of Greek and Hindu to Arabic (Euclid et al.) began with Al-Mansor, founder of Baghdad, grandfather of Al-Ma'mun.

Islamic Mathematics

- Caliph Al-Ma'mun appointed Al-Khwarizmi (780-850) court astronomer.
- His book, *Hisab al-jabr w'al-muqabala*, Solve linear or quadratic, 6 forms. Terms: Algebra and Algorithm.
- Al-Kindi (801-873) Arithmetic, 11 texts.
- Al-Bhattini (850-929)
  - Trigonometry, 1st Cotangent table.
- Thabit ibn Qurra (836-901) Theory of Numbers.
- Al-Kuhi (c940-1000) Archimedean and Apollonian math. Equations of degree > 2.
- Arabic numerals 1st in a book 874 and zero 2 yrs before Hindus.

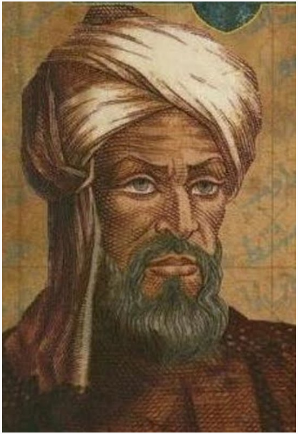


Figure 10: Al-Khwarizmi

International Year of Light 2015

Ibn al-Haytham's scientific method

Hasan Ibn al-Haytham (Latinized Alhazen)

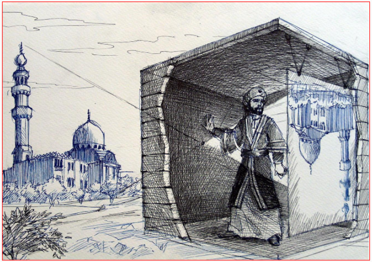
During the International Year of Light 2015, Ibn al-Haytham was celebrated at UNESCO as a pioneer of modern optics. He was a forerunner to Galileo as a physicist, almost five centuries earlier, according to Prof. S.M. Razaullah Ansari (India). Also known as Alhazen, this brilliant Arab scholar from the 10th – 11th century, made significant contributions to the principles of optics, astronomy and mathematics, and developed his own methodology: experimentation as another mode of proving the basic hypothesis or premise.

by Shaikh Mohammad Razaullah Ansari

Abū Ali al-Ḥasan Ibn al-Haytham al-Baṣrī (965-1040), known in European Middle Ages by the name of Alhazen, was called among Arab scholars as 'Second Ptolemy' (Baṭlamyūs Thānī). He was actually a scholar of many disciplines: Mathematics, physics, mechanics, astronomy, philosophy and medicine. He was one of the senior most member of the Muslim scholars' trio during 10th -11th centuries, the other two were al-Bīrūnī (973-1048) and Ibn Sīnā (980–1037).

From Basra, Ibn al- Haytham shifted to Cairo, where the Fatimid Caliph al-Ḥākim had invited him. The Caliph was a great patron of scientist-scholars, he got built an observatory for the astronomer Ibn Yūnus (d.1009) and he founded a library Dār al–'Ilm, whose fame almost equaled that of its precursor at Baghdad, Bayt al- Ḥikma(the House of Wisdom), established by the Abbasid Caliph al-Mā'mūn (reigned 813 – 833).

Ibn al-Haytham was a prolific writer. According to his own testimony, he wrote 25 works on mathematical sciences, 44 works on (Aristotelian) physics and metaphysics, also on meteorology and psychology. Moreover, his autobiographical sketch indicates clearly that he studied very thoroughly Aristotle's (natural) philosophy, logic and metaphysics of which he gave a concise account.



Ibn al-Haytham's pinhole camera and Al-Farisi's Rainbow

Later Islamic Mathematicians

- Ibn Al-Haytham (965-1039) Studied optics and visual perception.
  Used Euclid and Apollonius for Reflection, refraction, spherical mirrors, rainbows, eclipses, shadows.
  Optics works in Europe 12-13th century.
  Also, tackled 5th postulate.
- Golden Age: 9-10th centuries.
- Al-Karkhi (d. 1019-1029) Arithmetic, $\sum_n n^k$, $k = 1, 2, 3$.
- Omar Khayyam (1048-1131) studied cubic equations, and the intersection of conics, Khayyam's triangle, parallel postulate.
- Al-Kashi (1380-1429), fractions, π to 16 places, Law of Cosines.

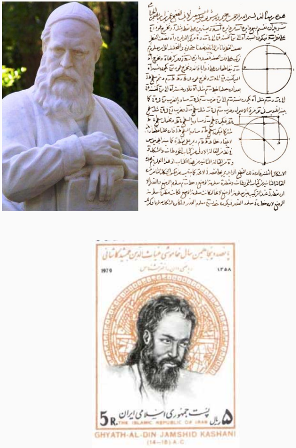


Figure 11: Omar Khayyam and Jamshīd al-Kashi stamp.

Rise of European Mathematics

- Fall of the Roman Empire
- Middle Ages, Medieval Period, 5th to the 15th century.
- Byzantine Empire (330-1453) - Church split,
- Preservation of Greek works.
- Al'Khwarizmi's work and Euclid translated into Latin.
- Crusades (1095-1291), The Plague (1347 to 1351).
- Mongols destroyed Islamic empire 1258.
- Johannes Gutenberg' printing press, 1440.
- Renaissance (1400-1600) and the Age of Discovery.
- Questioning of Aristotle
- Church of England (1534), Protestant vs Roman Catholic

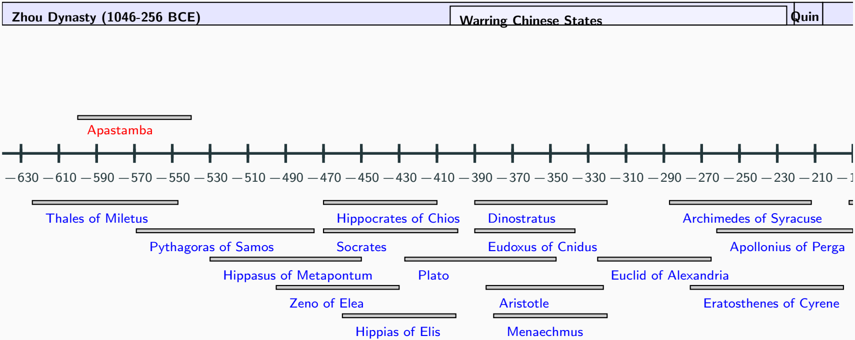
Medieval Mathematicians

- Leonardo of Pisa (Fibonacci) (1200).
- Nicole Oresme (1323-1382), coordinate geometry, fractional exponents, infinite series.
- Johann Müller Regiomontatus (1436-1476), separated trigonometry from astronomy.
- And others:

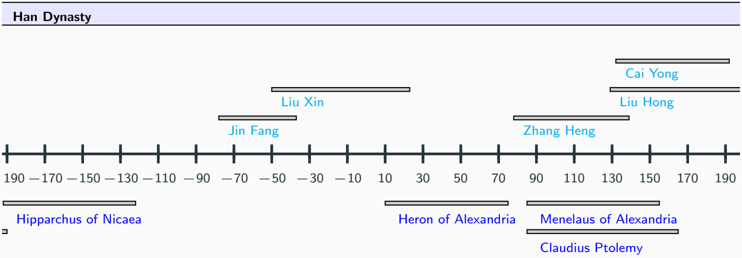
Roger Bacon (1214-1292)	William of Ockham (1288-1348)
Filippo Brunelleschi (1377-1446)	Leone Battista Alberti (1404-1472)
Nicholas of Cusa (1401-1464)	Piero della Francesca (1420 - 1492)
Leonardo da Vinci (1452-1519)	Luca Pacioli (1445-1517)
Nicolaus Copernicus (1473-1543)	Scipione del Ferro (1465-1526)

- Rise of European Mathematics .. beginning in Italy.

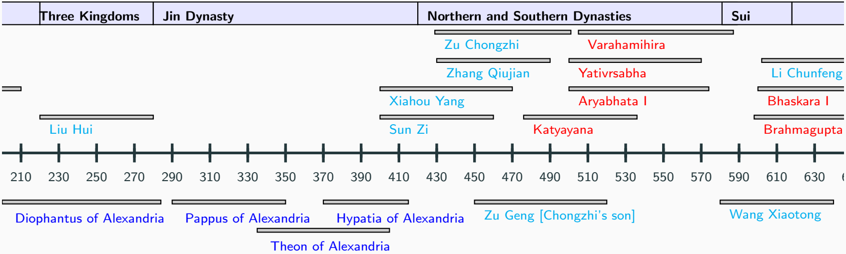
Timeline of Ancient Mathematicians i



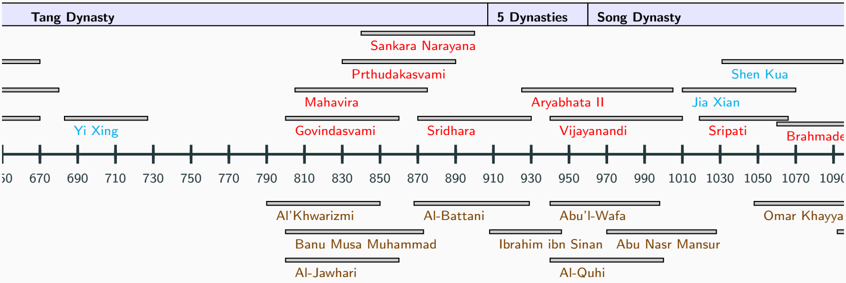
Timeline of Ancient Mathematicians ii



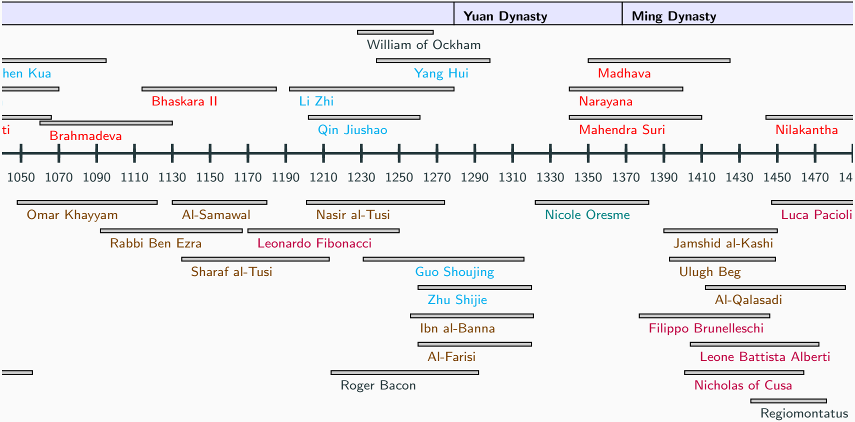
Timeline of Ancient Mathematicians iii



Timeline of Ancient Mathematicians iv



Timeline of Ancient Mathematicians v



Chinese Mathematical Classics

The Suàn shù shū (A Book on Numbers and Computations) is written on nearly 200 bamboo slips. The earliest known mathematical work in Chin, recently discovered in a tomb, dated to the early second century BCE.

The Jiu zhang suan shu (Nine Chapters on the Art of Mathematics) known from an imperfect copy printed in the Southern Song dynasty (1213 CE)

The Shi bu suan jing (Ten Books of Mathematical Classics)

- Zhoubi suanjing (Zhou Shadow Gauge Manual)
- Jiuzhang suanshu (Nine Chapters on the Mathematical Art)
- Haidao suanjing (Sea Island Mathematical Manual)
- Sunzi suanjing (Sun Zi's Mathematical Manual)
- Wucao suanjing (Mathematical Manual of the Five Administrative Departments)
- Xiahou Yang suanjing (Xiahou Yang's Mathematical Manual)
- Zhang Qiujian suanjing (Zhang Qiujian's Mathematical Manual)
- Wujing suanshu (Arithmetic methods in the Five Classics)
- Jigu suanjing (Continuation of Ancient Mathematics)
- Shushu jiyi (Notes on Traditions of Arithmetic Methods)
- Zhui shu (Method of Interpolation)
- Sandeng shu (Art of the Three Degrees; Notation of Large Numbers)

Cubic Equations

Fall 2023 - R. L. Herman

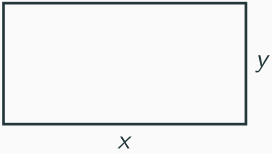

Solutions of Polynomial Equations

- Linear equations, known solutions.
- Chinese - Gaussian elimination: Systems of n linear equations and n unknowns.
- Quadratic equations: Need square roots.
- Cubic equations: Need square roots and cube roots. Solved in 16th century.
- Quintic equation: studied in 1820's. Eventually lead to group theory!

Figure 1: Leonardo da Vinci attempts Delian problem (Doubling cube).

Quadratic Equations

Babylonian Method (Modern Notation)



Find x and y for a given perimeter and area.

$$\begin{aligned} x+y &= p \\ xy &= q. \end{aligned}$$

Eliminate y, $x^2+q=px$. Then,

$$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

Method - Compute the following

1. $\frac{x+y}{2}$
2. $\left(\frac{x+y}{2}\right)^2$
3. $\left(\frac{x+y}{2}\right)^2 - xy = \frac{(x+y)^2-4xy}{4}$
4. $\sqrt{\frac{(x+y)^2-4xy}{4}} = \frac{x-y}{2}$
5. By inspection, get x, y from p and q, since

$$\begin{aligned} \frac{x-y}{2} &= \sqrt{\frac{p^2-4q}{4}} \\ &= \sqrt{\left(\frac{p}{2}\right)^2 - q}. \end{aligned}$$

Quadratic Equations (cont'd)

- Brahmagupta (628) - Explicit
  $ax^2 + bx = c$.

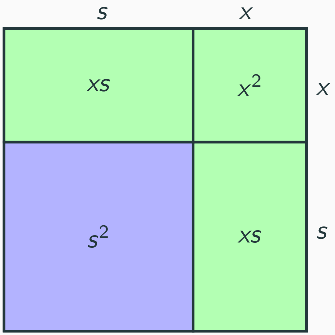
$$x = \frac{\sqrt{4ac + b^2} - b}{2a},$$

- Euclid - Prop. 28
- al'Khwarizimi

$$\begin{aligned} x^2 + 2xs &= n \\ x^2 + 2xs + s^2 &= n + s^2 \\ (x+s)^2 &= n + s^2 \end{aligned}$$

- Quadratic Irrationals

$$\frac{a + \sqrt{b}}{\sqrt{\sqrt{a} + \sqrt{b}}}$$



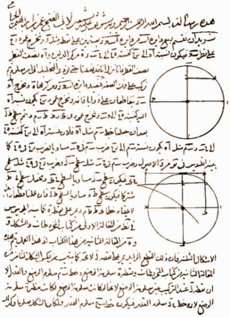
Note from the figure:
Green area $= x^2 + 2xs = n$.
No negative lengths (solutions).

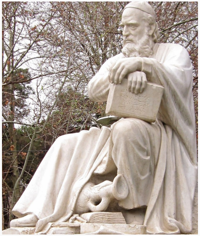
Cubic Equations

- Babylonians - Table of cubes
- Greeks - Geometric Problems
  - Duplicating cube (Delian Prob.).
  - Intersecting conics.
  - Cutting Sphere with plane.
- Omar Khayyam (1048-1131)
  - First general theory of cubics.
  - Provided 19 types of cubic.
  - Example

    $$x^3 + ax^2 = bx + c,$$

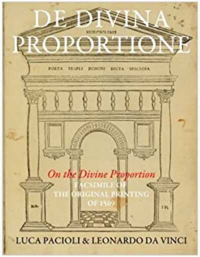
    Cube and square equals side plus number.
  - Geometric: Intersect two hyperbolae.





Pacioli, da Vinci and della Francesca

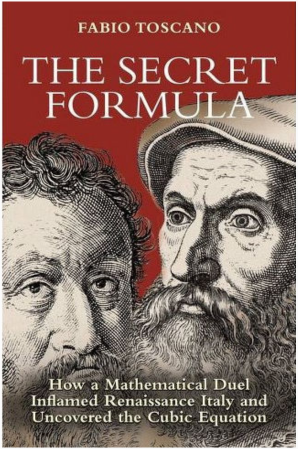
- Luca Pacioli (1445-1517)
  - Franciscan friar, tutor.
  - *Summa de arithmetica*, 1494. Father of Accounting.
  - *Divina proportione*, 1509.
  - "Solution to cubic is impossible!"
  - On table: slate, chalk, compass, dodecahedron. Hanging: Rhombicuboctahedron half-filled with water.
- Leonardo da Vinci (1452-1519)
- Piero della Francesca's (1420-1492) Painter/mathematician, met Alberti, 1451. Wrote books: algebra, perspective, Archimedean polyhedra. Pacioli used his work.



The Secret Formula

How a Mathematical Duel Inflamed Renaissance Italy and Uncovered the Cubic Equation, by Fabio Toscano, 2020.

- The Abbaco Master
  Italian Wars, Brescia, 1512, Tartaglia.
- The Rule of the Thing
  cosa (thing), *censo* (x^2), *numero*.
- The Venetian Challenge
  1535, Fior challenges Tartaglia.
- An Invitation to Milan
  Entrance of Gerolamo Cardano.
- The Old Professor's Notebook
  da Coi vs. Ferrari, del Ferro's priority:
  Solution: things and cube equal to number.
- The Final Dual
  Ars Magna, 1545. Ferrari vs. Tartaglia, 1547.



The Search for Solutions

- Scipio del Ferro (1465-1526)
  - University of Bologna, notebooks
  - Printing press - Guttenberg
  - 1506/1514, solution of **depressed cubic**: $x^3 + ax = b$.
  - Public Challenges led to secrecy.
- Gave to Antonio Maria Fior (Florido).
- Tartaglia (Nicolo Fontana) (1499-1557)
  - 1512, French attack - sabre wound led to stammer.
  - Self-educated
  - 1530 da Coi wrote to him $x^3+3x^2 = 5$, $x^3+6x^2+8x = 1000$.

Figure 2: Tartaglia

Sidenotes

Pacioli, *Summa de arithmetica* 1494, Not solvable:

$$\begin{aligned} n &= ax + bx^3 \\ n &= ax^2 + bx^3 \\ n &= ax^3 + bx^4 \end{aligned}$$

Tartaglia, Abaco teacher/master but engaged in other activities.

Challenges not uncommon, but expect challenger to know solutions to their questions.

Not happy with da Coi questions.

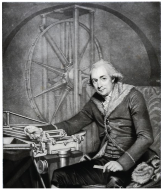
The Plot Thickens

- Tartaglia boasted he could solve $x^3 + ax^2 = c$.
- Florido challenged Tartaglia
  - Each posed 30 problems
  - Florido mostly gave problems of form $x^3 + ax^2 = c$.
  - Tartaglia won by solving depressed cubic 1535, but didn't publish.
- Girolamo Cardano (1501-1576)
  - Gambler, astronomer, physician, astrologer, heretic, father of murderer.
  - Begged for solution from Tartaglia. Finally, they met in Milan.
  - Tartaglia eventually gave solution in 1539 as a Poem if it was kept secret.
  - It was not in Cardano's book.

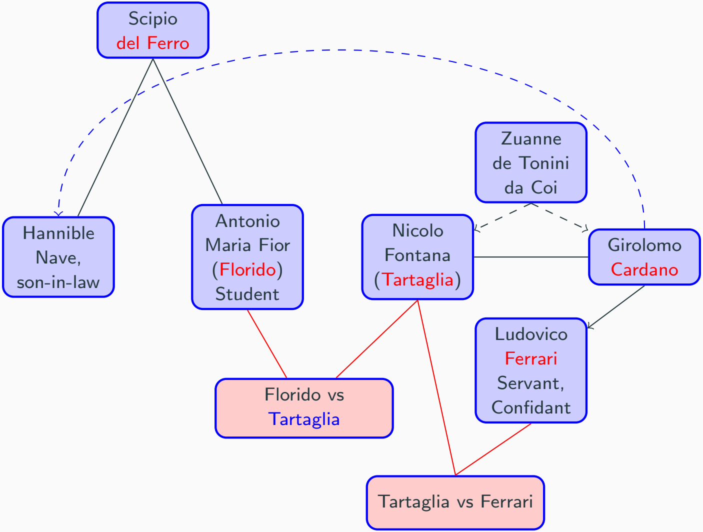
Figure 3: Cardano, *Ars Magna*.

Enter Ludovico Ferrari

- Ludovico Ferrari (1522-1565)
    - Servant at 14
    - Secretary, confidant
    - Worked on problems with Cardano
    - Cubic and biquadratic equations
- da Coi $\rightarrow$ Cardano $\rightarrow$ Ferrari
    - 4th degree polynomial
    - Ferrari solution involved solving cubic
    - Publishing was a problem.
- 1543 Trip to Florence, stopped in Bologna on the way.
    - Visitied Hannible Nave, del Ferro's son-in-law.
    - Saw del Ferro's notes.
    - Cardano believed he could publish in his *Ars Magna*, 1545.
- Barrage of letters from Tartaglia!



The Players in the Cubic Story



Tartaglia vs Ferrari - 1548

- Public debate in Milan, Ferrari's hometown.
- Cardano was absent.
- Tartaglia lost, blamed crowd.
- Tartaglia worked on arithmetic.
- Ferrari became professor in Bologna, 1565.
  Was poisoned 1565, white arsenic, possibly by sister.
- Cardano predicted exact date of his own death in 1576.

Figure 4: Tartaglia and Ferrari

Solution of the Quadratic $x^2 + ax + b = 0$

- Completing the square:
  $\left(x + \frac{a}{2}\right)^2 + b - \frac{a^2}{4} = 0.$
- Solution: $x + \frac{a}{2} = \pm\sqrt{\frac{a^2}{4} - b}.$
- Graph of parabola
  $y = x^2 + ax + b$
  Vertex $(-\frac{a}{2}, b - \frac{a^2}{4})$
- Number of real solutions?
- Substitute $x = u - \frac{a}{2}$:
  $0 = x^2 + ax + b$
  $0 = (u - \frac{a}{2})^2 + a(u - \frac{a}{2}) + b$
  $0 = u^2 + b - \frac{a^2}{4}.$
- Solve for u.

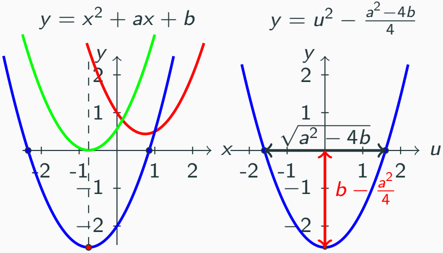


Figure 5: Plots of parabolae. Translating blue parabola on left by $\frac{a}{2}$ results in that on the right.

Plotting a Cubic Function $y = x^3 + ax^2 + bx + c$

- Set $y = 0$, then $x^3 + ax^2 + bx + c = 0$.
- Solutions are black points. Always have a real solution.
- $y' = 3x^2 + 2ax + b$ and $y'' = 6x + a$.
- Inflection point: $y'' = 0$ for $x_0 = -\frac{a}{3}$, $y_0 = c - \frac{ab}{3} + \frac{2a^3}{27}$.
- Slope of tangent at $(x_0, y_0 = q)$ is $p = b - \frac{a^2}{3}$.
- Extrema at $x_\pm = -\frac{a}{3} \pm \frac{\sqrt{a^2-3b}}{3}$.

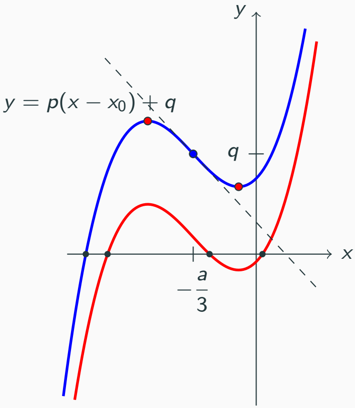


Figure 6: Plots of the cubic function exhibiting either one or three real roots.

Can you complete the cube: $(x + \alpha)^3 = x^3 + 3x^2\alpha + 3x\alpha^2 + \alpha^3$?

Solution of the Cubic $x^3 + ax^2 + bx + c = 0.$

Let $x = y - \frac{a}{3}.$

Then, $y^3 + py + q = 0,$ where

$$p = b - \frac{a^2}{3},$$
$$q = c - \frac{ab}{3} + \frac{2a^3}{27}.$$

Let $y = u + v:$

$$u^3 + v^3 + (p + 3uv)(u + v) + q = 0.$$

Let $p + 3uv = 0$, then

$$u^3 v^3 = -\frac{p^3}{27},$$
$$u^3 + v^3 = -q.$$

Now, define $X = u^3$. $Y = v^3$.

We obtain

$$X + Y = -q.$$
$$XY = -\frac{p^3}{27},$$

Does this look familiar?

The solution of Cubic:

$$X, Y = u^3, v^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \frac{p^3}{27}},$$

$y = u + v,$

$x = y - \frac{a}{3}.$

Example: $2x^3 - 30x^2 + 162x - 350 = 0.$

Let $x = y - \frac{b}{3a} = y + \frac{30}{6} = y + 5.$

We obtain a depressed cubic (del Ferro), $y^3 + 6y - 20 = 0.$

Letting $y = u + v$, $X, Y = u^3, v^3$, we solve

$$X + Y = -q = 20, \quad XY = -\frac{p^3}{27} = -\frac{6^3}{27} = -8.$$

Eliminating Y, $X^2 - 20X - 8 = 0.$

Solving, leads to $X = 10 \pm \sqrt{108}, \qquad Y = 20 - X = -10 \mp \sqrt{108}.$

So, $u = \sqrt[3]{10 \pm 6\sqrt{3}} \qquad v = \sqrt[3]{-10 \pm 6\sqrt{3}}$ and

$$y = u + v = \sqrt[3]{10 \pm 6\sqrt{3}} + \sqrt[3]{-10 \pm 6\sqrt{3}}$$

$$x = \sqrt[3]{10 \pm 6\sqrt{3}} - \sqrt[3]{-10 \pm 6\sqrt{3}} + 5$$

Depressed Cubic $y^3 + 6y - 20 = 0$.

Note: $y = 2$ is a solution. Therefore, $y^3 + 6y - 20 = (y-2)(y^2 + \alpha y + \beta)$.

Method 1: Expand and match coefficients.

$y^3 + 6y - 20 = y^3 + (\alpha - 2)y^2 + (\beta - 2\alpha)y - 2\beta$

$\alpha - 2 = 0, \qquad \beta - 2\alpha = 6, \qquad 2\beta = 20 \quad \Rightarrow \quad \alpha = 2, \beta = 10.$

Method 2: Long division.

$$
\begin{array}{r}
y^2 \quad + 2y + 10 \\
y - 2 \overline{\big) \; y^3 \qquad\qquad + 6y - 20} \\
\underline{- y^3 + 2y^2} \qquad\qquad\quad \\
2y^2 \quad + 6y \qquad\;\, \\
\underline{- 2y^2 \quad + 4y} \qquad\;\, \\
10y - 20 \\
\underline{- 10y + 20} \\
0
\end{array}
$$

Find other roots:

$$y^2 + 2y + 10 = 0$$

$$
\begin{aligned}
y &= \frac{-2 \pm \sqrt{4 - 40}}{2} \\
&= -1 \pm 3i.
\end{aligned}
$$

Nested Radicals

We have found that $y = \sqrt[3]{10 \pm 6\sqrt{3}} - \sqrt[3]{-10 \pm 6\sqrt{3}}$. Can one simplify this? We consider $\sqrt[3]{10 \pm 6\sqrt{3}} = \sqrt{x} \pm \sqrt{y}$ and $\sqrt[3]{-10 \pm 6\sqrt{3}} = -\sqrt{x} \pm \sqrt{y}$. Note:

$$\begin{aligned}
(\sqrt{x} \pm \sqrt{y})^3 &= x\sqrt{x} \pm 3x\sqrt{y} + 3y\sqrt{x} \pm y\sqrt{y} \\
&= (x+3y)\sqrt{x} \pm (3x+y)\sqrt{y} \\
&= 10 \pm 6\sqrt{3}. \qquad (1) \\
(-\sqrt{x} \pm \sqrt{y})^3 &= -x\sqrt{x} \pm 3x\sqrt{y} - 3y\sqrt{x} \pm y\sqrt{y} \\
&= -(x+3y)\sqrt{x} \pm (3x+y)\sqrt{y} \\
&= -10 \pm 6\sqrt{3}. \qquad (2)
\end{aligned}$$

From (1) $y = 3$, $x + 9 = 10$, $\quad 3x + 3 = 6$. Then, $x = 1$ and $\sqrt[3]{10 \pm 6\sqrt{3}} = 1 \pm \sqrt{3}$.

Similarly, from (2), $\sqrt[3]{-10 \pm 6\sqrt{3}} = -1 \pm \sqrt{3}$. So,

$$y = \sqrt[3]{10 \pm 6\sqrt{3}} + \sqrt[3]{-10 \pm 6\sqrt{3}} = 2.$$

Complex Solutions $y^2 + 2y + 10 = 0.$

We found $y^3 + 6y - 20 = (y-2)(y^2 + 2y + 10) = 0.$

we solved the quadratic: $y = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3\sqrt{-1}.$

- Cardano, complex numbers
  "as subtle as they are useless."
- Raphael Bombelli (1526-1572)
  First to take seriously.
- Ex: $x^3 = 15x + 4$
  $x = \sqrt[3]{2 + 11\sqrt{-11}} + \sqrt[3]{2 - 11\sqrt{-11}}$
- But, $x = 4$ is a solution!
- Complex numbers, $a + bi$, $i = \sqrt{-1}.$
- $x = \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i}$

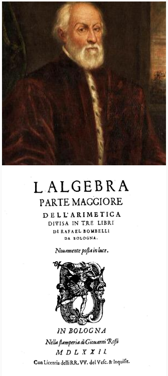


Figure 7: Bombelli

Cube Root of Complex Numbers

- Last Example: $x^3 = 15x + 4$
  $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$
- Seek: $\sqrt[3]{2+11i} = c + di$.

$$\begin{aligned}\sqrt[3]{2+11i} &= c+di\\ 2+11i &= (c+di)^3\\ &= c^3+3c^2di+3c(di)^2+(di)^3\\ &= c^3-3cd^2+i(3c^2d-d^3).\end{aligned}$$

Then

$$\begin{aligned}2 &= c^3-3cd^2 = c(c^2-3d^2),\\ 11 &= 3c^2d-d^3 = d(3c^2-d^2).\end{aligned}$$

- Bombelli: c, d, positive integers.

Since 2 is prime, $c = 1, 2$.

If $c = 1$, $2 = 1 - 3d^2$. No!

If $c = 2$, then $d = 1$.

$$\begin{aligned}2 &= 8-6d^2\\ 11 &= 12d-d^3\end{aligned}$$

Then,

$$\begin{aligned}x &= \sqrt[3]{2+11i} + \sqrt[3]{2-11i}\\ &= (2+i)+(2-i) = 4.\end{aligned}$$

Françios Viète (1540-1603)

- Counselor to Henry III, IV, France
- French Wars of Religion 1562-1598
- Tutored Catherine de Pathenay (1554-1631), noblewoman, mathematician
- 1596 - Adriaan van Roomen
  "No French mathematician could solve the 45th degree polynomial."
  $x^{45} - 45x^{43} + 945x^{41} + \cdots - 3795x^3 + 45x = A.$
- Viète solved quickly:
  $2\sin(45\alpha) = A, \quad x = 2\sin\alpha.$
- Trig identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
  Let $y = \cos\theta$. $4y^3 - 3y = c, |c| \le 1$. $c = \cos 3\theta$.
  Solve for θ given c. Solution, $y = \cos\theta$.
- Use identities to rewrite $2\sin(45\alpha) = A$ in terms of $2\sin\alpha$.

Figure 8: Viète, Henry IV, and van Roomen.

Viète's Solution

Define the quantities

$$c = 2\sin 45\theta, \qquad y = 2\sin 15\theta,$$
$$z = 2\sin 5\theta, \qquad x = 2\sin\theta. \tag{3}$$

Problem: Find x, given c.

Use the identities:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \tag{4}$$
$$\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha \tag{5}$$

Then,

$$c = 2\sin 45\theta = 6\sin 15\theta - 8\sin^3 15\theta = 3y - y^3. \tag{6}$$
$$y = 2\sin 15\theta = 6\sin 5\theta - 8\sin^3 5\theta = 3z - z^3. \tag{7}$$
$$z = 2\sin 5\theta = 10\sin\theta - 40\sin^3\theta + 32\sin^5\theta = 5x - 5x^3 + x^5. \tag{8}$$

Viète's Solution (cont'd)

Since $z = 5x - 5x^3 + x^5$., we write c in terms of x :

$$\begin{aligned}
y &= 3z - z^3 \\
&= 3[5x - 5x^3 + x^5] - [5x - 5x^3 + x^5]^3 \\
&= -x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x. \qquad (9) \\
c &= 3y - y^3 \\
&= 3[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x] \\
&\quad -[-x^{15} + 15x^{13} - 90x^{11} + 275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x]^3 \\
&= x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + 111150x^{37} - 740259x^{35} + 3764565x^{33} \\
&\quad -14945040x^{31} + 46955700x^{29} - 117679100x^{27} + 236030652x^{25} - 378658800x^{23} \\
&\quad +483841800x^{21} - 488494125x^{19} + 384942375x^{17} - 232676280x^{15} + 105306075x^{13} \\
&\quad -34512075x^{11} + 7811375x^9 - 1138500x^7 + 95634x^5 - 3795x^3 + 45x \\
&= P_{45}(x). \qquad (10)
\end{aligned}$$

Math Symbols

- $+$ Oresme (1300)
- $-$ Widman (1400)
- $=$ Recorde (1500)
- $\times$ Outred (1500)
- Letters Viète (1500)
- Descartes (1500-1600)
  unknowns, constants a, b, c
  variables x, y, z
- $<>$ Harriot (1600)
- ∞ Wallis (1700)
- imaginary, Descartes
- $x^{3/2}, x^{-1}$, Newton 1600
- $x^2 \to xx$, Gauss 1800
- π, i, Σ. Euler 1700
- $f(x)$
- $\frac{df}{dx}, \int$ Leibniz 1600

Analytic Geometry

- Fermat (1601-1665)
- Descartes (1596-1650)
- Newton (1642-1727)

Coordinates

- Hipparchus - sky
- Apollonius - conics
- Oresme (1300s) - position, velocity plots
- Fermat-Descartes described curves in coordinate systems
- Degree 1, Linear relations

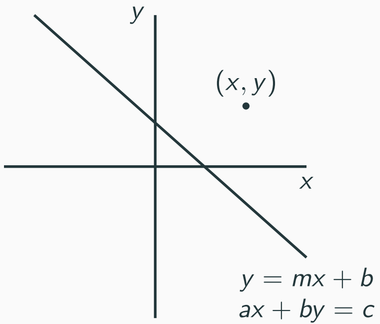


Figure 9: Cartesian system.

Curves of Degree 2 - Quadratics

$$ax^2 + \underbrace{bxy}_{rotation} + cy^2 + \underbrace{dx + ey}_{translation} + f = 0$$

- Describes Conics
- $b \neq 0$, rotation
- $d \neq 0$ or $e \neq 0$, translation
- Classification

$$D = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

$D > 0$ ellipse
$D < 0$ hyperbola
$D = 0$ parabola

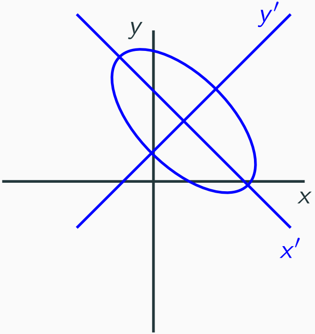


Figure 10: Rotated, translated ellipse.

Curves of Degree 3: $ax^3 + bx^2y + cxy^2 + dy^3 + \cdots = 0$, Cubics

- Newton classified cubic curves, 1710, 72 types (missed 6)
- $y = x^3$ and other types.
- Descarte's folium (leaf) $x^3 + y^3 = 3axy$
- Parametric Solutions

$$x = \frac{3at}{1+t^3}, \quad y = \frac{3at^2}{1+t^3}.$$

- Rational Points
  Ex. $x^3 + y^3 = 1$.
  Let $x = \frac{n}{p}$, $y = \frac{m}{p}$. $n^3 + m^3 = p^3$.
- Fermat - only trivial $(0,1), (1,0)$.
- Fermat's Last Theorem, 1637

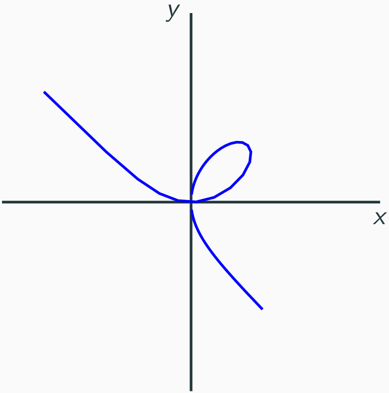


Figure 11: Descarte's Folium.

Fermat's and Bezout's Theorems

- Fermat's Last Theorem, 1637
  $x^n + y^n = z^n$
  Wiles proved in 1995.
- Bezout's Theorem. Let

$$p(x, y) = 0, \qquad \text{degree } n.$$
$$q(x, y) = 0, \qquad \text{degree } m.$$

  Then, p and q intersect in nm points.
- Elimination gives eq of degree nm.
- Need complex numbers,
  point at infinity.
  Next - Projective Geometry.

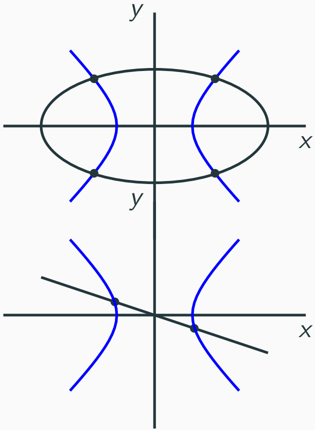


Figure 12: Intersecting Degree 2 curve (blue) with Degree 2 or 1 (black).


al-Haitham
965-1039

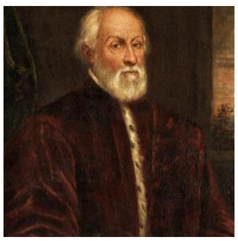

al-Kashi
1380-1429


al-Khwarizmi
780-850


Apollonius
262-190 BCE


Archimedes
287-212 BCE


Aryabhata
476-550


Bombelli
1526-1572


Cardano
1501-1576


Cavalieri
1598-1647

Diophantus
200-284


Eratosthenes
276-194 BCE

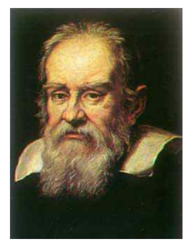

Euclid
325-265 BCE


Euler
1707-1783


Fermat
1601-1665

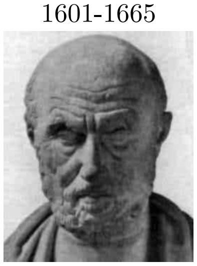

Ferrari
1522-1565


Fibonacci
1170-1250

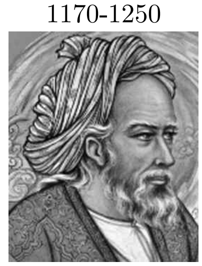

Galileo
1564-1642

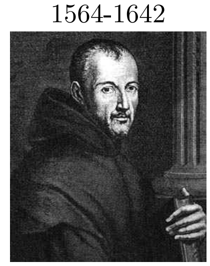

Gauss
1777-1855

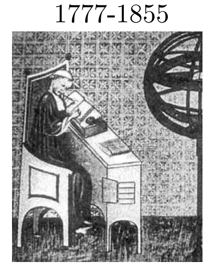

Harriot
1560-1621


Hippocrates
470-410 BCE

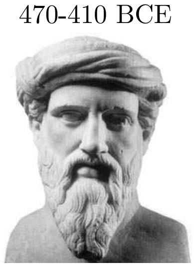

Hypatia
370-415


Khayyam
1048-1131


Mersenne
1588-1648

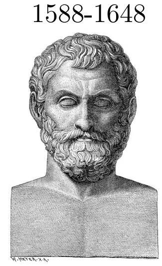

Oresme
1323-1382

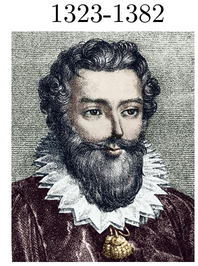

Ptolemy
85-165


Pythagorus
570-490 BCE


Seki
1642-1708


Tartaglia
1500-1557


Thales
624-547 BCE


Viete
1540-1603

Note: There are more from these times but the images were not available.

Projective Geometry

Fall 2023 - R. L. Herman

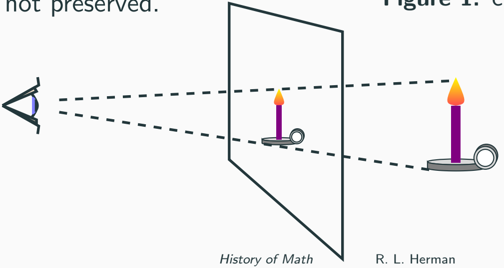
Perspective Drawing

- Art - Perspective Drawing
- Before Renaissance- no illusion of depth and space.
- 13th century Italian masters used shadowing.
- Mathematics of perspective

    Lengths not preserved.

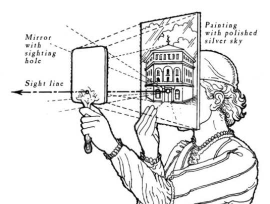
    Angles not preserved.

Figure 1: c.1308-1311



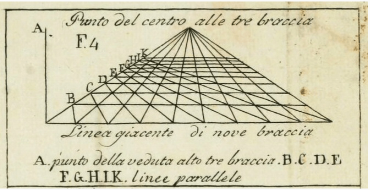
Filippo Brunelleschi (1377-1446) - Architect

- First to describe linear perspective.
- Experimented (1415-1420) using a panel with a grid of squares and a plaque with a hole at eye level.
- Drawings of the Baptistry in Florence, Place San Giovanni and other Florence landmarks.
- His method was studied by Alberti, Da Vinci & della Francesca's *The Perspective of Painting*.



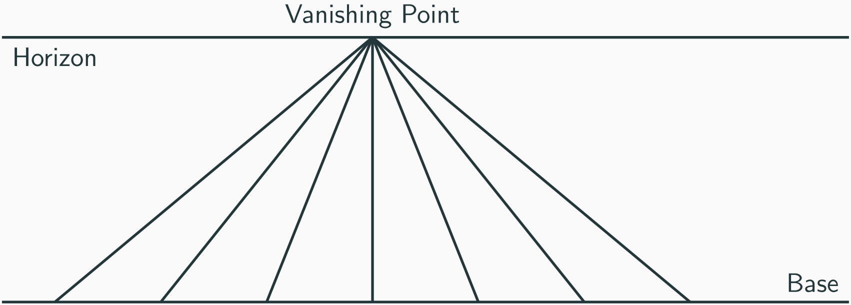
Leon Battista Alberti (1404-1472)

- Alberti's Veil
  Transparent cloth on a frame,
  Good for actual scenes not
  imaginary ones.
- Basic principles:
  1. A straight line in perspective
  remains straight.
  2. Parallel lines either remain
  parallel or converge to a point.
- Drawing a square-tiled floor, solved
  by Alberti (1436).



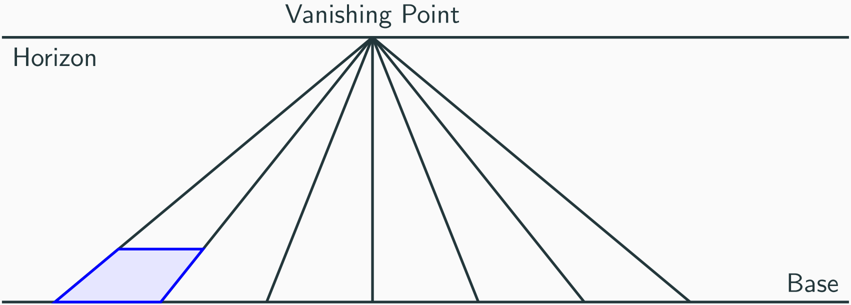
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.



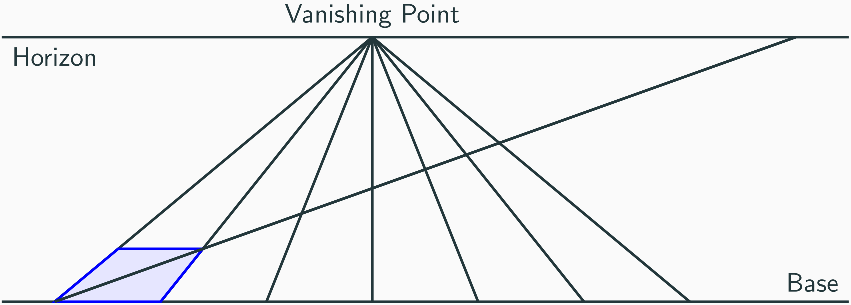
Alberti’s Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.



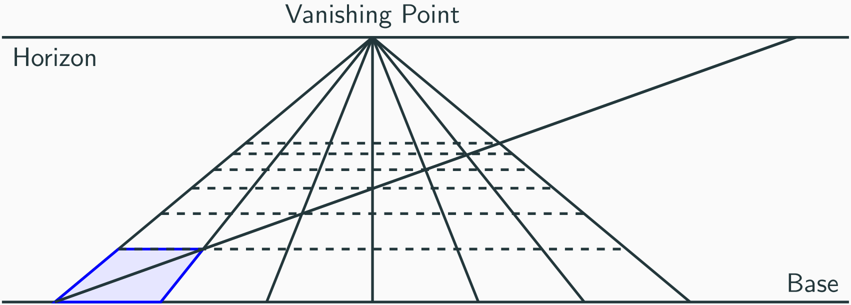
Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.



Alberti's Method

- Align nonhorizontal lines equally along base, converging to one point on the horizon.
- Choose one tile.
- Extend diagonal.
- Intersections determine the horizontals.



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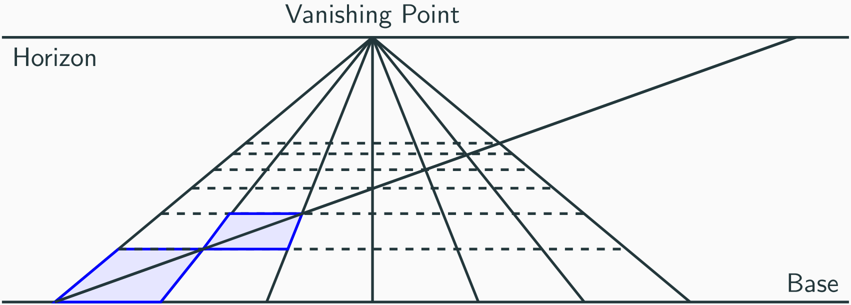

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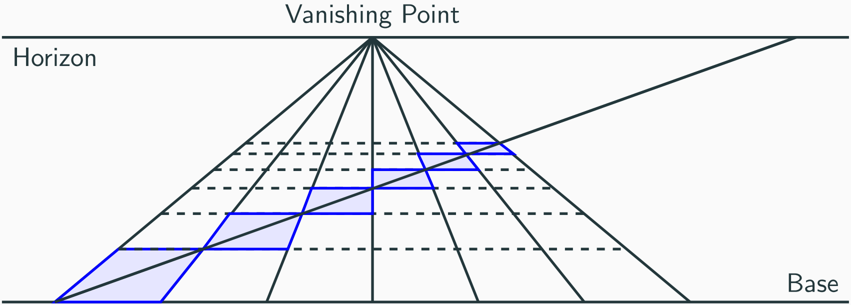

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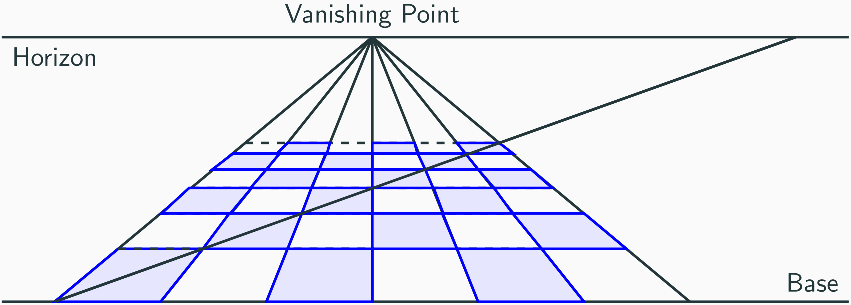


Desargues' Projective Geometry[1]

- Mathematics behind Alberti's Veil:
  Family of lines (light rays) through a point (eye) plus a plane (veil).
- Recall **Pappus' Theorem**:
  A_1, A_2, A_3, collinear;
  B_1, B_2, B_3, collinear;
  then, so are C_1, C_2, C_3.
- Blaise Pascal (1623-1662) at 16 generalized to conics.
- Desargues (1640) **Projective Geometry** only relies on a straight edge.
- Note: Piero della Francesca (c. 1415-1492) formalized rules of perspective, mid-1470s.

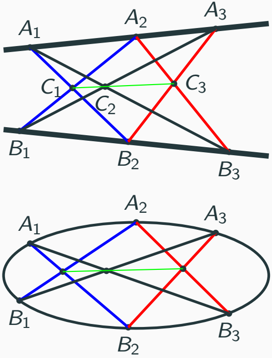
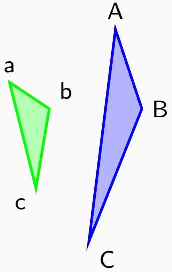


Figure 2: Pappus' and Pascal's Theorems.

[1] Two centuries ahead of his time.

Girard Desargues (1591-1661)

- Architect in Paris, Lyon and engineer.
- Desargues' Theorem in appendix of book on perspective, by friend Abraham Bosse (1602–1676).
  Two triangles are a) in ***perspective axially*** *if and only if they are b) in* ***perspective centrally***.



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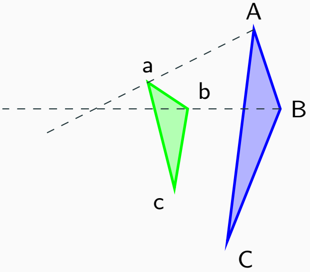
- b) Extend Aa, Bb, Cc **center of perspectivity** P.



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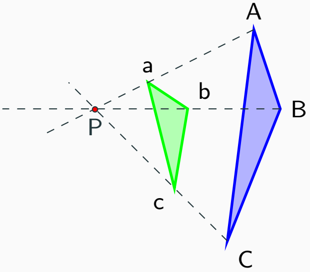

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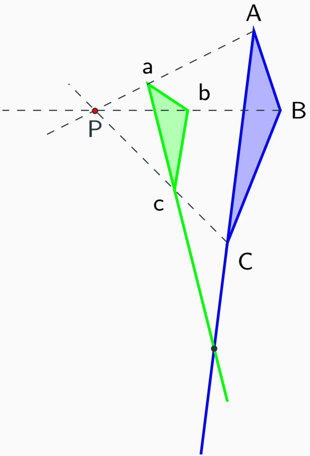

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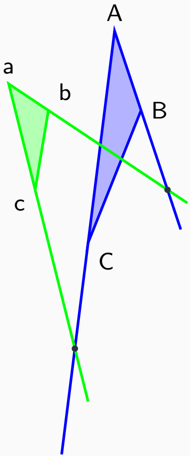

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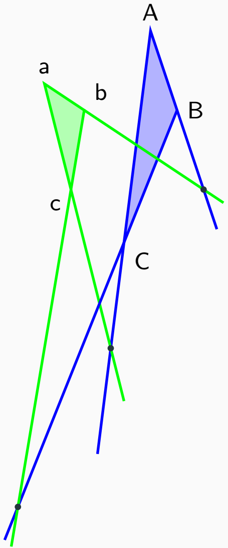

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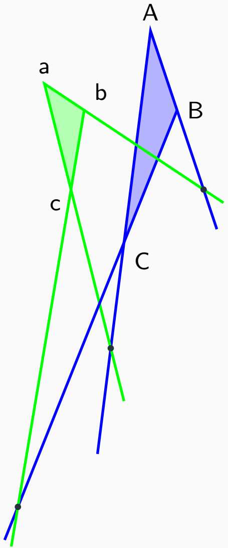

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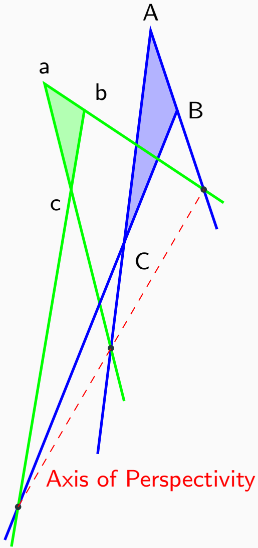
- b) Extend Aa, Bb, Cc **center of perspectivity** P.
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.



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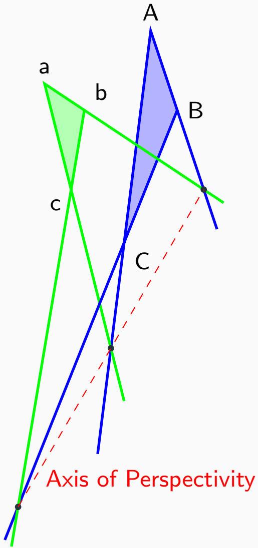
- Points are collinear.
- What if two sides are parallel?



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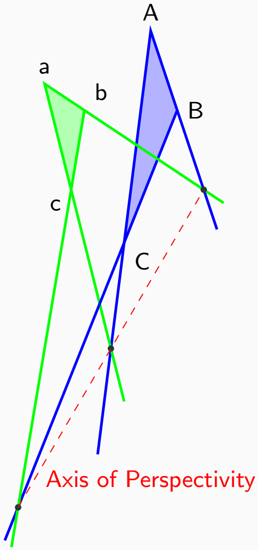
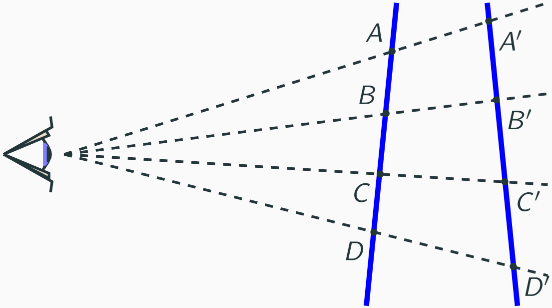
- a) Extend pairs, AC-ac, AB-ab, etc.
- Points are collinear.
- What if two sides are parallel?
- Need **Projective plane**.



Invariance of the Cross Ratio

Lengths and angles are not preserved under projection.



But, for any four points on a line, $\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}}$ is invariant. That is,

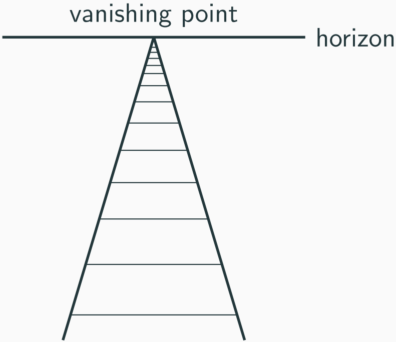
$$\frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}} = \frac{\overline{A'C'}}{\overline{B'C'}} : \frac{\overline{A'D'}}{\overline{B'D'}}.$$

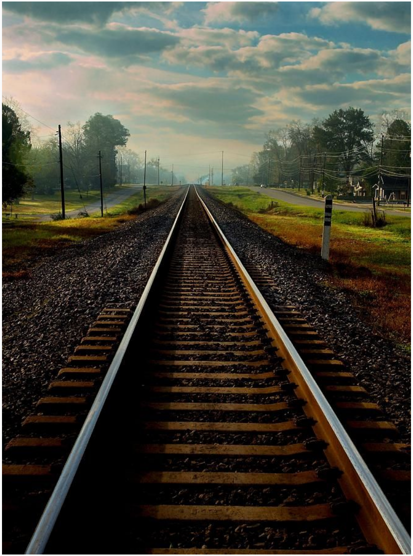
Projective Geometry Rebirth in 1800's.

Perspective

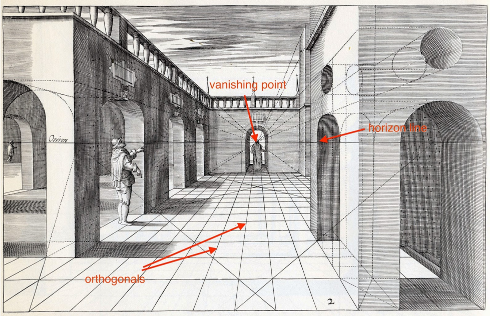
1. Parallel lines meet at a pt.
2. Lines map to lines.
3. Conics map to conics.

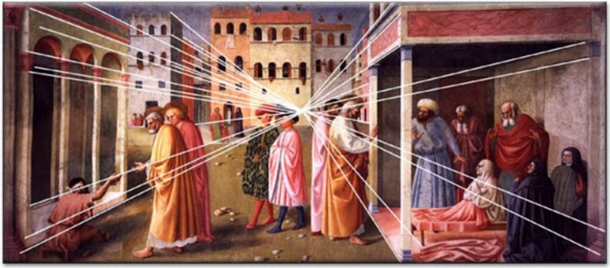
Example: Train tracks.





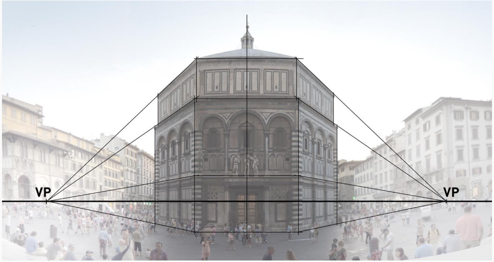
One Point Perspective - Find the Vanishing Points





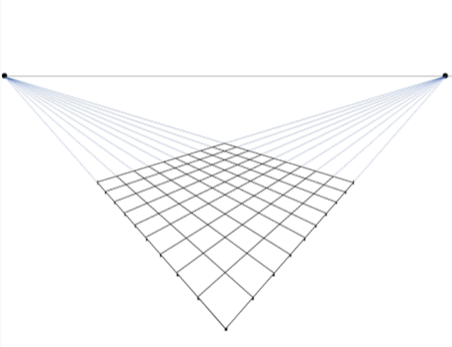
Two Point Perspective - Find the Vanishing Points

Two Point Perspective - Find the Vanishing Points



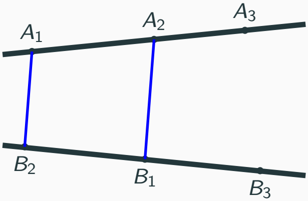
Two Point Perspective Vanishing Point(s)

Two Point Perspective Vanishing Point(s)



Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?



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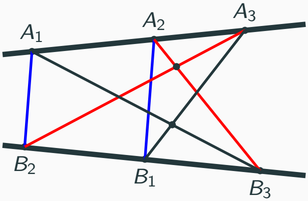

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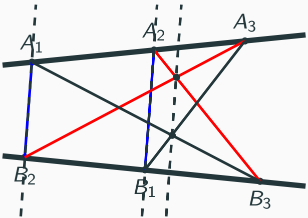

Points at Infinity

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- Pappus' Theorem -
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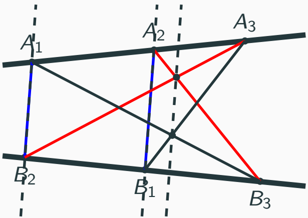
  Does the theorem hold?
- Desargues - **line at infinity**.



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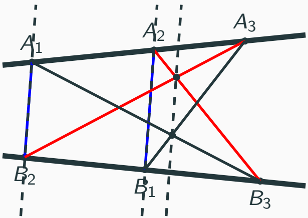
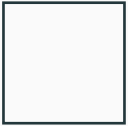
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane





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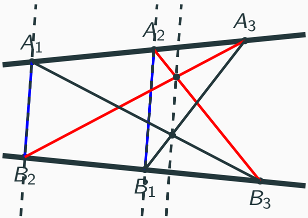
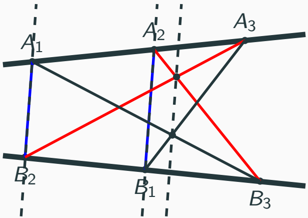
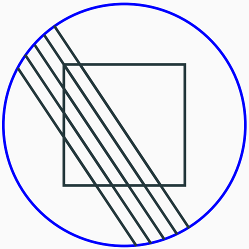
- Desargues - **line at infinity**.
- Look at a plane



Points at Infinity

- Artists' use vanishing points.
- Pappus' Theorem -
  Consider parallel lines A_1B_2, A_2B_1.
  Does the theorem hold?
- Desargues - **line at infinity**.
- Look at a plane
- Add parallel lines.
  Where do they go?
- Line at Infinity
- Plane + line at infinity =
  Projective Plane





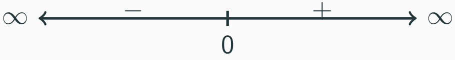
Line at infinity

Projective Line

- Consider the real line, $\mathbb{R}$.

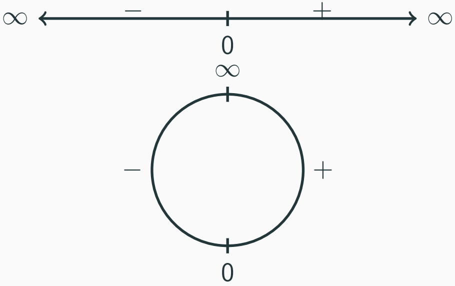
Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity,
  real projective line, $\mathbb{RP}^1$.



Projective Line

- Consider the real line, $\mathbb{R}$.
- Add point at infinity, real projective line, $\mathbb{RP}^1$.
- Topologically a circle!



Projective Line

- Consider the real line, $\mathbb{R}$.
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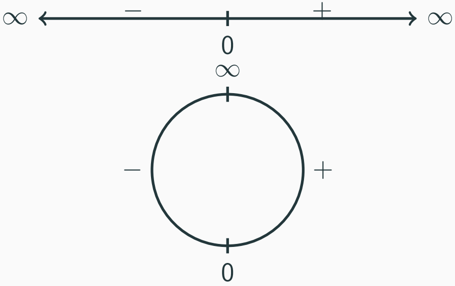
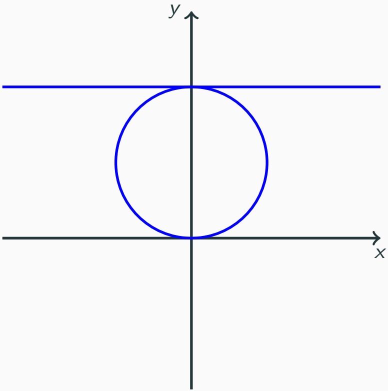
- Topologically a circle!
- We can map the circle to $\mathbb{R}$.





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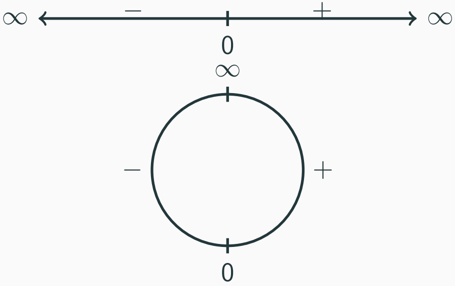

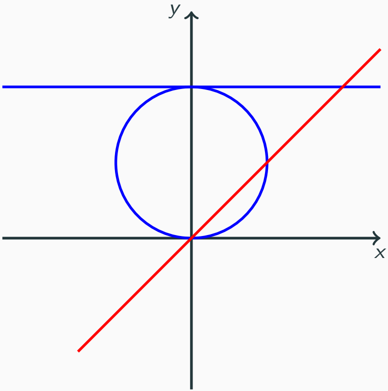

Intersection: $y = b$, $y = mx$: $x = \frac{b}{m}$.

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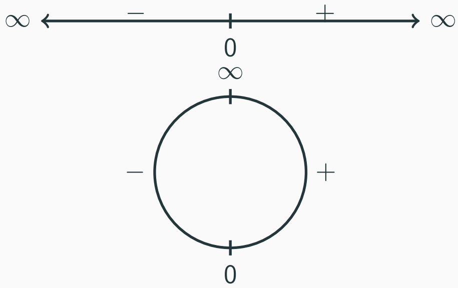

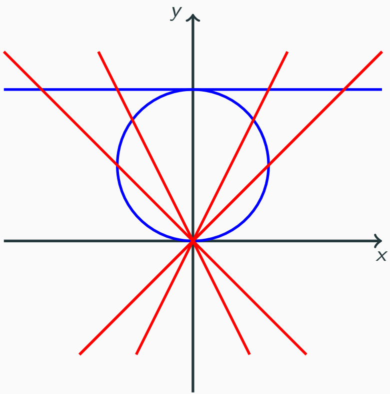

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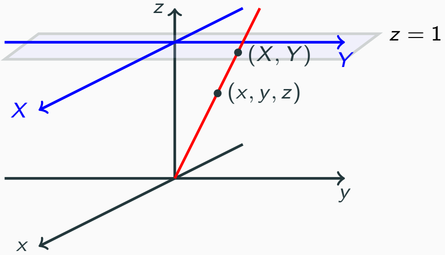
$x = \frac{b}{m}$.

Homogeneous Coordinates

- Point on line: (x, y, z)
- All points on line map to (X, Y) in the plane.
- (X, Y) are called homogeneous coordinates.
- Points on line are multiples, $(x', y', z') = \lambda(x, y, z)$.
- Point on plane: Let $\lambda = \frac{1}{z}$. Then, $(x', y', z') = (\frac{x}{z}, \frac{y}{z}, 1)$, or

$$X = \frac{x}{z}, \quad Y = \frac{y}{z}.$$



Curves: Given $Y = f(X)$, find $(x, y, z)-$surface.

- Curve in plane $z = 1$, $Y = X^2$.
- $X = \frac{x}{z}$, $Y = \frac{y}{z}$.
- Translates to

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2.$$

- Multiply by z^2.
- This is a surface in (x, y, z)-space,

$$x^2 = yz.$$

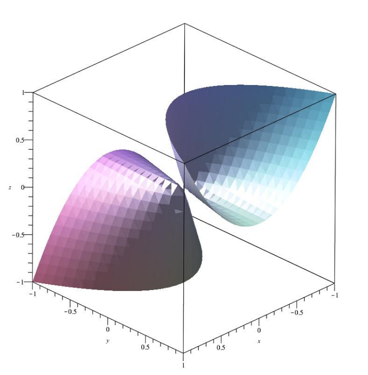


Figure 3: Surface $x^2 = yz$.

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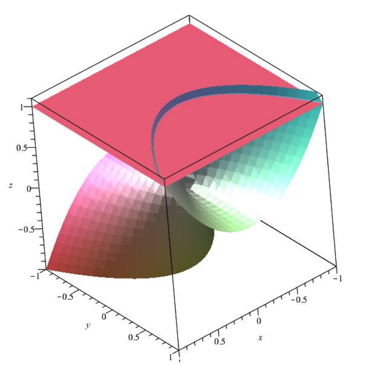

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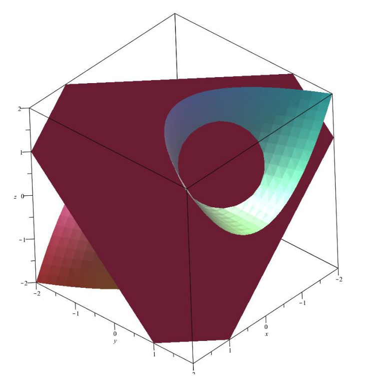


Figure 3: Surface $x^2 = yz$.

Projective Sphere: Extending $\mathbb{RP}^1$.

- Map points on a plane to points on surface of unit sphere, $\mathbb{S}^2$.
- Lines through South Pole uniquely intersect the plane and sphere.
- All points mapped except $(0,0,0)$. This point can be mapped to the line at infinity.
- Lines through origin are points of the real projective plane, $\mathbb{RP}^2$.

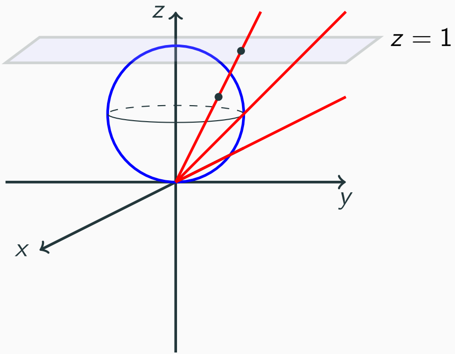
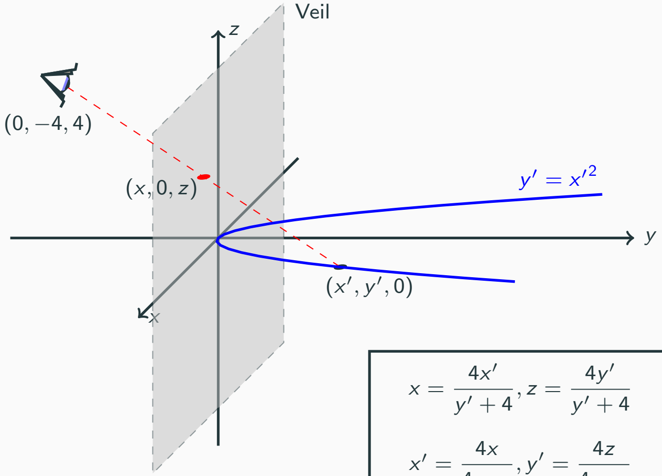


Figure 4: Stereographic Projection

Looking into the Veil - Parabola Projected



$$x = \frac{4x'}{y'+4}, z = \frac{4y'}{y'+4}$$

$$x' = \frac{4x}{4-z}, y' = \frac{4z}{4-z}$$

Figure 5: Problems 8.4.2-8.4.4

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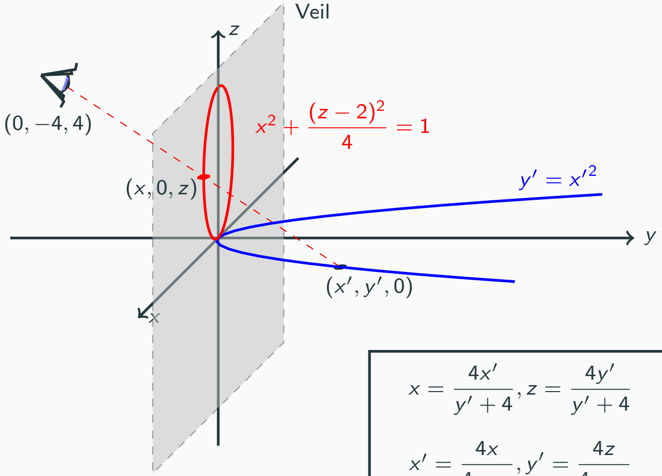

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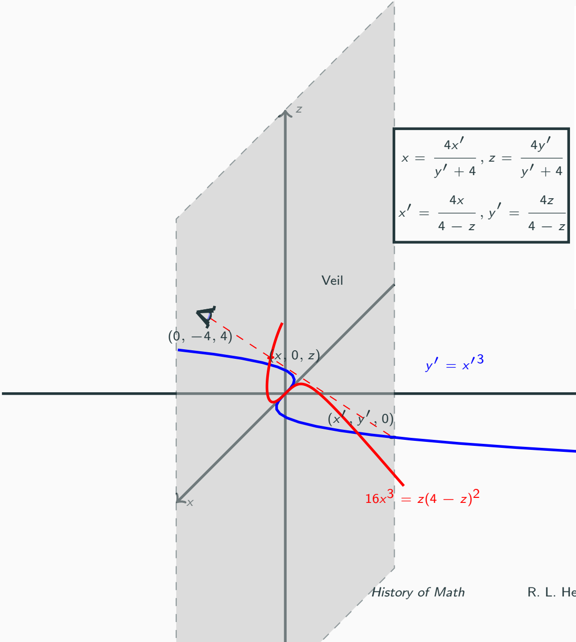
Viewing A Cubic in the Veil



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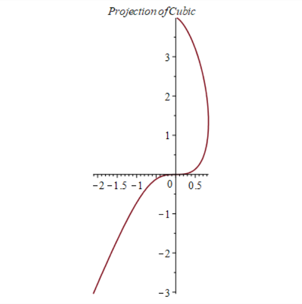
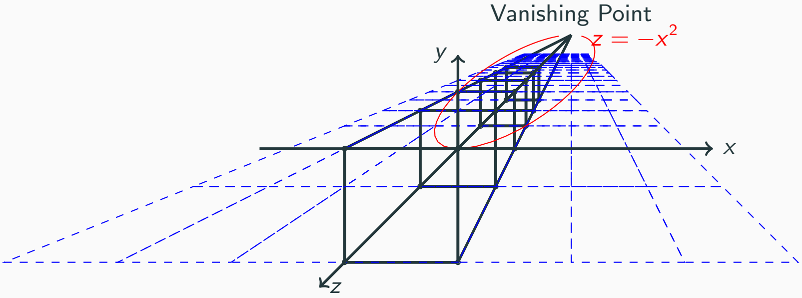


Above: Maple plot of projected cubic to the veil.

Perspective Drawing

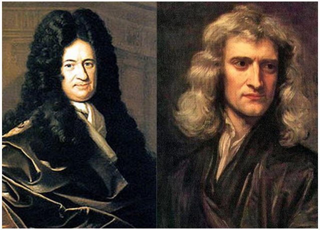
Looking at conics from a different perspective: The parabola $z = -x^2$ looks like an ellipse.



In the 1600's mathematicians had other mathematics to attend to. So, we return to geometry in the 1800's.

Emergence of Calculus

Fall 2023 - R. L. Herman



The Method of Exhaustion and the Infinite

- Zeno's Paradox of the Arrow

  "If a body moves from A to B, then before it reaches B it passes through the mid-point, say B_1 of AB. Now to move to B_1 it must first reach the mid-point B_2 of AB_1. Continue this argument to see that A must move through an infinite number of distances and so cannot move. " (450 BCE)
- Eudoxus - Method of Exhaustion.
- Archimedes - area of a segment of a parabola is $\frac{4}{3}$ the area of a triangle with the same base and vertex.
- Luca Valerio (1552-1618) published in 1608 *De quadratura parabolae.*
- Kepler (1571-1630): area as sum of lines. Inpsired Cavalieri's indivisibles.

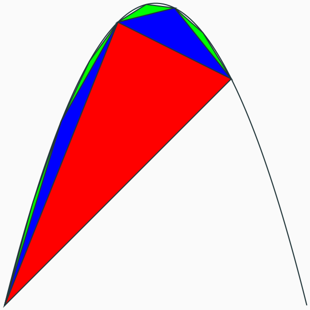


Figure 1: Archimedes: First known summation of series.

Area of blue $= \frac{1}{4}$ that of red, a. Then,

$$A = a + \frac{1}{4}a + \frac{1}{4^2}a + \dots.$$

Developments in the 1600's

Rapid developments first 60 years of 1600's based on Greek geometry, algebra, astronomy (Kepler, Galileo). Led to unification of geometry and algebra.

- Descartes (1596-1650)
- Cavalieri (1598-1647)
- Fermat (1601-1665)
- Roberval (1602-1675)
- Wallis (1616-1703)
- Barrow (1630-1677)
- Gregory (1638-1675)
- Newton (1642-1727)
- Leibniz (1646-1716)

Two main problems

- Tangents
- Areas

Need curves

- Conics
- Archimedean spiral
- Conchoid
- Cissoid
- Cycloid

Sixteenth Century Science

- Copernicus (1473-1543)
  Commentary - 1514
  Dē revolutionibus orbium coelestium, 1542 on death bed.
- Tycho Brahe (1546-1601)
- Galileo Galilei (1564-1642)
  1609 - Telescope
  Jupiter's Moons, Moon, Saturn, Phases of Venus.
  1633 trial and judgement for heresy, under house arrest.
- Johannes Kepler (1571-1630)
  1609 - Study on Mars orbit.
  Laws of Planetary Motion

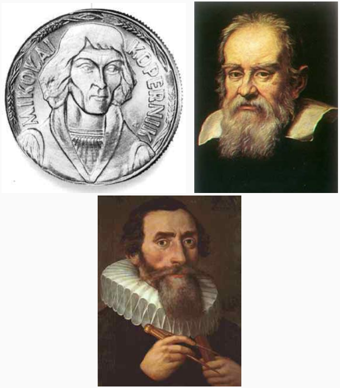


Figure 2: Copernicus, Galileo, Kepler

Seventeenth Century - French, German, English Mathematics

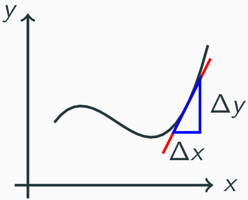
- 1590 Vièta, *The Analytic Art*
- Bonaventura Cavalieri (1598-1647)
- Evangelista Torricelli (1608-1647)
- John Napier (1550-1617) and Henry Briggs (1561-1631) - Introduced the logarithm
- French Mathematicians:
  René Descartes (1596-1650)
  Blaise Pascal (1623-1662)
  Pierre de Fermat (1601-1665)

- Descartes
  - philosopher, mathematician
  - *Discours de la méthode*, Marriage of algebra/geometry - analytic geometry
- Pascal
  - Wrote math before 16
  - Probability theory
  - Theology
- Fermat
  - Created analytic geometry
  - Contributions to Calculus
  - Number theory
  - Scribbled in Diophantus' *Arithmetica*

Tangents

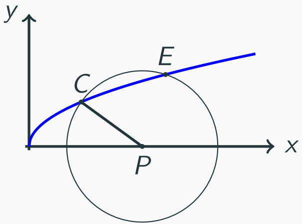
- Pierre de Fermat, René Descartes
- Both studied Apollonius' problem: construct a circle tangent to any three objects.
- Tangent line approximates curve at a point.
- Slope $\frac{\Delta y}{\Delta x}$.
- Infinitesimals - increments.
- Fermat:
  Method for maxima-minima
  1636 - Method of Tangents
- 1636 Letter: Descartes to Mersenne
  $dy = f(x + dx) - f(x) =?dx.$

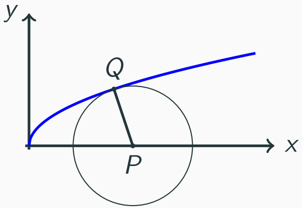
Figure 3: Fermat and Descartes



Descartes vs Fermat - Analytic Geometry, Tangents, Optics

- Decartes published *La Géométrie* - 1637
- Depicted $ax + by = c$ as a line.
- Introduced x, y.
- Fermat introduced analytic geometry earlier.
- Fermat interested in optimization.
- Fermat: lawyer in Toulouse, Math a hobby.
- Descartes denounced him and challenged him to find tangent to folium, $x^3 + y^3 = 3axy$.
- Descartes' Method of Tangents: Find circles tangent to curves.
- Fermat challenged Descartes to explain refraction. Fermat published in 1662.





Areas Under Curves

- First studied by Eudoxus, Archimedes
- Bonaventura Cavalieri (1598-1647) - *Geometria indivisibilibus continuorum nova quadam ratione promota,* 1635.

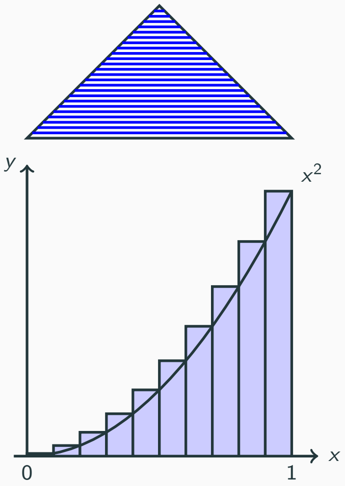
  Fill area with lines.
  But, an infinite number of lines sum to infinity.

- Archimedes, John Wallis (1616-1703):

$$\int_0^1 x^2\, dx.$$

N segments of width $\frac{1}{N}$. and height $\left(\frac{k}{N}\right)^2$, $k = 1, 2, \ldots, N$.

$$A \approx \sum_{k=1}^{N} \frac{1}{N} \left(\frac{k}{N}\right)^2.$$



Cavalieri's Method of Indivisibles

John Wallis (1616-1703) - A Side Note

- 1649, Savilian professor of geometry at the University of Oxford.
- *Arithmetica Infinitorum*, "The Arithmetic of Infinitesimals", 1655
- Extended Cavalieri's law of quadrature.
- Algebraic vs Geometric approach.
- Influenced Newton.
- *Mathesis Universalis* Developed notation: introduced ∞, negative and fractional exponential notation.
- Royal Society of London in 1662.
- *Tractatus de Sectionibus Conicis,* 1659; described as curves using algebra.
- Published colleagues work on quadratures.

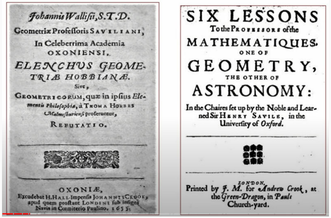


Figure 4: Wallace-Hobbes Rivalry

- Thomas Hobbes (1588-1679), called book a "scab of symbols," quarter century controversy.
- Controversies with Huygens, Descartes, Fermat, Pascal.

Savilian Chairs of Geometry and Astronomy

- The Savilian Chairs of Geometry and Astronomy, University of Oxford, 1619.
- By Henry Savile (1549 - 1622)
- Click to see list.
- 1570 Lectures on Ptolemy
- " he felt that mathematics at that time was not flourishing. Students did not understand the importance of the subject, Savile wrote, there were no teachers to explain the difficult points, the texts written by the leading mathematicians of the day were not studied, and no overall approach to the teaching of mathematics had been formulated."
- Read more at MacTutor.

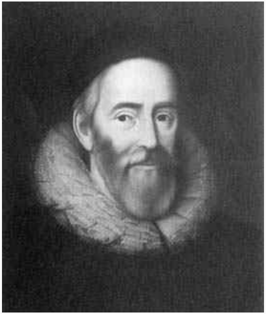


Figure 5: Henry Savile

Back to Wallis' Areas Under Curves

Find the sum

$$\begin{aligned} A &\approx \sum_{k=1}^{N} \frac{1}{N}\left(\frac{k}{N}\right)^2 \\ &= \frac{1}{N^3}\sum_{k=1}^{N} k^2 \\ &= \frac{1}{N^3}\frac{N(N+1)(2N+1)}{6} \\ &\sim \frac{2N^3}{6N^3} = \frac{1}{3}. \end{aligned}$$

Wallis showed

$$\boxed{\int_0^a x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_0^a = \frac{a^{n+1}}{n+1}}$$

for $k = 1, 2, \ldots, 9$.

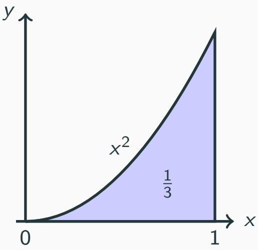
Note:

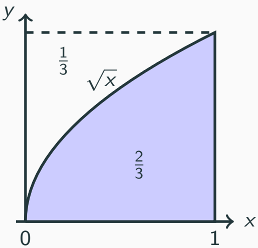
$$\begin{aligned} \sum_{k=1}^{N} k &= \underbrace{1 + \underbrace{2 + \cdots + (N-1)}_{2+(N-1)} + N}_{1+N} \\ &= N\frac{N+1}{2}. \end{aligned}$$

Figure 6: John Wallis

Integrating Powers, $\int x^k\,dx$,

- al Haytham (965-1039) $k = 1, 2, 3, 4$.
- Cavalieri (1635) knew for k up to 9.
- Proven in general by Fermat, Descartes, Roberval, 1630's.
- Fractional Powers (Fermat)
  Ex: $\int_0^1 \sqrt{x}\,dx$
  Use the symmetry in the figures.
- Areas under x^k, need sums $1^k + 2^k + \cdots n^k$.
- Volumes - use cylinders, $V = \pi r^2 h$.
  Sums needed: $1^{2k} + 2^{2k} + \cdots n^{2k}$.





Note: $1^3 + 2^3 + \cdots + k^3 = \frac{(N+1)^2 N^2}{4} = (1 + 2 + \cdots + k)^2$.

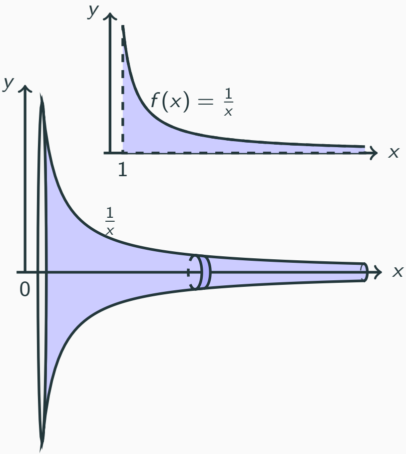
Evangelista Torricelli (1608-1647), barometer inventor

- Inverse Powers, x^{-1}
- Area under $y = \frac{1}{x}$.

$$\int_1^{\infty} \frac{1}{x}\, dx = \infty.$$

- 1641 Torricelli's trumpet (Gabriel's horn)

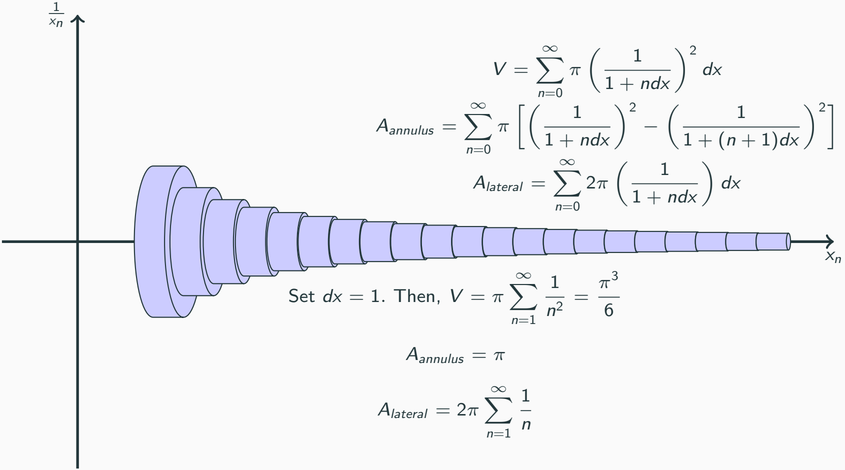
$$\begin{aligned} V &= \pi \int_1^{\infty} \frac{1}{x^2}\, dx = \pi. \\ A &= 2\pi \int_1^{\infty} \frac{dx}{x} \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \\ &> 2\pi \int_1^{\infty} \frac{1}{x}\, dx = \infty. \end{aligned}$$



What? You cannot paint the surface but can fill the trumpet with paint.

Hobbes - "to understand this for sense, it is not required that a man should be a geometrician or logician, but that he should be mad."

Gabriel's Wedding Cake - Discrete Case



$$V = \sum_{n=0}^{\infty} \pi \left(\frac{1}{1+ndx} \right)^2 dx$$

$$A_{annulus} = \sum_{n=0}^{\infty} \pi \left[\left(\frac{1}{1+ndx} \right)^2 - \left(\frac{1}{1+(n+1)dx} \right)^2 \right]$$

$$A_{lateral} = \sum_{n=0}^{\infty} 2\pi \left(\frac{1}{1+ndx} \right) dx$$

Set $dx = 1$. Then, $V = \pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^3}{6}$

$$A_{annulus} = \pi$$

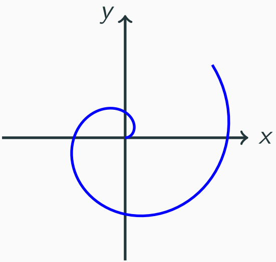
$$A_{lateral} = 2\pi \sum_{n=1}^{\infty} \frac{1}{n}$$

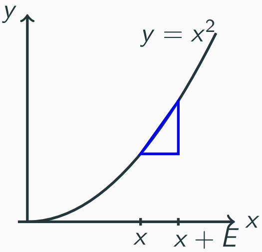
Tangents, Maxima, Minima

- Curves studied like Archimede's spiral, $r = a\theta$
- Fermat - studied polynomials
- Work simpler than Descartes
- Used infinitesimals, E
- **Example:** $y = x^2$

$$\frac{(x+E)^2 - x^2}{E} = 2x + E.$$

- Generalized to polynomials, $p(x, y) = 0$.





John Wallis' (1655) *Arithmetica Infinitorium*

- Combined Descartes' analytic geometry and Cavalieri's indivisibles.
- Some results already known.
- New approach to fractional powers.
- Ambivalent use of infinitesimals - attacked by Thomas Hobbes (1588-1679).
- 1632 Church banned infinitesimals.
- Formulae for π known by
  - Gregory, Newton, Leibniz
- Madhava (1350-1425) found π to 13 decimal places using series,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Wallis' Formulae:

$$\begin{aligned}
\frac{\pi}{4} &= \frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots \\
\frac{\pi}{2} &= \left(\frac{2}{1}\cdot\frac{2}{3}\right)\cdot\left(\frac{4}{3}\cdot\frac{4}{5}\right)\cdot\cdots \\
\frac{4}{\pi} &= 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}}
\end{aligned}$$

Already known formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Isaac Newton (1642-1727)

- Major use of infinite series
- *A Treatise of the Methods of Series and Fluxions*
- *Quadrature of the Hyperbola*
  Written in 1665,
  1st publication in 1668 by Mercator
- Akin to decimal expansions - powers of $\frac{1}{10}$ replaced by x^n.
- Example:

$$\log(1+x) = \int_0^x \frac{dt}{1+t}$$

[Note: Here $\log x = \ln x$.]

Note: Geometric series

$$\begin{aligned} 1 + t + t^2 + \cdots &= \frac{1}{1-t}, |t| < 1. \\ 1 - t + t^2 - \cdots &= \frac{1}{1+t}, |t| < 1. \end{aligned}$$

Then,

$$\begin{aligned} y &= \log(1+x) \\ &= \int_0^x (1 - t + t^2 - \cdots)\, dt \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots . \end{aligned}$$

Invert Power Series

We have for $y = \log(1+x)$,

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

In order to invert the series, let $x = a_0 + a_1 y + a_2 y^2 + \cdots$. Then,

$$\begin{aligned} y &= \left(a_0 + a_1 y + a_2 y^2 + \cdots\right) - \frac{1}{2}\left(a_0 + a_1 y + a_2 y^2 + \cdots\right)^2 + \cdots \\ &= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 + a_1\left(a_0^2 - a_0 + 1\right) y \\ &\quad + \left[a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right)\right] y^2 \\ &\quad + \left[\frac{a_1^3}{3} + a_1 a_2 (2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right)\right] y^3 + \ldots . \end{aligned}$$

Equate coefficients of powers of y, then ...

Series Inversion (cont'd)

We solve the resulting system of equations:

$$\begin{aligned}
0 &= a_0 - \frac{1}{2}a_0^2 + \frac{1}{3}a_0^3 \\
1 &= a_1\left(a_0^2 - a_0 + 1\right) \\
0 &= a_2\left(a_0^2 - a_0 + 1\right) + \left(a_0 - \frac{1}{2}\right) \\
0 &= \frac{a_1^3}{3} + a_1 a_2(2a_0 - 1) + a_3\left(a_0^2 - a_0 + 1\right).
\end{aligned}$$

The first equation gives $a_0 = 0$. The next two give $a_1 = 1$ and $a_2 = \frac{1}{2}$.

Continuing Newton found that

$$a_0 = 0, a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{6}, a_4 = \frac{1}{24}, \ldots, a_n = \frac{1}{n!.}$$

Newton's Series for Exponential

So far, inversion of

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots .$$

led to

$$x = y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

However,

$$y = \log(1+x) \Rightarrow x = e^y - 1.$$

So, we found the series expansion

$$e^y = 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \cdots .$$

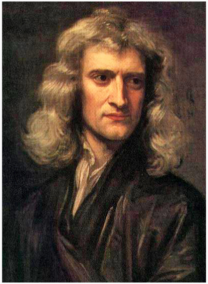


Figure 7: Newton

Newton's Series for Sine

Newton knew $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Recall binomial series: $(a+b)^n = \sum_{k=0}^n C_{n,k} a^{n-k} b^k$, where the coefficients are $C_{n,k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$. Then,

$$
\begin{aligned}
(1+a)^p &= 1 + pa + \frac{p(p-1)}{2!} a^2 + \frac{p(p-1)(p-2)}{3!} a^3 + \cdots . \\
\sin^{-1} x &= \int_0^x \frac{dt}{\sqrt{1-t^2}}, \quad a = -t^2, p = -\frac{1}{2}, \\
&= \int_0^x \left(1 + \frac{1}{2} t^2 + \frac{3}{8} t^4 + \cdots + \frac{-\frac{1}{2}(-\frac{3}{2})\cdots(\frac{1}{2}-k)}{k!} (-t^2)^k + \cdots \right) \\
&= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \cdots + \frac{1\cdot 3\cdots(2k-1)}{2\cdot 4\cdots 2k}\frac{x^{2k+1}}{2k+1} + \cdots
\end{aligned}
$$

Inverting, Newton found $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$.

Gottfried Wilhelm Leibniz (1646-1716)

- Librarian, philosopher, diplomat, doctorate in law.
- First papers in calculus (1684).
- Led to long dispute.
- Better notation, $\frac{dy}{dx}, \int\, dx$.
- Sum, product, quotient rules.
- Proved Fundamental Theorem of Calculus, $\frac{d}{dx}\int f(x)\,dx = f(x)$.

Figure 8: Leibniz

Infinite Series

- Geometric series,
  Known to Euclid (Zeno's paradox)
  Leonhard Euler (1707-1783)

  $$a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}, |r| < 1.$$

- Harmonic Series - Oresme (1350)

$$\begin{aligned} & 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \\ = & (1) + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ \geq & \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty. \end{aligned}$$

- Power series - 17th Century,
  Gregory, Wallis, Taylor, Mclaurin, ...

Figure 9: Euler

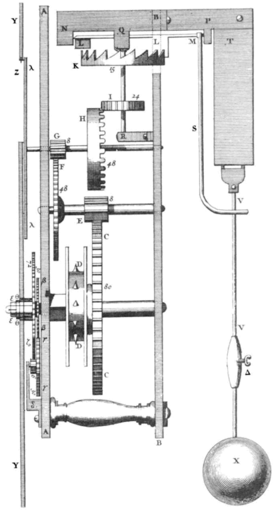
James Gregory (1638 - 1675)

- Uncle to David Gregory (1659-1708), Professor of Mathematics, University of Edinburgh at 24, filling the chair previously held by James Gregory. Savilian Professor of Astronomy, Oxford, Supported Newton in controversy and first to teach *Principia*.
- First to publish and prove Fundamental Theorem of Calculus, *Geometriae Pars Universalis* (1668) .
- Discovered 7 series before Taylor.
- *Optica Promota,* first practical reflecting (Gregorian) telescope.
- Worked with Angeli at Padua.

Figure 10: James and David Gregorie

Problems of the Day

- Pendulum clock of Galileo - Thought isochronous, time-independent period of swing. Son Vincenzio, worked on it, died 1649.
- Huygens built first pendulum driven clock, 1656.
- Tautochrone - Time taken independent of starting point, Huygens, 1659.
- Brachistochrone - Curve of fastest descent, posed by Johann Bernoulli, 1696.
- Isochrone - Connects points of equal time travel, Leibniz 1687, Jacob Bernoulli 1690.

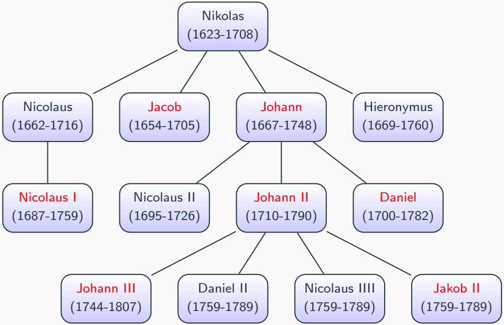


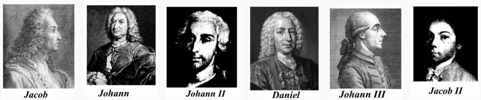
Calculus Wars

- Newton: 'the method of fluxions'.
- Paper on fluxions in 1666, but not published for decades.
- *Philosophiae naturalis principia mathematica*, published 1687.
- Little explicit calculus.
- Method of fluxions appeared in 1693.
- Leibniz, published first paper on calculus, 1684.
- Said he discovered calculus in 1670s.
- In 1695, Wallis: Leibniz learned about calculus from Newton.
- Nicolas Fatio de Duillier (1664-1753) in 1699 book, Newton's absolute priority.
- Angry responses from Johann Bernoulli and Leibniz.
- John Keill accused Leibniz of plagiarism, 1711.
- Royal Society in England gave report that Leibniz had concealed knowledge of Newton's work, 1712.
- Leibniz accused Newton and followers of stealing his calculus.
- Debate ended when Leibniz died, 1716.

The Bernoulli Family

- In three generations, there were 8 mathematicians.
- Dominated mathematics and physics, 17-18th centuries - with Newton, Leibniz, Euler, Lagrange, etc.
- Contributions: calculus, geometry, mechanics, probability, ballistics, thermodynamics, hydrodynamics, optics, elasticity, magnetism, astronomy.





Basel Problem (1644)

- Posed by Pietro Mengoli (1626-1686).

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

- Jacob and Johann Bernoulli (1704) tackled. $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=?$

Figure 11: Jacob and Johann

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n(n+1)} &= \sum_{n=1}^{N}\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\left(\frac{1}{N}-\frac{1}{N+1}\right)\\ &= \left(1-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots\left(\cancel{\frac{1}{N}}-\frac{1}{N+1}\right)\\ &= 1-\frac{1}{N+1}\underset{N\to\infty}{\Longrightarrow}1.\end{aligned}$$

Euler's Solution of Basel Problem - 1734

- Descartes' Factor Theorem
- $p(x)$ - polynomial
- $p(r) = 0$ implies
  $p(x) = (x - r)q(x)$,
  $q(x)$ - polynomial

Proof:

$$\begin{aligned} p(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ p(y) &= a_0 + a_1 y + \cdots + a_n y^n \\ p(x) - p(x) &= a_1(x - y) + \cdots + a_n(x^n - y^n) \end{aligned}$$

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\right)$$

Let $y = r$,

$$\begin{aligned} p(x) &= (x - r)[a_1 + a_2(x + r) + \cdots + a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1})] \\ &= (x - r)q(x). \end{aligned}$$

Leonhard Euler's Solution of Basel Problem

$\sin x$ has roots $n\pi,\ n = 0, \pm 1, \pm 2, \ldots$. - Generalize Factor Theorem:

$$\begin{aligned}
\sin x &= Ax\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\cdots \\
&= Ax\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots \\
&= A\left[x-\frac{x^3}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)+x^5(\cdots)-\cdots\right].
\end{aligned}$$

Compare to

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$

Then $A = 1$, and

$$\frac{1}{3!} = \frac{1}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right) \Rightarrow \zeta(2) \equiv \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

What are the next coefficients?

We need the x^5 terms in the expansion

$$\begin{aligned}
\sin x &= x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right)\\
&= x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\cdots\left(1-\frac{x^2}{m^2\pi^2}\right)\cdots
\end{aligned}$$

We multiply $\frac{x^2}{m^2\pi^2}$ times the factors $\frac{x^2}{n^2\pi^2}$, $n\neq m$, and summing:

$$x\sum_{m=1}^{\infty}\frac{x^2}{m^2\pi^2}\sum_{n=1,n\neq m}^{\infty}\frac{x^2}{n^2\pi^2}=\frac{x^5}{\pi^4}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n=1,n\neq m}^{\infty}\frac{1}{n^2}.$$

$$\frac{\pi^4}{5!}=\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m^2}\left[\zeta(2)-\frac{1}{m^2}\right]=\frac{1}{2}[\zeta(2)^2-\zeta(4)].$$

So, $\zeta(4)\equiv\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{36}-\frac{\pi^4}{60}=\frac{1}{12}\left(\frac{\pi^4}{3}-\frac{\pi^4}{5}\right)=\frac{\pi^4}{90}.$

Another Approach to Obtain $\zeta(4)$

Noting that $\frac{d}{dx}(\ln \sin x) = \cot x$ and using the known series expansion for $x \cot x$ in terms of Bernoulli numbers,

$$\begin{aligned}
\sin x &= x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
\ln \sin x &= \ln x + \sum_{n=1}^{\infty} \ln \left(1 - \frac{x^2}{n^2\pi^2}\right) \\
x \cot x &= 1 - \sum_{n=1}^{\infty} \frac{2x^2}{n^2\pi^2 - x^2} \\
&= 1 - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{x^2}{n^2} \sum_{k=0}^{\infty} \left(\frac{x^2}{n^2\pi^2}\right)^k \\
1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945} + \dots &= 1 - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{\pi}\right)^{2k+2} \zeta(2k+2) \\
x \cot x &= 1 + \sum_{k=0}^{\infty} (-1)^k B_{2k} (2x)^{2k}
\end{aligned}$$

Results for the Riemann Zeta Function, $\zeta(s)$

Therefore, we have

$$\begin{aligned}\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots &= \frac{\pi^2}{6}, \\ \zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots &= (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n},\end{aligned}$$

where B_{2n} are Bernoulli numbers,[1] $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots,$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Euler (1748) - Zeta function can be defined for p prime as

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\ &= \left(1 - \frac{1}{2^s}\right)^{-1}\left(1 - \frac{1}{3^s}\right)^{-1}\left(1 - \frac{1}{5^s}\right)^{-1} \cdots \left(1 - \frac{1}{p^s}\right)^{-1} \cdot .\end{aligned}$$

[1] Jacob Bernoulli, 1713, Seki Takakazu, 1712, published posthumously. $B_0 = 1, B_1 = -\frac{1}{2}$.

Georg Friedrich Bernhard Riemann[2] (1826-1866)

- Riemann extended Euler's zeta function, $s \in \mathbb{C}$

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

- Values

$$\begin{aligned} &\zeta(1) = \infty, \text{ harmonic series} \\ &\zeta(2) = \tfrac{\pi^2}{6} \\ &\zeta(3) \quad \text{irrational, Apery (1981)} \end{aligned}$$

- Zeros

$$\zeta(-2n) = 0,\ n \text{ integer} > 0.$$

Riemann Hypothesis:

$$\zeta(\sigma + it) = 0 \text{ when } \sigma = \tfrac{1}{2}.$$

- Connection to primes?

Figure 12: Bernhard Riemann

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(2s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx \\
\zeta(2n) &= \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!} \\
\zeta(s) &= 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s) \\
\Gamma(s) &= \int_0^\infty x^{s-1} e^{-x}\, dx
\end{aligned}$$

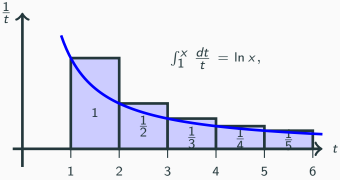
[2] *On the Number of Primes Less Than a Given Magnitude*, 1859

Connection to Primes and Other Tidbits

$$\begin{aligned}\zeta(s) &= \left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{3^s}\right)^{-1}\cdots\left(1-\frac{1}{p^s}\right)^{-1}\cdots \\ &= \prod_{p=\text{prime}}\left(1-\frac{1}{p^s}\right)^{-1} \\ &= \prod_{p=\text{prime}}\left[1+\frac{1}{p^s}+\left(\frac{1}{p^s}\right)^2+\cdots\right]\end{aligned}$$

- Primes less than $x \sim \int_2^x \frac{dt}{\log t}$
- Euler-Mascheroni Constant $\gamma \approx 0.577218\ldots.$
- Generalizing $n!$

$$\Gamma(n+1) = n\Gamma(n), \Gamma(0) = 1.$$



$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n\right) = \gamma.$$

Euler Eta Function

The Euler, or Dirichlet, eta function is

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad \text{Re}(s) > 0. \tag{1}$$

It is related to the zeta function by

$$\eta(s) = \left(1 - 2^{1-s}\right) \zeta(s).$$

Special values of the eta function are

$$\begin{aligned} \eta(0) &= 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}, \\ \eta(-1) &= 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}, \\ \eta(1) &= 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \cdots = \ln 2. \end{aligned} \tag{2}$$

$1 + 2 + 3 + \cdots = -\frac{1}{12}$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1. \tag{3}$$

Note that

$$\zeta(-1) = 1 + 2 + 3 + \cdots .$$

But, the Riemann zeta function is not defined for $s = -1$.

So, we can use $\eta(s)$ to analytically continue $\zeta(s) = \dfrac{\eta(s)}{1 - 2^{1-s}}$.

Setting $s = -1$, we obtain

$$1 + 2 + 3 + \cdots = \frac{\eta(-1)}{1 - 2^2} = -\frac{1}{12},$$

assuming that $\eta(-1) = \dfrac{1}{4}$, using Abel summation.

Leonhard Euler (1707-1783)

Euler (at 14) studied under Johann Bernoulli, graduated in 1723.

Went to St. Petersburg in 1727, Berlin in 1741, and back to St. Petersburg in 1766.

By 1730's - lost vision in right eye and blind by 1771.

866 books and papers - 228 after death. *Opera Omnia* - over 25,000 pgs

First appearance of e - letter to Goldbach in 1731.

Euler published *Introductio in Analysin infinitorum* - 1748

Euler's Formula, $e^{ix} = \cos x + i \sin x$.

Euler's Identity, $e^{i\pi} + 1 = 0$.

Euler's constant, γ

Euler's Polyhedral Formula, $V + F = E + 2$.

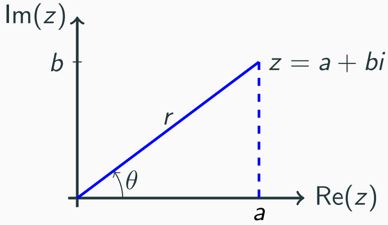
Amicable Numbers

- Recall Greeks knew 220 and 284; i.e., sum of proper factors of 220 = 284 and vice versa.
- Thabit ibn Qurra (836-901) discovered the next amicable pairs, for example 17296, 18416.
- Pierre Fermat rediscovered this pair in 1636.
- René Descartes discovered Qurra's pair 9,363,584 and 9,437,056 in 1638.
- 1747, Euler published [E100] giving 30 amicable pairs.
- By 1750 - Euler found 61 pairs!

Euler's Formula - Exponentiate $i\theta$.

- Complex numbers, polar form.
  $z = a + bi$, $a = r\cos\theta, b = r\sin\theta$

$$\begin{aligned} z &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta). \end{aligned}$$



- Exponential of imaginary number

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + i\theta - \frac{(\theta)^2}{2!} - i\frac{(\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{(\theta)^2}{2!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \cdots\right) \\ e^{i\theta} &= \cos\theta + i\sin\theta. \end{aligned}$$

Euler's Formula Applications $e^{i\theta} = \cos\theta + i\sin\theta.$

- $\theta = \pi,\ e^{i\pi} = -1$, or $e^{i\pi} + 1 = 0.$
- $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n\ e^{in\theta} = \cos n\theta + i\sin n\theta$ implies **de Moivre's Theorem**

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n\,.$$

- **Example:** $n = 2$

$$\begin{aligned}\cos 2\theta + i\sin 2\theta &= (\cos\theta + i\sin\theta)^2\\ &= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.\end{aligned}$$

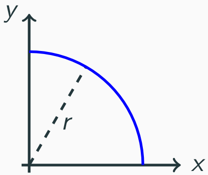
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\cos\theta.\end{aligned}$$

Rectification of a Circle - Recalling Calculus II

- Rectification = Finding arclengths
- The length of the curve $y = y(x)$:

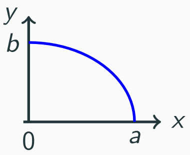
$$L = \int_a^b \sqrt{1 + y'^2}\, dx.$$

- **Example:** Circle: $x^2 + y^2 = r^2$
  $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$



$$\begin{aligned} L &= 4\int_0^r \sqrt{1 + \frac{x^2}{y^2}}\, dx \\ &= 4\int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= 4r\int_0^r \frac{dx}{\sqrt{r^2 - x^2}} = 4r \sin^{-1} 1 = 4r\left(\frac{\pi}{2}\right) = 2\pi r. \end{aligned}$$

Arclength of an Ellipse



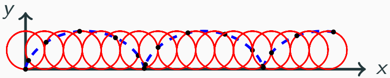
- **Example:** Ellipse
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad x \geq 0, y \geq 0.$

$$
\begin{aligned}
y &= \frac{b}{a}\sqrt{a^2 - x^2} \\
y' &= \frac{bx}{a\sqrt{a^2 - x^2}} \\
1 + y'^2 &= \frac{a^2 - k^2x^2}{a^2 - x^2}, \quad k = \frac{a^2 - b^2}{a^2} \\
L &= 4\int_0^a \sqrt{\frac{a^2 - k^2x^2}{a^2 - x^2}}\, dx.
\end{aligned}
$$

Historical Curves

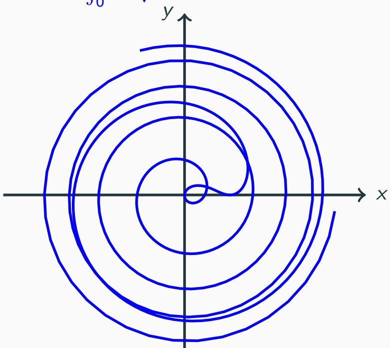
- 1609 - Kepler - Mars' orbit is an ellipse.
- 1659 - Pascal *Dimensions des lignes courbes de toutes les Roulettes.* Roulette curves.
- 1658 Proof by Wren published by Wallis in 1659 - On the rectification of the cycloid.
- 1676 - Newton - infinite series.
- 1742 - Maclaurin - expansion in eccentricities.
- 1691 - Jacob Bernoulli - parabolic spiral.

Cycloid, Parabolic Spiral, and Lemniscate



$x = r(t - \sin t),\ y = r(1 - \cos t)$

$L = \int_0^{2\pi} r\sqrt{2 - 2\cos t}\, dt = 8r$

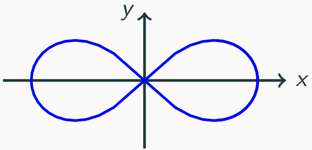


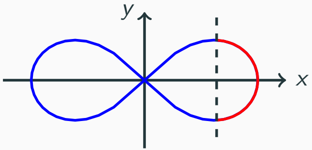
$(a - r)^2 = 2ab\theta$

$s = \int \sqrt{1 + \frac{r^2(a-r)^2}{a^2 b^2}}\, dr$

- **Example:** Lemniscate, $r^2 = \cos 2\theta$

$$L = 4\int_0^1 \frac{dr}{\sqrt{1 - r^4}}$$





Elastica

Sep 1694, Jacob Bernoulli

Oct 1694, Johann Bernoulli

Elliptic Functions

- Lemniscate integral leads to new functions, $u = \int_0^x \frac{dt}{\sqrt{1-t^4}}$.
- Compare to $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.
- Elliptic Integrals: $\int R(t, \sqrt{p(t)})\, dt$, R is rational function, $p(t)$ is polynomial of degree 3 or 4.
- Bernoulli (1694) - geometry, mechanics.
- Fagnano (1682-1766) - Doubling arc of lemniscate, 1718.
- Carl Friedrich Gauss (1777-1855) ~1800 studied inverse $x = sl(u)$ Doubly periodic functions

$$sl(u + 2\bar{\omega}) = sl(u), \quad sl(u + 2i\bar{\omega}) = sl(u)$$

$$\bar{\omega} = 2\int_0^1 \frac{dt}{\sqrt{1-t^4}} = 2.62205\ldots.$$

- Rediscovered by Niel Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851) in 1820's.

Addition Theorem for Circle

- **Example** Circle

$$\begin{aligned} \sin 2u &= 2\sin u \cos u \\ &= 2\sin u\sqrt{1-\sin^2 u} \end{aligned}$$

- Let $u = \sin^{-1} x$. Then,

$$\begin{aligned} 2u &= 2\int_0^x \frac{dt}{\sqrt{1-t^2}} \\ &= \sin^{-1}\left(2\sin u\sqrt{1-\sin^2 u}\right) \\ &= \sin^{-1}\left(2x\sqrt{1-x^2}\right) \\ 2\int_0^x \frac{dt}{\sqrt{1-t^2}} &= \int_0^{2x\sqrt{1-x^2}} \frac{dt}{\sqrt{1-t^2}}. \end{aligned}$$

Elliptic Integral Addition Theorem for Lemniscate

In 1718 Fagnano found formula for doubling arclength of lemniscate.

He solved differential equation

$$\frac{dt}{\sqrt{1-t^4}} = \frac{2dx}{\sqrt{1-x^4}}, \Rightarrow t = \frac{2x\sqrt{1-x^2}}{1+x^4}.$$

So, if the arclength of lemniscate is

$$\int_0^x \frac{dt}{\sqrt{1-t^4}},$$

then double the arclength is

$$\int_0^{\frac{2x\sqrt{1-x^2}}{1+x^4}} \frac{dt}{\sqrt{1-t^4}}.$$

Led Euler to write extensively on elliptic integrals starting in 1752.

Elliptic Integrals

- Study of Inversions
  Gauss 1790s - $\int \frac{dt}{\sqrt{1-t^3}}$,
  Abel 1823 (pub 1827)
  Jacobi 1829 book
- Legendre - two papers 1786 (40 yrs earlier)
  Legendre classified elliptic integrals into 3 cases,
  Produced 3 volumes, 1811-1816. Examples:

$$F(\phi) = \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad E(\phi) = \int_0^\phi \sqrt{1-k^2\sin^2\theta}\, d\theta.$$

- Riemann's geometric setting, Riemann surfaces, 1851 - torus.

Gauss' *AGM* - Arithmetic-geometric mean

- Gauss's constant $G = \frac{1}{AGM(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 0.8346268\dots$
- Between 1 and $\sqrt{2}$ is $\dfrac{\pi}{\bar{\omega}} = \dfrac{1}{G}$.
- Arithmetic mean $\dfrac{a+b}{2}$.
- Geometric mean $\dfrac{a}{g} = \dfrac{g}{b} \Rightarrow g = \sqrt{ab}$.
- $AGM(a,b)$ algorithm: Start with $a_0 = a, \quad b_0 = b$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \dots.$$

- Gauss - $AGM(1,\sqrt{2}) = \dfrac{\pi}{\bar{\omega}}$ to 11 decimal places.
- Led to study of general theory, modular functions, theta functions - Ramanujan (early 1900s).

Application of $AGM(a, b)$

Example: $AGM(1, 2)$. Start with $a_0 = 1, \quad b_0 = 2,$

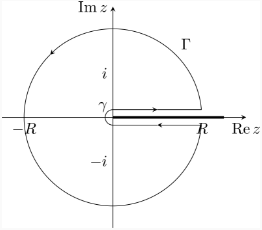
$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 0, 1, \ldots.$$

a_n	b_n
1.0000	2.0000
1.5000	1.4142
1.4571	1.4565
1.4568	1.4568
$\vdots$	$\vdots$

$$AGM(a, b) = \frac{\pi}{4} \frac{a + b}{K\left(\frac{a-b}{a+b}\right)}, \quad K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Complex Analysis

Fall 2023 - R. L. Herman



History of Complex Analysis

- Before 1600
  - Cardano 1545, quadratic
  - Bombelli 1572, cubic
  - Harriot 1600, quartic
  - Negative roots - false
  - Complex roots - impossible
- 1600s
  - Descartes, 1637, $a+b\sqrt{-1}$
  - Wallis 1685
  - Insights from geometry trigonometry, conics - justified
- 1700s
  - Bernoulli - integral transformation
  - Euler - Euler's formula, i
  - Gauss (1799, 1815) FTA, quadratic forms
  - Wessel (1797), Argand (1806) Geometric Visualization
  - Cauchy (1814, 1825) Complex Analysis
  - Riemann (1826-1866) Surfaces

Complex Numbers, $\mathbb{C}$

- $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
- Quadratic Equation,
  $ax^2 + bx + c = 0$,
  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
  If $b^2 - 4ac < 0$,
  complex conjugate roots.
- Cubics - Role was clearer
  $y^3 = py + q$
  $y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}$.

Example: $x^3 = 15x + 4$

$$\begin{aligned} x &= \sqrt[3]{2 + 11i} + \sqrt[3]{2 - 11i} \\ &= 2 + i + 2 - i = 4. \end{aligned}$$

Bombelli (1572)

$$\begin{aligned} (2+i)^3 &= (2+i)(4 + 4i + i^2) \\ &= (2+i)(3+4i) \\ &= 2 + 11i. \end{aligned}$$

Bernoulli's Transformations

- Johann Bernoulli (1712)

$$\begin{aligned}
\frac{1}{1+z^2} &= \frac{1}{(1+iz)(1-iz)} \\
&= \frac{1}{2}\left(\frac{1}{1-iz}+\frac{1}{1+iz}\right) \\
\int \frac{dz}{1+z^2} &= \frac{1}{2}\int\left(\frac{1}{1-iz}+\frac{1}{1+iz}\right)
\end{aligned}$$

- Note:

$$\int \frac{dz}{a+bz} = \frac{1}{b}\ln(a+bz).$$

So, integrating $(1+t^2)^{-1}$ gives

$$\tan^{-1} z = \frac{1}{2i}\left[\ln(1+iz) - \ln(1-iz)\right].$$

Examples: Tangent Identities

- Bernoulli studied $y = \tan n\theta$ in terms of $x = \tan\theta$.
- *Example:* $n = 2$
  $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.
- Let $y = \tan n\theta$, Then, $n\theta = \tan^{-1} y, \theta = \tan^{-1} x$.

$$\begin{aligned}
\int \frac{dy}{1+y^2} &= n\int \frac{dx}{1+x^2} \\
\ln\frac{y+i}{y-i} &= n\ln\frac{x+i}{x-i} \\
\frac{y+i}{y-i} &= A\left(\frac{x+i}{x-i}\right)^n
\end{aligned}$$

$A = (-1)^{n+1}$. Solve for y.

Ex: $n = 2$:

$$\tan 2\theta = \frac{2x}{1-x^2}.$$

Ex: $n = 3$:

$$\tan 3\theta = \frac{x^3-3x}{3x^2-1}.$$

Ex: $n = 4$:

$$\tan 4\theta = \frac{4x-4x^3}{x^4-6x^2+1}.$$

Ex: $n = 5$:

$$\tan 5\theta = \frac{x^5-10x^3+5x}{5x^4-10x^2+1}.$$

The Fundamental Theorem of Algebra

- Integration of $\frac{p(x)}{q(x)}$ for $p(x), q(x)$ polynomials
- Need Integration by parts. Assumes $q(x)$ can be factored
  - Fundamental Theorem of Algebra (FTA)
- Albert Girard (1629), *L'invention en algèbre*,
  First to claim there are always n roots of degree n polynomial.
- By 1750 - Any polynomial with real coefficients can be factored into real linear and quadratic factors.
- Nicolas II Bernoulli (1687-1759) gave a counterexample:
  $p(x) = x^4 - 4x^3 + 2x^2 + 4x + 4.$
- Euler found the factors:

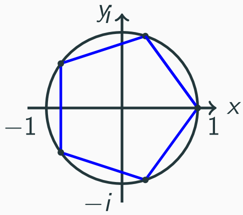
$$x^2 - \left(2 \pm \sqrt{4 + 2\sqrt{7}}\right) x + \left(1 \pm \sqrt{4 + 2\sqrt{7}} + \sqrt{7}\right)$$

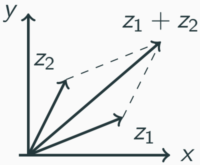
He gave incorrect proof for any quartic.
His was followed by proofs from d'Alembert and Gauss.

Roots of Unity

- Cotes, de Moivre, Euler
  - $x^n - 1 = 0$. Seems $x = \sqrt[n]{1}$.
  - $x = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$, $k = 0, 1, \ldots, n-1$. **Roots of unity.**
  - Geometric Interpretation
- Caspar Wessel, surveyor.
  - Complex number = point in the complex plane, 1797.
  - Also, proposed vectors.
- Argand, 1806, visual representation, operational (translation, rotation, reflection)
- Gauss also rediscovered, 1831.

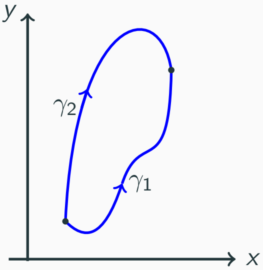
$e^{2k\pi i/5}, k = 0, 1, \ldots, 4$





Representing Complex Numbers

- Gauss (1777-1855) adopted "complex number," used i.
- Integration in $\mathbb{C}$-plane.
- $\int_\gamma \phi(z)\,dz$ is path independent for "nice" $\phi(z)$.
- Cauchy proved later, in 1814 talk, published 1827. - Now called *Cauchy's Theorem.*



Path Independence

$$\int_{\gamma_1} \phi(z)\,dz = \int_{\gamma_2} \phi(z)\,dz$$

Equivalently, for a simple, closed loop Γ, $\int_\Gamma \phi(z)\,dz = 0$.

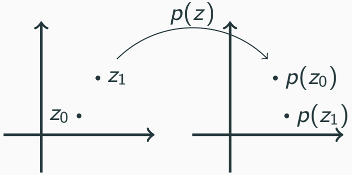
Augustin Louis Cauchy (1789-1857)

- Father of Complex Analysis
- Does $f(x) \to f(z)$ make sense?
- Integration along paths (1814) pub 1827.
- Cauchy's Theorem, Cauchy-Riemann Equations.
- Calculus of Residues (1826) - dealing with singularities.
- Convergence of infinite series.
- Use complex integration to integrate real functions.
- Path Independence (1825).
- Complex function of complex variable (1828).

Fundamental Theorem of Algebra I

Every polynomial $p(x)$ can be written as a product of linear complex factors. (Contains 1750 version)

- d'Alembert (1717-1783)
- **Lemma** $p(z_0) \neq 0$, $p(z) \neq$ constant. There exists a $z_1 = z_0 + w$ such that $|p(z_1)| < |p(z_0)|$ where $|a + bi| = \sqrt{a^2 + b^2}$.



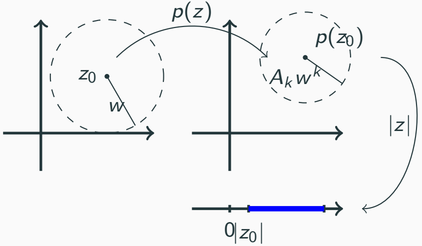
Proof

$$
\begin{aligned}
p(z) &= a_0 z^n + a_1 z^{n-1} + \cdots + a_n. \\
p(z_0 + w) &= a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_1 w + A_2 w^2 + \cdots + A_n w^n.
\end{aligned}
$$

Fundamental Theorem of Algebra II

$$p(z_0 + w) \quad = \quad a_0 z_0^n + a_1 z_0^{n-1} + \cdots + a_n + A_k w^k + \epsilon$$

Here $A_k w^k$ is the first nonzero, lowest power of w term and ϵ contains the higher powers terms in w and is small for large $|z|$.

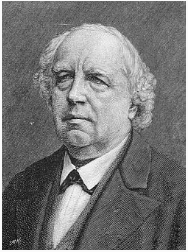


$\exists w$ such that $p(z_0) + A_k w^k$ is closer to the origin.

Let $p(z) \neq 0$. By the lemma, $\exists$ a point closer than z_0 to the origin.
$\therefore$ there exists a zero of $p(z)$.

Fundamental Theorem of Algebra III

- Gauss attempted several proofs.
- Karl Weierstrauss (1815-1897) - continuous functions on closed, bounded regions which assume maximum and minimum values.
- Gauss (1799 Thesis) considered curves $\text{Re}(p(z)) = 0$, $\text{Im}(p(z)) = 0$, $z = x + iy$.
- For $|z|$ large, $\text{Re}(a_0 z^n) = 0$, $\text{Im}(a_0 z^n) = 0$, curves are asymptotic to lines through the origin.
- Curves $\text{Re}(p(z)) = 0$, $\text{Im}(p(z)) = 0$, entering $|z| = r$ must come out and intersect inside disk. [See examples.]



Examples

Plotting $Re(p(z)$ and $Im(p(z))$, outside a large circle one gets alternating lines. Inside the circle they must intersect for $p(z) = 0$.

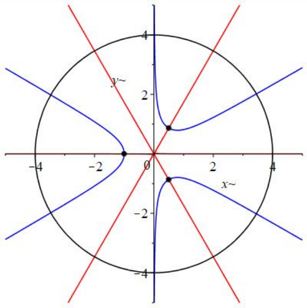


Figure 1: $p(z) = z^3 + 1$.

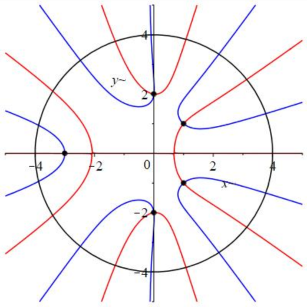


Figure 2: $p(z) = z^5 + z^4 + 10z^2 - 16z + 24 = (z - 1 - i)(z - 1 + i)(z^2 + 4)(z + 3)$.

Theory of Curves, $p(x, y) = 0$

- Descartes - linear/lines
  - quadratic/conics
- Newton - cubics
- Recall Bezout's Intersection Thm
  - ◁ Count multiplicities.
  - ◁ Intersection with ∞.
- 19th Century
  - Projective Geometry
    homogeneus coordinates
    Möbius, Plücker - 1830
  - Complex Numbers
    Gauss - FTA
  - Topological ideas
    - Riemann surfaces, 1850's

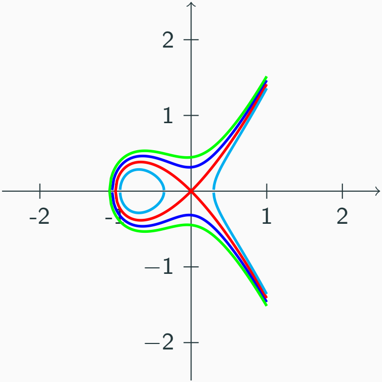
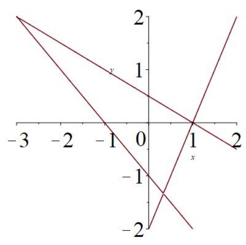


Figure 3: Cubic curves of form $y^2 = x^3 + x^2 + bx + 2b$

Cubic Curves

Consider products of linear factors or lines
$p(x, y) = (a_1x + b_1y + c_1)(a_2x + b_2y + c_2)(a_3x + b_3y + c_3)$

- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$



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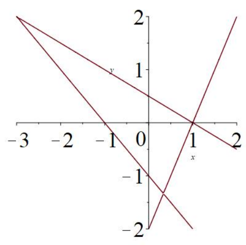
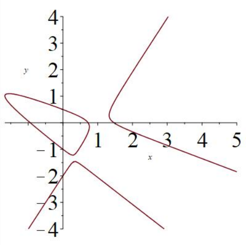
- Ex: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2)$
- Modify: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$





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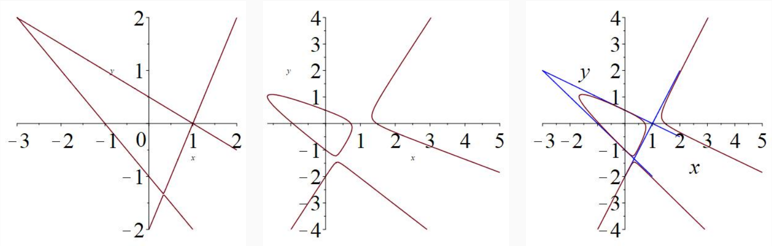
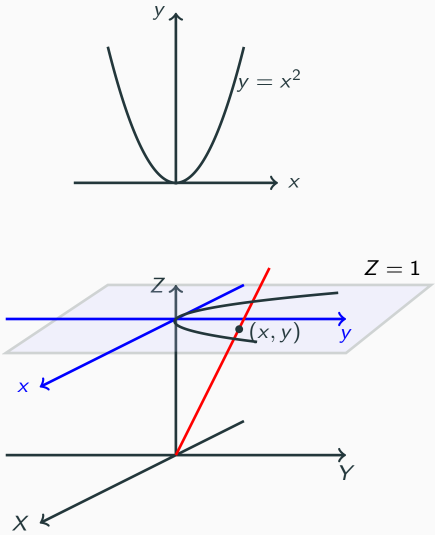
- Modify: $p(x, y) = (x + y + 1)(x + 2y - 1)(-2x + y + 2) + \frac{x^2}{2}$
- Branches go to **points at infinity**. Consider projective geometry.



Projective Geometry

- Homogeneous coordinates:
  $x = \frac{X}{Z}, y = \frac{Y}{Z}$.
- Introduced by Möbius, Pücker.
- **Example:** $y = x^2$ gives $X^2 = YZ$



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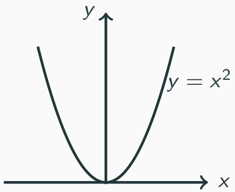
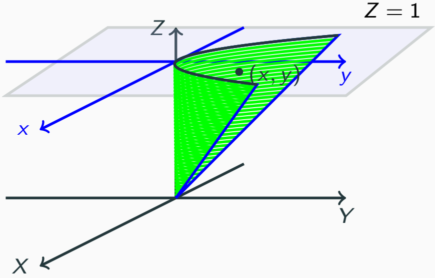
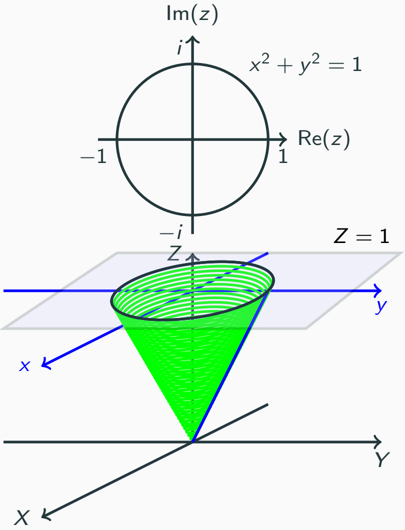
- **Example:** $y = x^2$ gives $X^2 = YZ$
- Lines thru origin (projective plane).
- $X^2 = YZ$ is a "cone"
- Points at Infinity: $Z = 0 \Rightarrow X = 0$,
- These points, $[0, Y, 0]$, lie on horizon.





Projective Plane and Complex Numbers

- **Example:** $x^2 + y^2 = 1$
- Projective curve: $X^2 + Y^2 = Z^2$
- Pts at infinity,
  $Z = 0 \Rightarrow X^2 + Y^2 = 0.$
- In $\mathbb{C}$, Circular pts at infinity.
  $X = 1, Y = i : I_1 = (1, i, 0)$
  $X = 1, Y = -i : I_2 = (1, -i, 0)$



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  $X = 1, Y = i : I_1 = (1, i, 0)$
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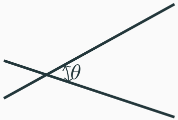
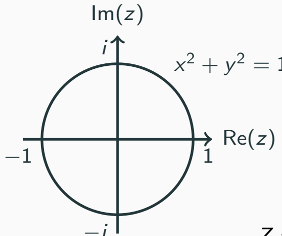
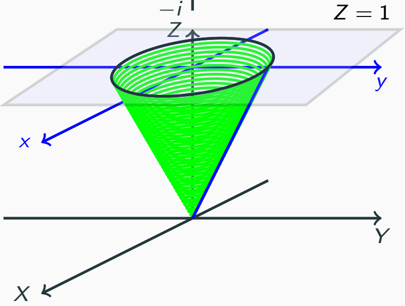
- Edmund Laguerre (1834-1886)
  - Angles, $\theta = i \log R$.
- R - Cross ratio

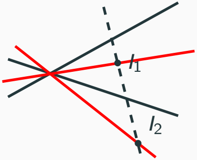


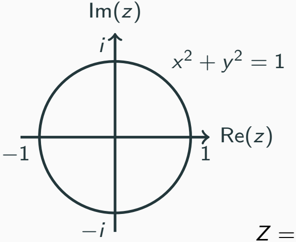


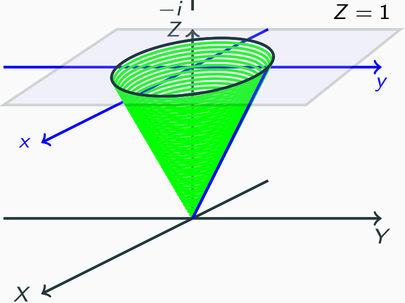


Projective Plane and Complex Numbers

- **Example:** $x^2 + y^2 = 1$
- Projective curve: $X^2 + Y^2 = Z^2$
- Pts at infinity,
  $Z = 0 \Rightarrow X^2 + Y^2 = 0.$
- In $\mathbb{C}$, Circular pts at infinity.
  $X = 1, Y = i : I_1 = (1, i, 0)$
  $X = 1, Y = -i : I_2 = (1, -i, 0)$
- Edmund Laguerre (1834-1886)
  - Angles, $\theta = i \log R$.
- R - Cross ratio



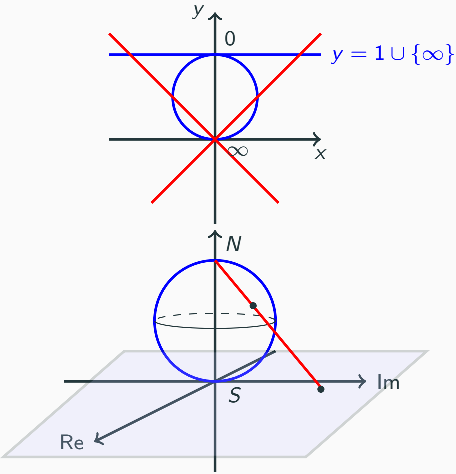




Stereographic Projection

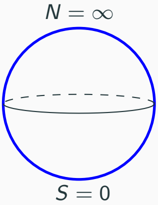
What do complex curves look like?

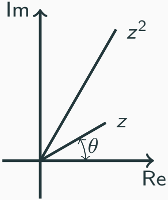
- Projective lines:
  Lines thru origin
  Topologically looks like a circle,
  S^1, after adding point at infinity
- Extend to $\mathbb{C}$ - topologically, S^2
- Stereographic Projection
  Connect pts in $\mathbb{C}$ to North Pole.
- N mapped to pt at ∞.
- Möbius (1790-1868) Image of circle = circle.



Mapping Functions onto Surfaces

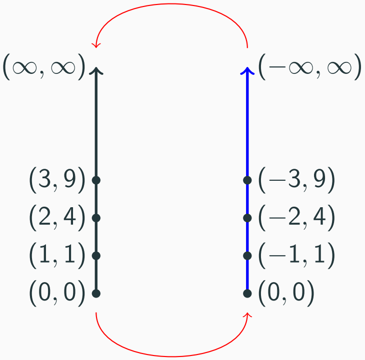
- Riemann (1826-1866)
  - Riemannian manifolds
  - Curved spaces - Gauss
  - Complex Analysis - Riemann surfaces [Cauchy (1788-1857)]
  - Number theory - $\zeta(s)$
- Start with a Sphere
- Extend $f : \mathbb{C} \to \mathbb{C}$ to $g : S^2 \to S^2$.
- Complex function, $f(z) = z^2$
  Let $z = re^{i\theta}$.
  [$\theta =$ argument, $r =$ modulus, $|z|$.]
  Then, $f(z) = r^2 e^{2i\theta}$.

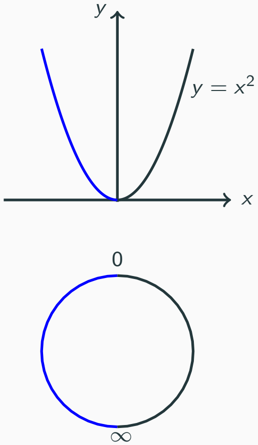




Real Function, $y = f(x)$, Mapped to S^1

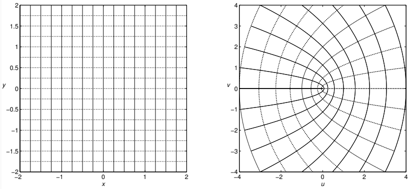
- **Example** $f(x) = x^2$.

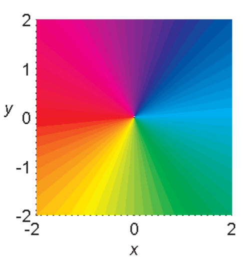


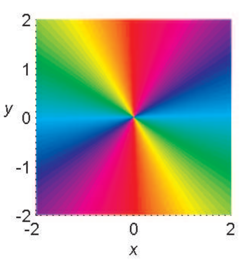


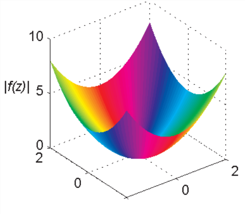
Visualizing Complex Functions: $w = f(z) = u(x, y) + iv(x, y)$

- What is $f(z) = z^2$?
- Map xy-plane to uv-plane.
- $(x + iy)^2 = x^2 - y^2 + 2ixy$.
- $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$
- Domain Coloring



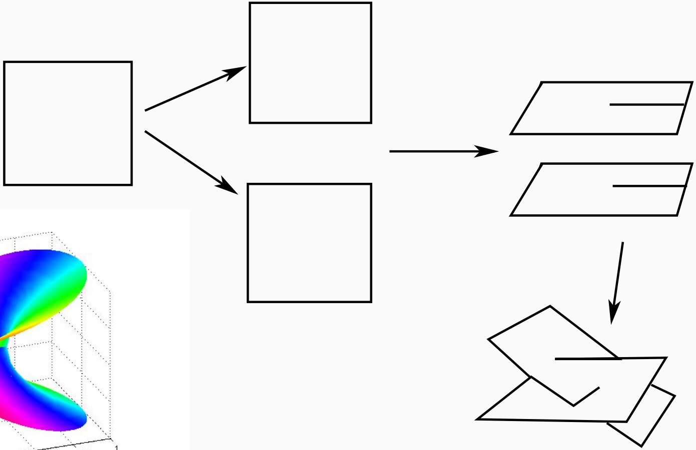


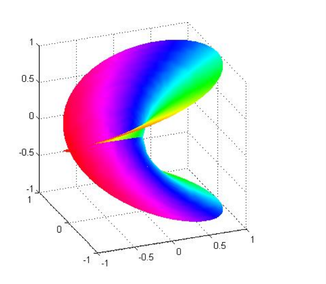




Riemann Surfaces and the Square Root Function

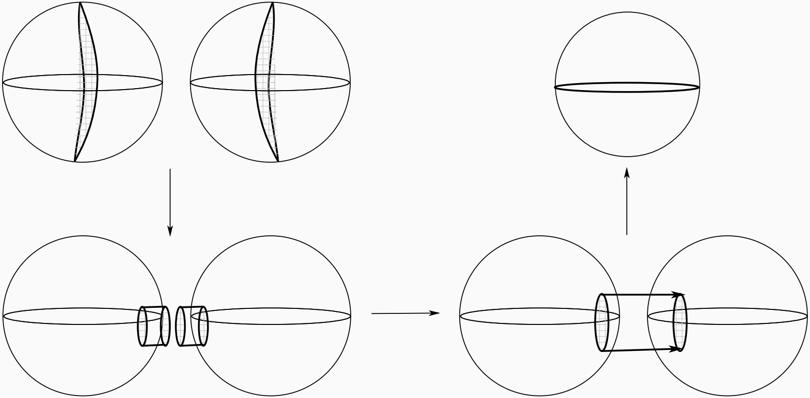
- Riemann Sheets - Two copies of $\mathbb{C}$. Riemann's Dissertation, 1851.
- **Example:** $w = \sqrt{z}$.





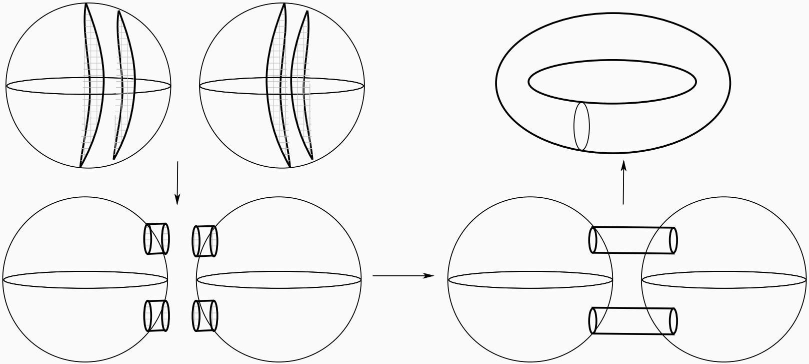
Mapping $f(z) = z^2$ to S^2.

- **Example:** $f(z) = z^2$.

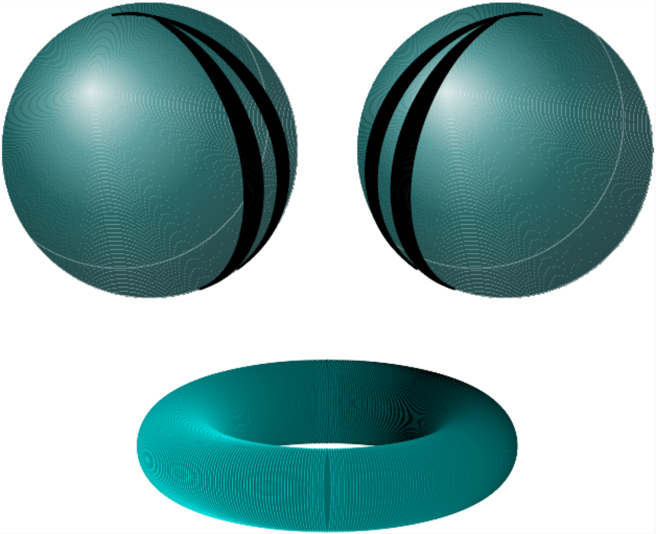


Riemann Surfaces and Elliptic Integrals

- **Example:** $\int_0^x \frac{dz}{\sqrt{z(z-a)(z-b)(z-c)}}$.
- $w^2 = z(z-a)(z-b)(z-c)$
- Beginning of topology.

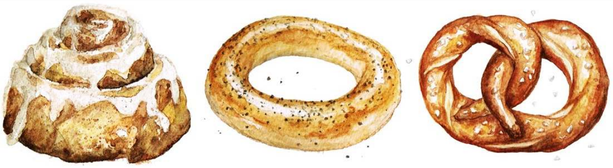


Merging Two Cut Riemann Spheres



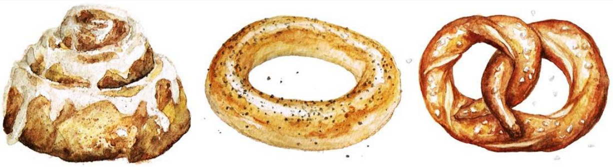
Beginnings of Topology ...

Figure 4: Genus $g = 1, 2, 3$.



Beginnings of Topology ...

Figure 4: Genus $g = 1, 2, 3$.



Women in Mathematics in the 1800s

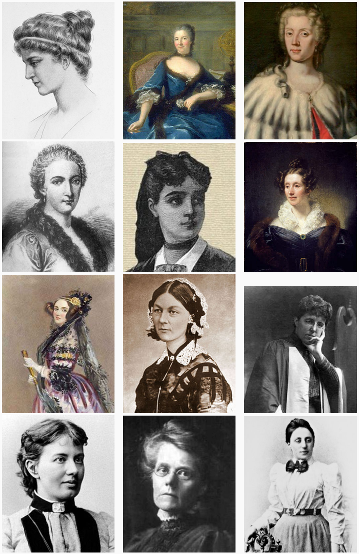
Fall 2023 - R. L. Herman





Famous Women Mathematicians Before 1900

- Hypatia of Alexandria (c. 350-415)
- Émilie du Châtelet (1706-1749)
- Laura Bassi (1711-1788)
- Maria Agnesi (1718-1799)
- Sophie Germain (1776-1831)
- Mary Fairfax Somerville (1780-1872)
- Ada Lovelace (1815-1852)
  (Augusta Byron, Countess of Lovelace)
- Florence Nightingale (1820-1910)
- Charlotte Angas Scott (1848-1931)
- Sofia Kovalevskaya (1850-1891)
- Alicia Boole Stott (1860-1940)
- Amalie 'Emmy' Noether (1882-1935)



Émilie du Châtelet (1706-1749)

- Gabrielle-Émilie Le Tonnelier de Breteuil
- Father - official at the Court of Louis XIV at Versailles.
- Husband - Marquis Florent-Claude Chastellet, military man, governor of Semur-en-Auxois in Burgundy.
- Lovers: Pierre Louis Moreau de Maupertuis (1698-1759), Alexis Clairaut (1713-1765) and François-Marie Arouet (Voltaire) (1694-1778).
- Wrote on Newton, Leibniz, and the propagation of fire.
- Translation of the *Principia* into French.
- Debated Euler and others over *vis viva*, "living force," or kinetic energy Σmv^2.

Figure 1: Émilie du Châtelet

Laura Bassi and Marie Agnessi

- Laura Bassi (1711-1788)
    - 1st female physics professor. Studied Newton, electricity.
    - Second in the world: Ph.D., 1732.

        1st - philosopher Elena Cornaro Piscopia, 1678.
    - First woman: doctorate in science.
- Maria Agnessi (1718-99).
    - First woman: mathematics handbook.
    - First woman appointed: mathematics professor.
    - First book on both differential and integral calculus
    - Witch of Agnesi curve.

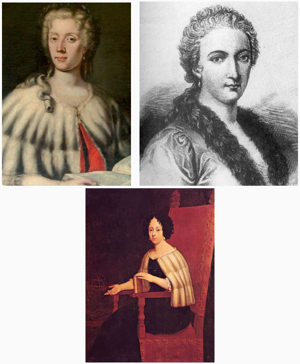


Figure 2: Bassi, Agnessi, Piscopia.

Marie Sophie Germain (1776-1831)

- Self-taught, French revolution
- 1794 - École Polytechnique opened
  - for men, obtained notes
  Signed HW - Monsieur Le Blanc
- Joseph-Louis Lagrange (1736-1813)
- Adrien-Marie Legendre (1752-1833)
- Gauss (1777-1855) - letters 1804-12; saved his life.
- Germain Primes - If p is prime, then so is $2p+1$ Ex: $5 = 2(2)+1$, $7 = 2(3)+1$, ~~$9 = 2(4)+1$~~
- Elasticity work did not get her name on Eiffel Tower.

Figure 3: Sophie Germain

- Fermat's Last Theorem
- Chladni Plates, elasticity.
- Competitions 1811, 1813, 1815.

Mary Fairfax Somerville (1780-1872)

- Mathematics and astronomy
- Wrote books
- Jointly - the first female member of the Royal Astronomical Society with Caroline Herschel.
- First to sign petition to Parliament to give women the right to vote.
- experiments to explore the relationship between light and magnetism
- Translated/expanded Laplace's work, 1831, *The Mechanism of the Heavens*.
- First Geography text, 1848.

Figure 4: Mary Sommerville.

Ada Lovelace (1815-1852)

- Daughter of Lord Byron, (poet, died 1824) and
- Mathematician Anne Isabelle Milbanke, self-named as "princess of parallelograms."
- She wrote papers and first computer programs.

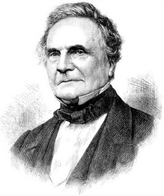
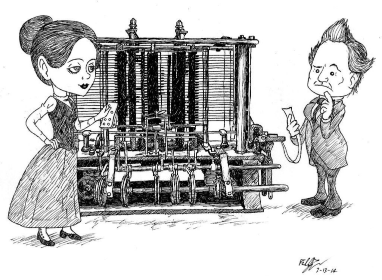


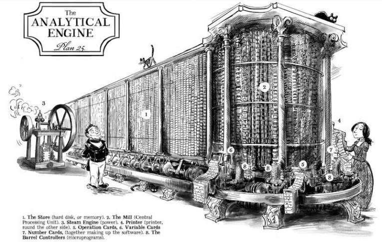
Figure 5: Charles Babbage

Figure 6: Augusta Ada Byron

Ada Lovelace and Charles Babbage

- Charles Babbage (1791-1881)
    - English mathematician, philosopher, engineer.
    - 1833 Difference Engine.
    - 1844 Analytical Engine.
    - Designed, never Built.
- Lovelace first algorithm for a machine.
- 1842-1843, Translated an article Luigi Menabrea on the engine. added notes containing first computer program.
- Loops, recursion - Bernoulli numbers, systems of linear equations.
- 1980's - Ada, programming language.





Florence Nightingale (1820-1910)

- Crimean War (1853-1856)
- Supervised nurses.
- Studied under famous mathematicians.
- Used statistics - mortality rates
- Pioneer in data visualization, polar area diagrams.
- National heroine, 1883 recipient of the Royal Red Cross, and later others.

Figure 7: Florence Nightingale

Sofia Kovalevskaya (1850-1891)

- Born Sofya Vasilyevna Korvin-Krukovskaya in Moscow.
- Education in Europe.
- Teachers - Hermann von Helmholtz, Gustav Kirchhoff and Robert Bunsen.
- Advisor - Weierstrass (1874) - 3 papers PDEs, elliptic integrals, Saturn's rings.
- 1st woman to get doctorate in math outside Italy. - not enrolled! 1874.
- 1883 Teaching position, U. Stockholm.
- 1889 1st to hold chair in European university since Laura Bassi and Maria Agnessi.

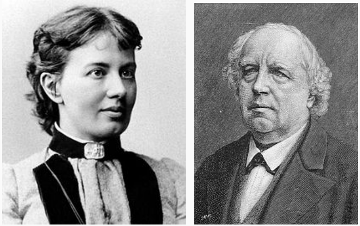


Figure 8: Sofia Kovalevskaya and Karl Weierstrass (1815-1897)

Sofia Kovalevskaya (1850-1891)

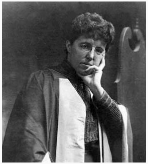
- Light waves, tops, wrote books.
- 1886 - French Competition - spinning tops.
- 1889 - Swedish Academy of Science Prize Chebyshev got her membership in Imperial Academy of Sciences
- 1891 - On vacation, Influenza - pneumonia.
- Cauchy–Kovalevskya Theorem: local existence and uniqueness theorem for Cauchy problem in PDEs.
- Kowalevski top - a symmetric top with a particular ratio of the moments of inertia: $I_1 = I_2 = 2I_3$.

Figure 9: Sofia Kovalevskaya

Turn of Century - Charlotte Scott and Alicia Stott

Charlotte Angas Scott (1848-1931)

- One of 1st woman to obtain a doctorate in England.
- Studied under Arthur Cayley.
- Algebraic curves of degree higher than two.
- 1885 - 1st mathematician at Bryn Mawr College, dept head.
- A founder of AMS.



Alicia Boole Stott (1860-1940)

- Parents: George Boole (1815-1864) and Mary Everest Boole (1832-1916).
- Four-dimensional polytopes.
- Exactly six regular polytopes in four dimensions
- Worked with Harold Coxeter,(1907–2003).

Amalie 'Emmy' Noether (1882-1935)

- German mathematician
- Abstract algebra - theories of rings, fields, and algebras.
- Noether's theorem - connects symmetry and conservation laws.
- Mathematical Institute of Erlangen, 1908–1915 - without pay.
- University of Göttingen, 1915-1933, First four years lecturing under Hilbert's name.
- Bryn Mawr - 1933-5.
- Lectured at Institute for Advanced Study in Princeton.

Figure 10: Emmy Noether

That’s *Not* All Folks! (Click on the Links)

- Ellen Amanda Hayes (1851-1930)
- Christine Ladd-Franklin (1847-1930)
- Elizaveta Fedorovna Litvinova (1845-1919)
- Ada Isabell Maddison (1869-1950)
- Helen Abbot Merrill (1864-1949)
- Mary Frances Winston Newson (1869-1959)
- Mary Emily Sinclair (1878-1955)
- Pauline Sperry (1885-1967)
- Anna Johnson Pell Wheeler (1883-1966)
- Grace Chisholm Young (1868-1944)







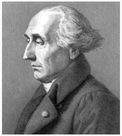



Non-Euclidean Geometry and Group Theory

Fall 2023 - R. L. Herman








Euclidean Geometry

- 300 BCE - Euclid's *Elements*
- Five Postulates.
- 5th Postulate - not needed in first 28 propositions.

If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

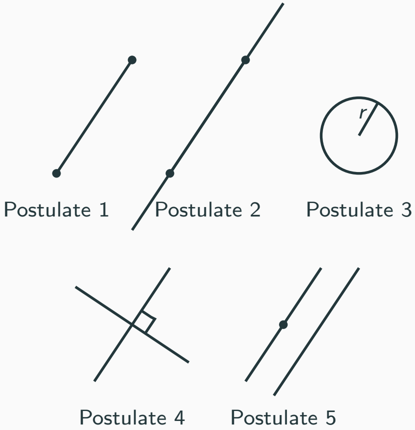


Figure 1: Euclid's 5 Postulates.

Statements of Parallel Axiom in Text

P$_1$ For each straight line L and point P outside L there is exactly one line through P that does not meet L.

Equivalent statements

> The angle sum of a triangle $= \pi$. - Euclid.
>
> The locus of points equidistant from a straight line is a straight line. - al-Haytham.
>
> Similar triangles of different sizes exist. - Wallis

Saccheri (1733) - provided two alternatives to arrive at proof by contradiction.

P$_0$ There is not line through P that does not meet L.

P$_2$ There are at least two lines through P that do not meet L.

Parallel Postulate

- Proclus (410-485) Equivalent postulate. Revived as Playfair axiom.
- William Ludlam (1785):
  Two straight lines, meeting at a point, are not both parallel to a third line.
- John Playfair, *Elements of Geometry* (1795):
  Playfair's axiom: Two straight lines which intersect one another cannot be both parallel to the same straight line.
- Many false attempts to prove based on other four postulates.
- 1663 John Wallis "To each triangle, there exists a similar triangle of arbitrary magnitude."
- Giralomo Saccheri (1667-1733) Assume 5th postulate false and get contradiction.
- Used assumption - lines are infinite. Led to contradiction of $\mathbf{P}_1$, almost $\mathbf{P}_2$.
- d'Alembert, 1767 - "The scandal of elementary geometry."

Spherical Geometry

- Lines = geodesics, Lie on great circles.
- Euclidean triangles, $a + b + c = \pi$.
- Spherical triangles, $a + b + c > \pi$.
- Thomas Harriot (1560-1621), astronomy, mathematics, and navigation
- Johann Heinrich Lambert (1726-1777)
    - General properties of map projections.
    - hyperbolic functions
    - π is irrational
    - optics

$a + b + c = \pi + \frac{A}{R^2}$.

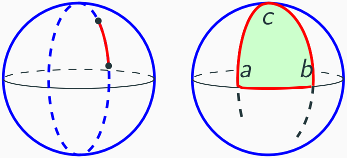


Figure 2: Harriot and Lambert.

Other Geometries

- Ferdinand Karl Schweikart (1780) Astral geometry, sum of three angles of a triangle is less than two right angles.
- Wrote to Gauss, 1818, via student Christian Ludwig Gerling (1788-1864).
- Franz Taurinus (1784-1854), Schweikart's nephew. Proposed geometry on a sphere of imaginary radius, logarithmic-spherical geometry.
- 1826, hyperbolic law of cosines in *Geometriae prima elementa.*
- Wrote to Gauss. after being encouraged, he sent copies of his works with no reply.
- Later he burned copies of his book.

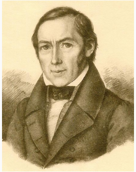


Figure 3: Gerling and Schweikart

Parallel Postulate Revisited

- Carl Friedrich Gauss (1777-1885) started on it in 1799; was convinced it was independent of first 4.
- Discussed with Farkas Bolyai (1775 - 1856) - told his son no to waste his time.
- János Bolyai (1802-1860) - Believed a non-Euclidean geometry existed.
- Nikolai Lobachevsky (1792-1856) - independently 1840 new 5th postulate: There exists two lines parallel to a given line through a given point not on the line.
  Developed trig identities, hyperbolic geometry.

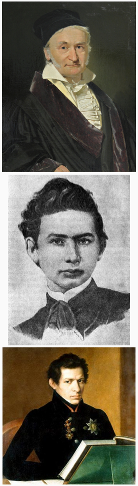
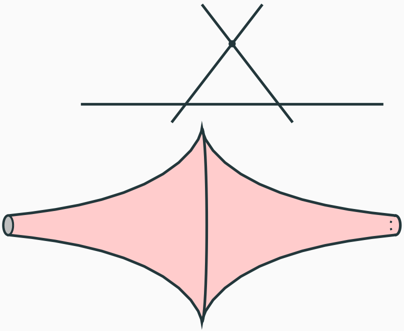


Figure 4: Gauss, Bolyai, Lobachevsky

Riemannian Geometry

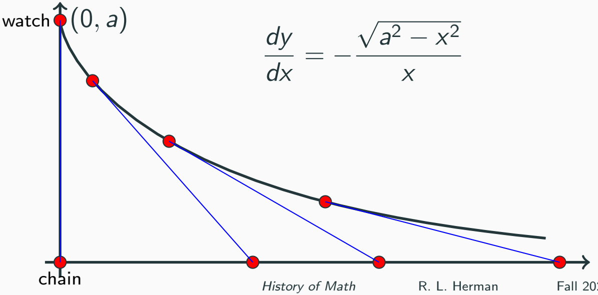
- Georg Friedrich Bernhard Riemann (1826-1866)
  - Published in 1868 Lecture
  - Spherical geometry
  - Riemannian geometry → differential geometry
  - Every line through a point not on a given line meets the line.
- Eugenio Beltrami (1835-1900) Published interpretations of non-Euclidean geometry - introduced pseudosphere in 1868 using a **tractrix**.



tractrix $(a(t - \tanh t), a\,\mathrm{sech}\, t)$

Aside: The Tractrix

- Claude Perrault [brother Charles author of *Cinderella*, *Puss-in-Boots*] in 1693, Paris, placed a watch in the middle of a table and pulled its chain along the edge of the table. What was the curve traced out ?
- Studied by Newton (1676), Huygens (1692) and Leibniz (1693). Euler gave complete theory in 1788. [Am. Math. Monthly, 72(10) (1965), 1065-1071.]
- Huygens coined name from Latin, *tractus*.



$$\frac{dy}{dx} = -\frac{\sqrt{a^2 - x^2}}{x}$$

Curvature

- $k = 0,\ k > 0,\ k < 0.$
- sums of angles of triangles $a + b + c - \pi = kA.$

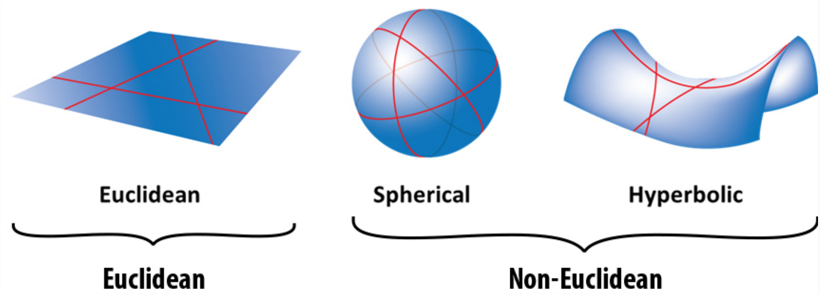


Figure 5: Surfaces of Constant Curvature.

Hyperbolic Geometry

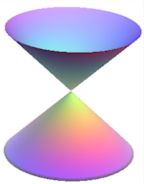
- Sphere

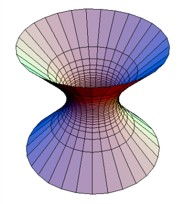
$$x^2 + y^2 + z^2 = \text{const}$$

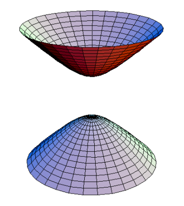
- Modify

$$x^2 + y^2 - z^2 = K$$

- $K = 0$, $z^2 = x^2 + y^2$. Cones.
- $K = 1$, $x^2 + y^2 - z^2 = 1$.
  Hyperboloid of one sheet
- $K = 1$, $z^2 - x^2 - y^2 = 1$.
  Hyperboloid of two sheets.





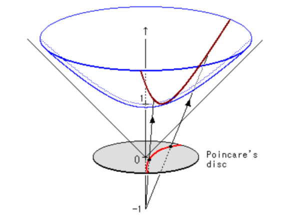


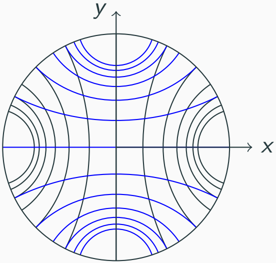
Beltrami-Poincaré Model

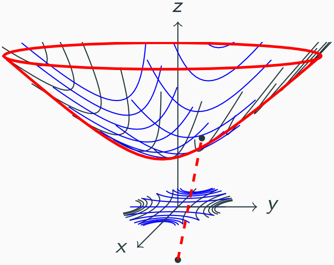
- Poincaré's Disks

$$(x, y, z) = (c \cosh t, \sinh t, \sqrt{1 + c^2} \cosh t)$$

- Stereographic Projection thru $(0, 0, -1)$ to $z = 0$: $(x, y, z) \rightarrow \frac{(x,y)}{1+z}$.
- Hyperbolic geometry.



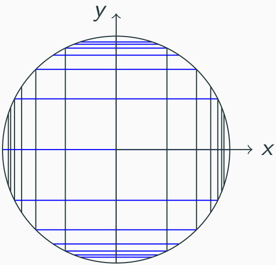


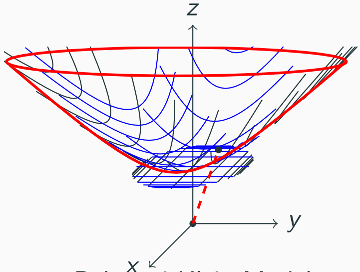


Beltrami-Poincaré Model

Beltrami-Klein Model

- Stereographic Projection thru $(0,0,0)$ to $z=1:(x,y,z)\to\frac{(x,y)}{z}$.
- Klein's Disks
  Projection to $(0,0,1)$

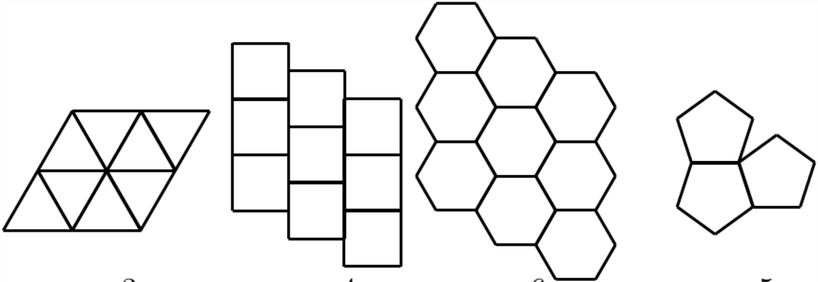




Beltrami-Klein Model

Tiling the Plane

One can tile the plane with a single polygon with sides 3, 4, and 6. However, one cannot fit pentagons together. As seen below, the angles do not allow for a fit. For large *n*, the interior angles are too small.



$n = 3$ $n = 4$ $n = 6$ $n = 5$

Other Tilings

- Johannes Kepler (1571-1630)
  - Studied Tilings
  - *Harmonicae Mundi* (Harmony of the World).
  - Planned in 1599.
  - Published 1619 - delay by Tycho Brahe to look as orbit of Mars.
- Roger Penrose (1931-)
  - 2020 Nobel Prize
  - 70's Inspired by Tilings - Penrose tilings. In 80's found in nature.
  - and M. C. Escher (1889-1972)
  - Circle Limit - Tiling Hyperbolic Plane.
- Others - Polyominoes and Pentominoes.

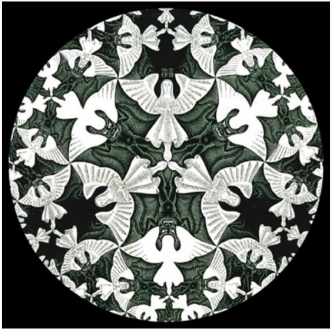


Figure 6: Circle Limit IV

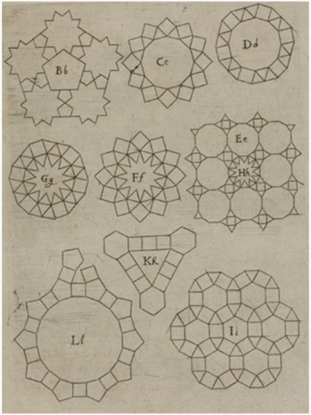
Hyperbolic Tessellations

- Tessellation = cover plane by tiles, no tiles overlap and no space empty.
- Schläfli symbol: $\{n, m\}$,
  $n =$ number of sides on the tile,
  $m =$ number of tiles that meet at a vertex.
  Euclidean: $\dfrac{1}{n} + \dfrac{1}{m} = \dfrac{1}{2}$,
  Hyperbolic: $\dfrac{1}{n} + \dfrac{1}{m} < \dfrac{1}{2}$.

Figure 7: Circle Limits I-IV.

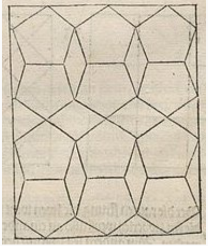
Kepler's Tiling

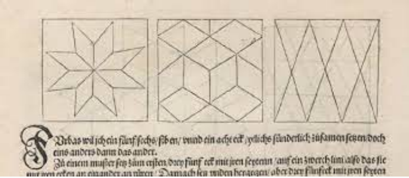
- 1619 Johannes Kepler published the first classification regular polygon tilings, Book II of *Harmonices Mundi*.
- First catalog of the 11 uniform tilings of the plane.
- See online discussion.



Albrecht Dürer's Tilings

- 1525, *Underweysung der Messung mit dem Zirckel und Richtscheyt* (A Course in the Art of Measurement with Compass and Ruler), the Painter's Manual.
- Constructed various curves and regular polygons with a ruler and compass.
- Illustrates three regular tilings (squares, triangles and hexagons), octagon tiling, uniform tiling with a six pointed star pattern, and rhomb tiling.





Aperiodic Tiling

- Non-periodic tiling that does not contain arbitrarily large periodic regions.
- 1964 Robert Berger, 20,426 Wang tiles.Later reduced his set to 104.
- 1966 Hans Läuchli, 40 Wang tiles.
- 1967 Raphael M. Robinson, 104.
- 1968, Donald Knuth, 96.
- 1971, Robinson, 6 tiles.
- 1974 Penrose, 6 tiles. P1.

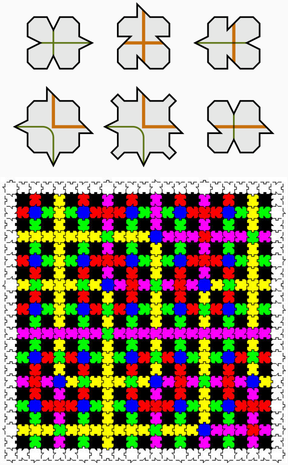
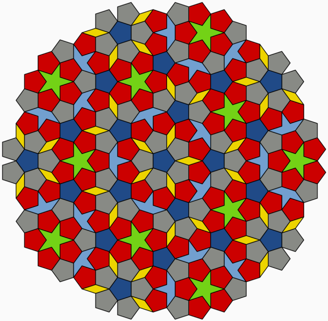


Figure 8: Robinson's tiles.

Penrose Tiling P1

- Penrose's first tiling, 1978.
- Uses pentagons and three other shapes:
  a five-pointed star,
  a "boat"
  and a "diamond".
- Need matching rules specifying how tiles meet each other to give non-periodic tilings.
- There are three different types of matching rules for the pentagonal tiles.
- Treating these as different prototiles gives a set of six.
- Indicate the three different pentagonal tiles using different colors.



Penrose Tiling P2

- Penrose introduced aperiodic tiling with two tiles.
- P2: Used quadrilaterals, "kite" and "dart." Can be combined to make a rhombus.
- Need matching rules.
- A. Color the vertices and require that adjacent tiles have matching vertices.
- B. Use circular arcs to constrain the placement of tiles. The patterns must match at these edges.

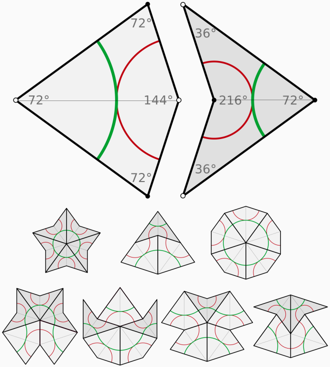


Figure 9: Penrose Kites and Darts. Use to create shown shapes: star, ace, sun, king, jack, queen, deuce.

Penrose Tiling P3

- Rhomus tiles.
- Thin rhombus with angles of 36, 144, 36, and 144 degrees.
- Thick rhombus with angles of 72, 108, 72, and 108 degrees.
- Tiles must be assembled such that the curves on the faces match in color and position across an edge.

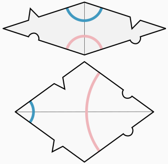


Figure 10: Thin and thick rhombs.

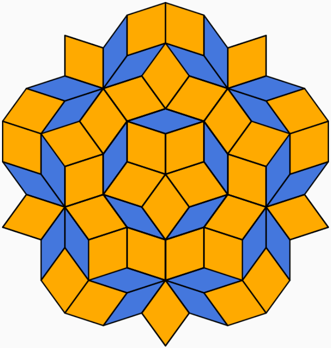
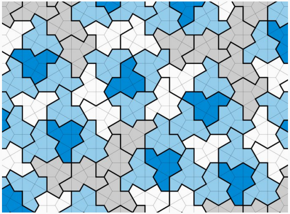
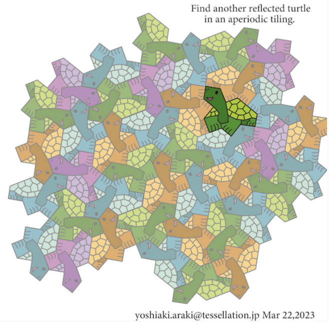


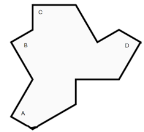
Figure 11: Penrose Rhombus Tiling.

Einstein Tiles

- The search for tiling with one tile.
- David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss, `https://arxiv.org/abs/2303.10798`, March 20, 2023.
- *An Aperiodic Monotile*
- Proved that "the hat" is an aperiodic monotile, called an einstein (one stone).
- Involves the hat and its mirror image, noted by Yoshiaki Araki.

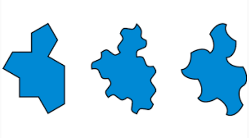


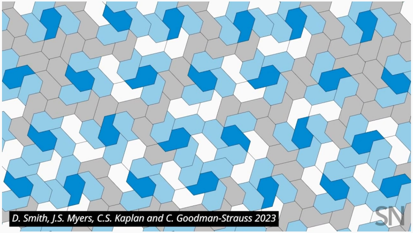




Spectre Tiles

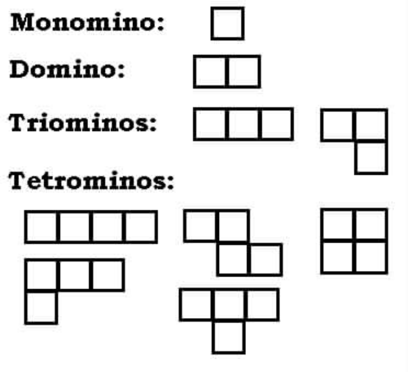
- David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss found a new tile `https://arxiv.org/abs/2305.17743`, May 28, 2023.
- *A Chiral Aperiodic Monotile*
- Is not accompanied by its reflection, a "vampire einstein."

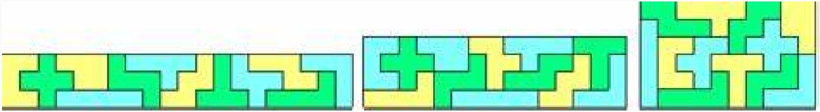




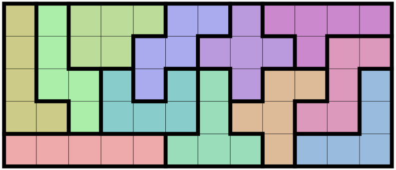
Polyominoes

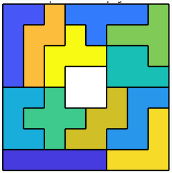
- A plane geometric figure formed by joining one or more equal squares edge to edge.
- Used in puzzles since at least 1907.
- Name *polyomino* invented by Solomon W. Golomb in 1953.
- Some types: domino, triomino, tetromino, pentomino, etc,
- Wikipedia page

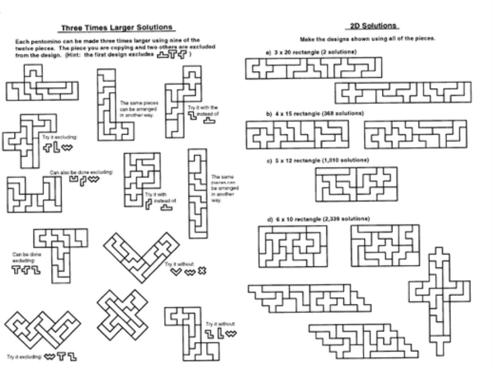




Pentomino Puzzles







Return to the Quintic

- In the meantime, the search for solving the quintic continues.
- The general quintic cannot be solved algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions.
- Malfatti (1731-1807) was the first to solve a solvable quintic using a resolvent of sixth degree, 1771.
- The general quintic was solved in terms of Jacobi theta functions by Hermite in 1858. See story.
- Our story begins with Gauss and Lagrange.

Carl Friedrich Gauss (1777-1855)

- *Disquisitions Arithmeticae* - 1801
- Summary and Extension on Number Theory.
- Initiated finite Abelian groups.
  - closed, identity, inverse, associative,
  plus commutative.
- Proved Fermat's Little Theorem: If p is prime, then for any integer a, $a^{p-1} \equiv 1 \mod(a)$.
- Represented integers as quadratic forms, like Fermat Primes ($4n + 1 = x^2 + y^2$.) for x and y integers.
- Binary quadratic forms - $ax^2 + bxy + cy^2$ - for a, b, c integers.
  - composition has properties of an abelian group.
- Did not have a general theory of groups.

Figure 12: List of things named after Gauss

Joseph Louis Lagrange (1736-1813)

- Born in Turin, Italy.
- Professor at 19 (artillery school).
- 1766 Frederick the Great (Prussia) sought a great mathematician to replace Euler.
- Lagrange went to Berlin for 20 yrs.
- Invited by Louis XVI to Paris, 1786.
- 1793, Reign of Terror, saved by Lavoisier.
- 1795 - established dept. at École Normal.
- 1797 - established dept. at École Polytechnique.
- Napoleon made him senator, count, and he received many other honors.
- Sought solution of quintic by studying cubic and quartic.

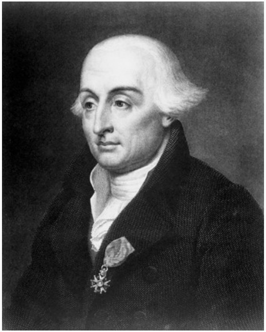
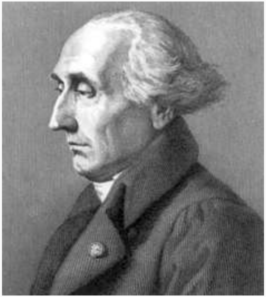


Figure 13: List of things named after Lagrange

Resolvents

- Consider $x^3 + nx + p = 0$. Let $x = y - \frac{n}{3y}$.
- Yields 6^{th} degree polynomial,
  $y^6 + py^3 - \frac{n^3}{27} = 0$, the resolvent.
- Let $r = y^3$, $r^2 + pr - \frac{n^3}{27} = 0$.
- Has roots r_1, r_2, where $r_2 = -\left(\frac{n}{3}\right)^3 \frac{1}{r_1}$.
- Then, $x = \sqrt[3]{r_1} + \sqrt[3]{r_2}$
- Cardano got this real root but did not seek complex solutions.
- Lagrange knew there should be 3 roots.
  $\sqrt[3]{r}, \omega\sqrt[3]{r}, \omega^2\sqrt[3]{r}$, where ω is a cube root of unity, $\omega^3 = 1$. Then,

$$
\begin{aligned}
x_1 &= \sqrt[3]{r_1} + \sqrt[3]{r_2} \\
x_2 &= \omega\sqrt[3]{r_1} + \omega\sqrt[3]{r_2} \\
x_3 &= \omega^2\sqrt[3]{r_1} + \omega^2\sqrt[3]{r_2}
\end{aligned}
$$



Permutation of Roots

- Lagrange then wrote roots of the resolvent
  $y = x_i + \omega x_j + \omega^2 x_k, \quad i, j, k = 1, 2, 3, \quad i \neq j \neq k.$
- 3!=6 permutations of cubic roots.
- In $y^6 + py^3 - \frac{n^3}{27} = 0$, the coefficients of y^5, y^4, y^2, y are
  $x_1 + x_2 + x_3$, $p = x_1x_2x_3$, and $\dfrac{n^3}{27} = \dfrac{(x_1x_2 + x_1x_3 + x_2x_3)^3}{27}$.
- Resolvent coefficients are rational functions of the cubic roots.
- Lagrange obtained similar results for the quartic.
- Lagrange sought solutions of higher order equations using symmetric functions of the roots and permutations.
- Paola Ruffini (1765 – 1822) - 1802, 1805, 1813 - gave proofs that quintic can't be solved. Proofs not understood.

Niels Henrick Abel (1802-1829)

- Born in Norway into poverty and had a pulmonary condition.
- Mathematical ability discovered by his teacher.
- Toured Europe after college and published 5 papers in *Journal für die reine und angewandte Mathematik*.
- Studied convergence of infinite series, the theory of doubly periodic functions, elliptic functions, elliptic functions and the theory of equations.
- Could not get employment, so tutored.
- At university, thought he had solution of quintic. Then, proved no solution existed.
- Died of tuberculosis before completing work.

Évariste Galois (1811-1832)

- Born Oct 25, 1811
- Interest in math at 14.
- Read Adrien-Marie Legendre (1752-1833).
- 1828 Failed to get into École Polytechnique.
- 1829 Paper on continued fractions.
- Studied polynomial equations.
- Wrote two papers.
  Reviewed by Arthur Cayley (1821-1895)
  Entered competition.
- 1830 Submitted to Joseph Fourier (1768–1830) - got lost.
  Winners - Niels Henrik Abel (1802-1829) and Carl Gustav Jacobi (1804-1851).
- Published 3 papers.

Figure 14: Évariste Galois

Évariste Galois (cont'd)

- Political turmoil in France.
- Student uprising - Galois left school.
- He was arrested and acquitted.
- Arrested Jul 1831 - April 29, 1832.
- Siméon Denis Poisson (1781-1840) asked him to submit work 1831.
- July 4 - declared work incomprehensible.
- Galois found out in October.
- Stayed up all night; wrote letters and note to Auguste Chevalier.
- On May 30, fought in duel and lost.
- Chevalier forwarded papers for publication by Joseph Liouville.

Figure 15: Legendre, Cayley, Fourier, Jacobi, Poisson, Liouville

Group Theory

1843 - Joseph Louiville (1809-1882) reviewed Galois' delayed manuscript, published 1846. - introduction of groups and fields.

- Multiplicative group modulo n.
- Euler - Fermat's Little Theorem
  p prime, $(a, p) = 1$,
  $$a^{p-1} \equiv 1 (\bmod\ p).$$
- Euler's ϕ function:
  $$\phi(n) = \#\{k \in \{1, 2, \ldots, n-1\} | (k, n) = 1\}.$$
  $\phi(5) = 4$, $\{1, 2, 3, 4\}$,
  $\phi(8) = 4$, $\{1, 3, 5, 7\}$.
- Group Properties:

  closed, identity, inverse, associative

Examples: Mod 5 and 8.

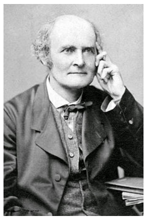
x	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

x	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Symmetry Groups

- Levi ben Gorshun (1321)
  Number of permutations of n objects $= n!$
- Leads to Symmetric Group.
- Felix Klein (1872) extended groups to geometry - studied invariants of groups of transformations.
- Sophus Lie (1842-1899) continuous groups of transformations, applied to differential equations.
- Emmy Noether (1882-1935) related symmetries to constants of motion in physics.

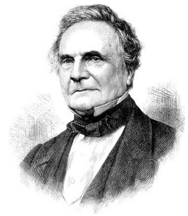
Figure 16: Sophus Lie and Emmy Noether.


Cayley
1821-1895


Beltrami
1835-1900


Riemann
1826-1866


Jacobi
1804-1851


Babbage
1791-1871


Chatelet
1706-1749


Daniel Bernoulli
1700-1782


Litvinova
1845–1919


Euler
1707-1783


Galois
1811-1832

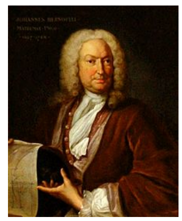

Germain
1776-1831


Huygens
1629-1695


Jacob Bernoulli
1655-1705


Bolyai
1802-1860


Fourier
1768-1830


Johann Bernoulli
1667-1748


Bassi
1711-1778


Legendre
1752-1833

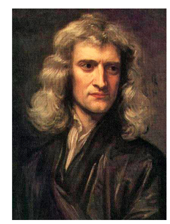

Leibniz
1646-1716


Liouville
1809-1882


Lobachevski
1792-1856


Lovelace
1815-1852


Agnesi
1718-1799

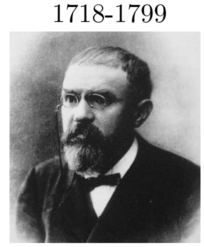

Newton
1643-1727


Nightingale
1820-1910


Noether
1882-1935


Pacioli
1445-1517


Fermat
1601-1665


Poincaré
1854 - 1912


Descartes
1596-1650


Poisson
1781-1840

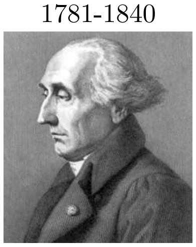

Somerville
1780-1872

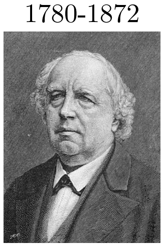

Lie
1842-1899


Tschirnhaus
1651-1708

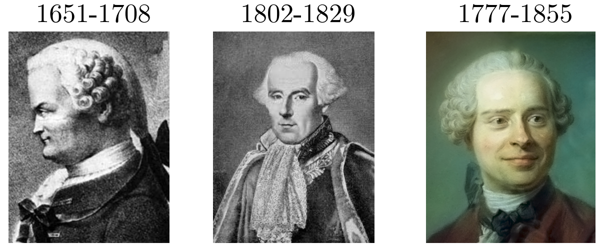

Abel
1802-1829


Gauss
1777-1855


Lagrange
1736-1813


Weierstrass
1815-1897


Wallis
1616-1703

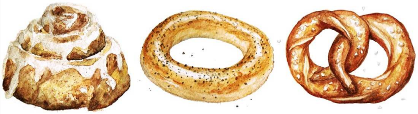

Lambert
1728-1777

Laplace
1749-1827

d'Alembert
1717-1783

Topology and Knot Theory

Fall 2023 - R. L. Herman

What is Topology?

From Wikipedia

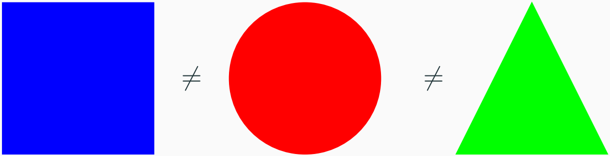
“In mathematics, topology (from the Greek *t*opos, 'place', and *l*ogos, 'study') is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.”

Nobel Prize in Physics 2016

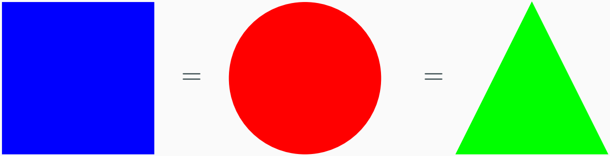
Figure 1: Nobel Prize in Physics 2016

Geometry vs Topology

Geometry



Topology



Types of Topology

General topology (Point Set Topology) Study of basic topological properties derived from properties such as connectivity, compactness, and continuity.

Metric topology Study of distance in different spaces.

Algebraic topology (Combinatorial Topology) Study of topologies using abstract algebra like constructing complex spaces from simpler ones and the search for algebraic invariants to classify topological spaces.

Geometric topology Study of manifolds and their embeddings.

Network topology Study of topology discrete math. Network topologies are graphs consisting of nodes and edges.

Differential Topology Study of manifolds with smoothness at each point to allow calculus.

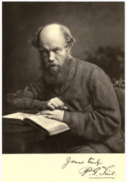
Origins of Topology

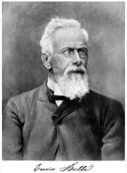
The search for a type of geometry where distance is not relevant.

- Euler - Graphs, Polyhedra
- Gauss, Maxwell - Physics
- Thomson, Tait - Knot Theory
- Riemann - 2D Surfaces in 3D
- Betti - Higher Dimensions
- Klein - Geometry and Groups
- Poincaré - Algebraic Topology
- Noether - Homology Groups









Leonhard Euler (1707-1783) - Königsberg Bridges

- 1736 Correspondences with Carl Gottlieb Ehler (1685-1753)
- Ehler's Letter

  "You would render to me and our friend Kuhn a most valuable service, putting us greatly in your debt, most learned sir, if you would send us the solution, which you know well, to the problem of the **seven Königsberg bridges** together with a proof. It would prove to an outstanding example of the calculus of position [calculi situs] worthy of your great genius. I have added a sketch of the said bridges."

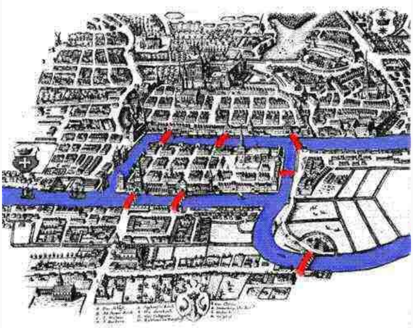
Leonhard Euler (1707-1783) - Königsberg Bridges

- 1736 Correspondences with Carl Gottlieb Ehler (1685-1753)
- Euler's reply

  "Thus you see, most noble sir, how this type of solution bears little relationship to mathematics and I do not understand why you expect a mathematician to produce it rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others. In the meantime most noble sir, you have a assigned this question to the geometry of position but I am ignorant as to what this new discipline involves, and as to which types of problem Leibniz and Wolff expected to see expressed this way."
- Based on Liebniz's *geometria situs* and *analysis situs*.
- Geometry of position: concerned only with the determination of position and does not involve using distances.

Königsberg Bridges Problem

Can the seven bridges over the river Preger in the city of Königsberg (formerly in Prussia but now known as Kaliningrad, Russia) all be traversed in a single trip without doubling back and ending where you started?



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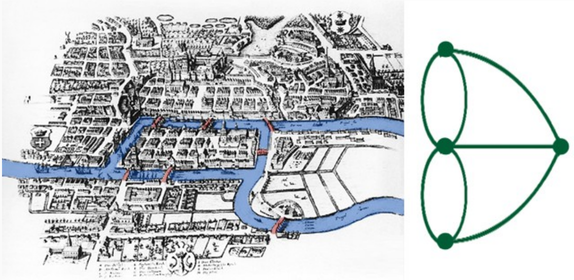

- Euler's result did not depend on the lengths of the bridges or on their distance from one another, but only on connectivity.

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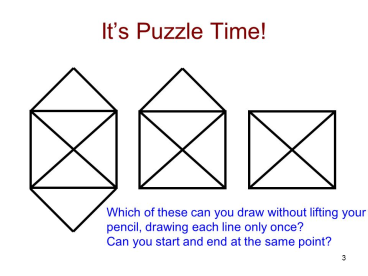

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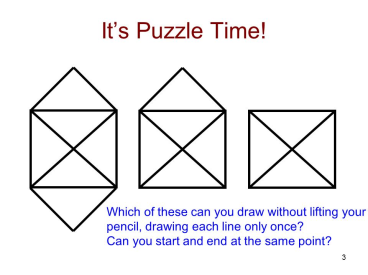


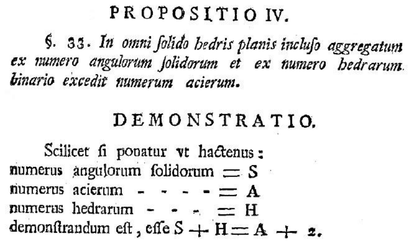
- A connected graph has an Euler cycle
  $\Leftrightarrow$ every vertex has even degree.

Euler's Polyhedron Formula

- 1750, Euler wrote Christian Goldbach (1690-1764)
- For polyhedron, like Platonic solids,

$$V - E + F = 2.$$

- Published papers in 1752.
- Not known before.
  - Descartes was close (1676).
- Euler characteristic: $\chi = V - E + F$.

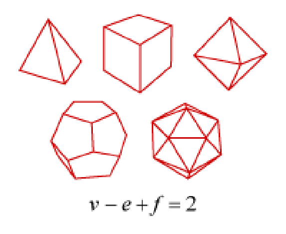


PROPOSITIO IV.

§. 33. In omni ſolido hedris planis incluſo aggregatum ex numero angulorum ſolidorum et ex numero hedrarum binario excedit numerum acierum.

DEMONSTRATIO.

Scilicet ſi ponatur vt hactenus:
numerus angulorum ſolidorum = S
numerus acierum - - - - = A
numerus hedrarum - - - = H
demonſtrandum eſt, eſſe S + H = A + 2.



Shape	Vertices	Edges	Faces
Tetrahedron	4	6	4
Cube/Hexahedron	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12

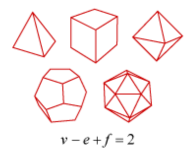
There Are Exactly Five Regular Polyhedra

Proof:

- Let $n = \#$ of sides of each face.
- Let $m = \#$ of faces meeting each vertex.
- $E = \frac{1}{2}Fn$ and $V = \frac{1}{m}Fn$.
- Since $V - E + F = 2$,

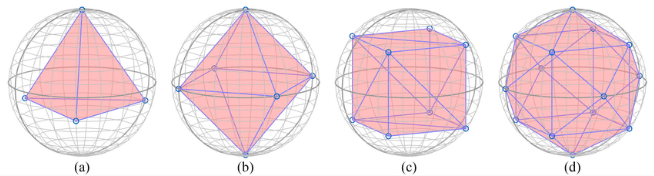
$$F = \frac{4m}{2n - mn + 2m.}$$

- $2n - mn + 2m > 0$ and $n \geq 3$.
  $n = 3$: $2n - mn + 2m = 6 - m$.
  So, $m = 3, 4, 5$.
  $n = 4$: $2n - mn + 2m = 8 - 2m$.
  So, $m = 3$.
  $n = 5$: $2n - mn + 2m = 10 - 3m,\ m = 3$.



Solutions (n, m)
$(3, 3)$ tetrahedron,
$(3, 4)$ octahedron,
$(3, 5)$ icosahedron,
$(4, 3)$ cube,
$(5, 3)$ docecahedron.

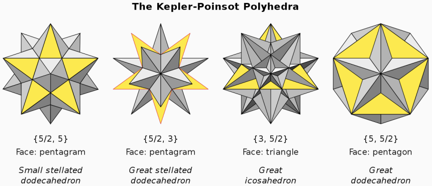
Euler Characteristic of a Sphere

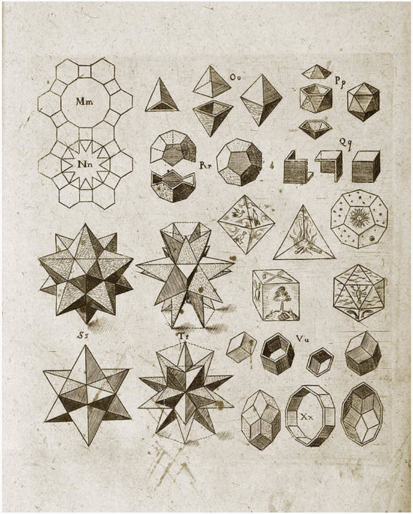


- Inscribe Platonic solids in a sphere.
- a to b: Add 2 vertices, 4 faces, 6 edges. $\Delta\chi = 2 - 6 + 4 = 0$
- Push faces to sphere surface.
- Euler characteristic, $\chi = V - E + F = 2$.

Kepler's Polyhedra - 1619 *Harmonice Mundi*

- Johannes Kepler (1571-1630) systematized and extended polyhedra.
- He defined classes of polyhedra and proved that his set was complete.
- Kepler–Poinsot polyhedron - regular star polyhedra.
- Stellated polyhedra - extend edges or faces until they meet to form a new polyhedron.





Kepler's Mysterium Cosmographicum

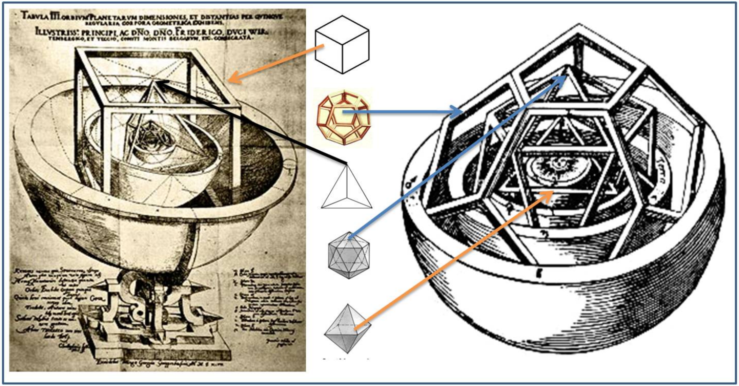


Figure 2: Published at Tübingen in 1596. Second edition, 1621. The orbits of the planets (Mercury, Venus, Earth, Mars, Jupiter and Saturn) were arranged in spheres nested around the five Platonic solids: octahedron, icosahedron, dodecahedron, tetrahedron and cube.

The Thirteen Archimedean Solids: $V - E + F = ?$

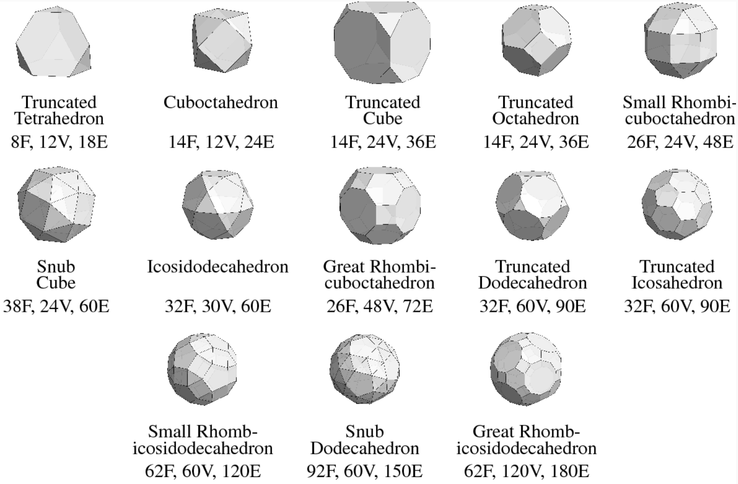
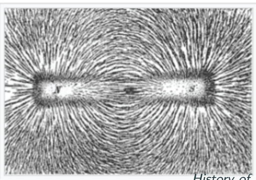


Figure 3: The 13 Archimedean solids. The duals are called Catalan Solids.

The Birth of Electromagnetism

- 1785, Coulomb's Law.
- 1820, New discoveries:
- Ørsted: Electric current deflects compass.
- Biot-Savart: Currents produce magnetic fields.
- Amperè: Parallel wires carrying currents attract or repel. 1827 *électrodynamiques.*
- 1821, Faraday: Electromagnetic rotation.
- 1831, Electromagnetic induction, Faraday's Law.
- Field lines.



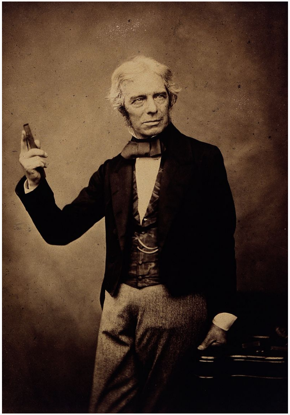
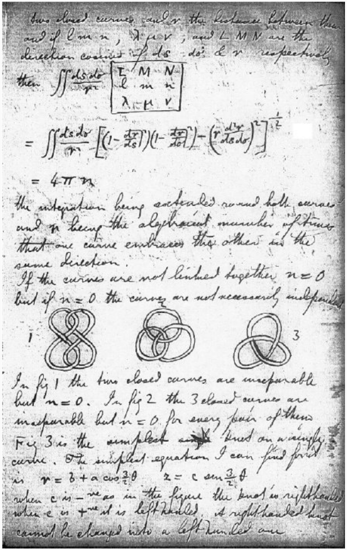


Figure 4: Michael Faraday (1791 – 1867)

C F Gauss (1777-1855) and Wilhelm Weber (1804-1891)

Carl Friedrich Gauss - 1831 - EM Induction.

- Gauss and Weber.
  - First telegraph 1833 to communicate 3 km.
  - Mapped Earth's Magnetic Field.
  - Weber - 1856, $c =$ speed of light.
- Gauss introduced formula for two intertwining curves.


two closed curves and r the distance between them
and if l m n, λ μ ν, and L M N are the
direction cosines of ds, dσ & r respectively

then $\iint \frac{ds\,d\sigma}{r^2} \begin{vmatrix} L & M & N \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix}$

$= \iint \frac{ds\,d\sigma}{r^2} \left[\left(1-\frac{dr}{ds}^2\right)\left(1-\frac{dr}{d\sigma}^2\right) - \left(r\frac{d^2r}{ds\,d\sigma}\right)^2\right]^{\frac{1}{2}}$

$= 4\pi n$

the integration being extended round both curves
and n being the algebraical number of times
that one curve embraces the other in the
same direction.
If the curves are not linked together $n = 0$
but if $n = 0$ the curves are not necessarily independent

1 3

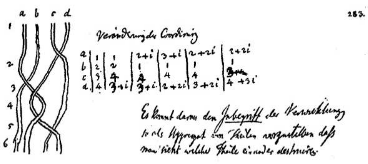
In fig 1 the two closed curves are inseparable
but $n = 0$. In fig 2 the 3 closed curves are
inseparable but $n = 0$ for every pair of them
Fig 3 is the simplest knot on a single
curve. The simplest equation I can find for it
is $r = b + a\cos\frac{3}{2}\theta$ $z = c\sin\frac{3}{2}\theta$
when c is −ve as in the figure the knot is righthanded
when c is +ve it is lefthanded. A righthanded knot
cannot be changed into a lefthanded one

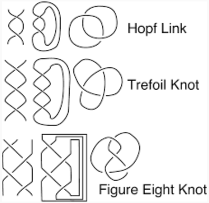
http://mathematiciancfgauss.blogspot.com/2016/11/contribution-of-math-gauss-published-in.html

Gauss' Linking Number and Braids

$$\int\int \frac{(x'-x)\,dydz' - dzdy') + (y'-y)\,(dzdx' - dzdz') + (z'-z)\,(dxdy' - dydx'))}{((x-x')^2 + (y-y')^2 + (z-z')^2)^{-3/2}} = 4m\pi$$

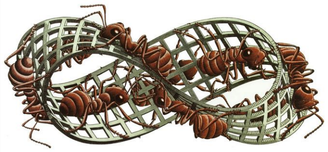
- 1833 Entered in notebook.
- Published in 1867.
- No proof or reason given.
- Possibly from E&M or astronomy. Orbits of Ceres and Pallas 1801
- Gauss studied linked orbits.
- Maxwell sent postcard to Tait.
- Braids (Artin, 1926) drawn in unpublished notebooks.





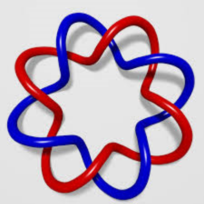
Two of Gauss' Students - Möbius and Listing

Möbius Band - 1858 - independently discovered by both. Möbius went on to study under Pfaff, Gauss' advisor.

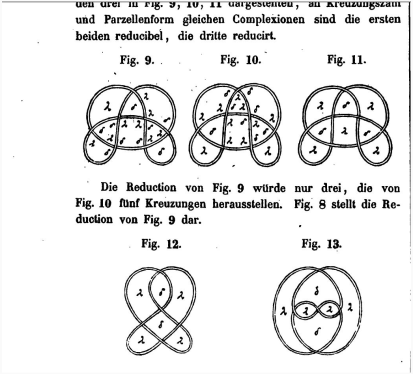


First Use of 'Topologie'

- Johann Listing gave Gauss' *geometria situs* a new name:
- 1847 -*Vorstudien zur Topologie*.
- Studied Connectivity and
- Link Invariants.



Inversion (rotation) and perversion (reflection)



den drei in Fig. 9, 10, 11 dargestellten, an Kreuzungszahl und Parzellenform gleichen Complexionen sind die ersten beiden reducibel, die dritte reducirt.

Fig. 9. Fig. 10. Fig. 11.

Die Reduction von Fig. 9 würde nur drei, die von Fig. 10 fünf Kreuzungen herausstellen. Fig. 8 stellt die Reduction von Fig. 9 dar.

Fig. 12. Fig. 13.

How do you distinguish knots and their symmetries?

Hermann Ludwig Ferdinand Helmhotz (1821-1894)

- German Physicist, Physician.
- Mathematics of the eye, theories of vision, perception of sound, electrodynamics, thermodynamics.
- In 1858 Helmholtz wrote on **vortex dynamics**, translated by Tait into English. *On Integrals of the Hydrodynamical Equations, which Express Vortex-motion*

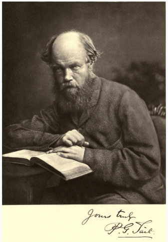
The evolution of a magnetic field $\boldsymbol{B}$ is similar to the evolution of vorticity ω, the curl of the flow velocity, **u.**

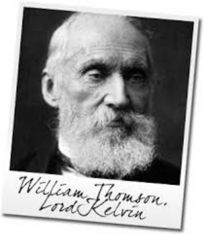
$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}.$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$$

Scottish Physics and Knots

James Clerk Maxwell (1831-1879), Peter Guthrie Tait (1831-1901), and William Thomson (1824-1907)





https://www.gutenberg.org/files/39373/39373-h/39373-h.htm

Maxwell and Tait met at Edinburgh Academy, went to University 1847.

Thomson (22) elected to Glasgow College Chair of Natural Philosophy.

James Clerk Maxwell and Helmholtz's Water Vortices

Maxwell read Gauss' work and referred to the work of Liebniz, Euler, and Vandermonde on *geometria situs.*

Maxwell wrote to Tait about Helmhotz's paper.

Tait's interest in Helmholtz was from recalling reading Hamilton's *Lectures* in 1853 and remembering formulae.

Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen.

(Von Herrn *H. Helmholtz.*)

Es sind bisher Integrale der hydrodynamischen Gleichungen fast nur unter der Voraussetzung gesucht worden, dafs die rechtwinkligen Componenten der Geschwindigkeit jedes Wassertheilchens gleich gesetzt werden können den nach den entsprechenden Richtungen genommenen Differentialquotienten einer bestimmten Function, welche wir das ***Geschwindigkeitspotential*** nennen wollen. Allerdings hat schon ***Lagrange**** *) nachgewiesen, dafs diese Voraussetzung zulässig ist, so oft die Bewegung der Wassermasse unter dem Einflusse von Kräften entstanden ist und fortgesetzt wird, welche selbst als Differentialquotienten eines ***Kräftepotentials*** dargestellt werden können, und dafs auch der Einflufs bewegter fester Körper, welche mit der Flüssigkeit in

GLENLAIR
DALBEATTIE,
Nov. 13, 1867.

Dear Tait

If you have any spare copies of your translation of Helmholtz on "Water Twists" I should be obliged if you could send me one.

I set [sic] the Helmholtz dogma to the Senate House in '66, and got it very nearly done by some men, completely as to the calculation, nearly as to the interpretation.

Thomson has set himself to spin the chains of destiny out of a fluid plenum as M. Scott set an eminent person to spin ropes from the sea sand, and I saw you had put your calculus in it too. May you both prosper and disentangle your formulae in proportion as you entangle your worbles. But I fear the simplest indivisible whirl is either two embracing worbles or a worble embracing itself.

For a simple closed worble may be easily split and the parts separated

but two embracing worbles preserve each others solidarity thus

though each may split into many, every one of the one set must embrace every one of the other. So does a knotted one.

yours truly
J. CLERK MAXWELL

Sir William Rowan Hamilton (1805–1865)

This led Tait to work on quaternions.

Hamilton discovered **quaternions** in 1843, an extension of complex numbers: $w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where w, x, y, z are real and $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ satisfy the bridge equations.

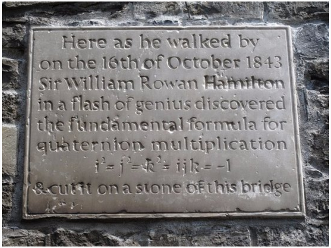


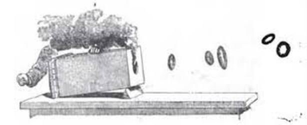
Figure 5: Hamilton carved his equations into the stone of the Brougham Bridge while on a walk.

Back to vortex rings ...

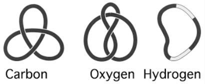
From Vortex Rings to Vortex Atoms

https://i.imgur.com/Y64h8ol.mp4 Movie

Tait experimented with smoke rings in 1867.

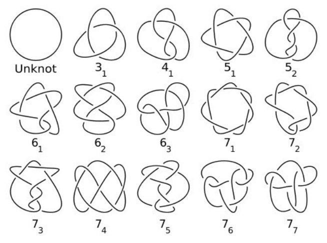


Thomson wrote to Helmholtz ...
later wrote *On vortex atoms*
as vortex rings in the aether.



Tait - Classification of Knots

- Knots up to 7 crossings reduced to 8 different knots.
- Periodic table of knot elements [Mendeleev - 1869].
- Tables up to 10 alternating crossings.
- Aether disproved in 1887 by Michelson Morley Experiment.



Put ideas in envelope for the Royal Society of Edinburgh [Open by 15/10/1987].

Tait Conjectures[1]

1. Reduced alternating diagrams have minimal link crossing number.
2. Any two reduced alternating diagrams of a given knot have equal writhe.
3. The flyping conjecture, which states that the number of crossings is the same for any reduced diagram of an alternating knot.

In 1987 one of Tait's conjectures was found in the envelope.

1,2 proved by Kauffman, Murasugi, and Thistlethwaite 1987.

3 proved by Menasco and Thistlethwaite, 1991 using Jones polynomials, 1984.

Perko Pair 1974

Links to Papers. and Gresham College Lecture about Tait.

[1]From Mathworld

More Knots - Reidemeister Moves

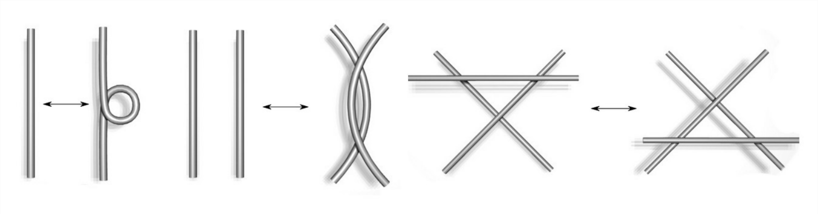


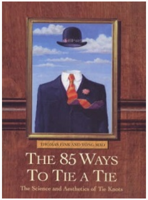
Figure 6: Reidemeister Moves: Untwist, Poke, Slide.

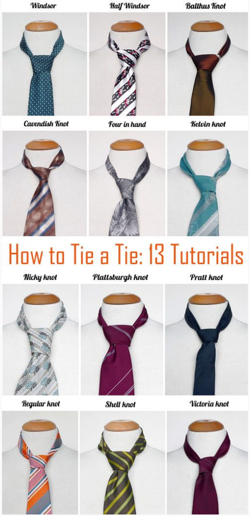
John Conway (1937-2020) - the theory of finite groups, knot theory, number theory, combinatorial game theory, coding theory, recreational mathematics - The Game of Life.

Graduate Student Solves Decades-Old Conway Knot Problem

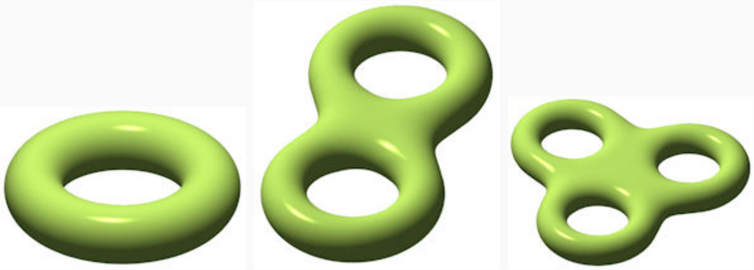
How to Tie a Tie - Recent Application

- Two physicists, “The 85 Ways to Tie a Tie,” 1999.
- The Man Who Invented Fifteen Hundred Necktie Knots, The New Yorker. Nov. 2023.
- Boris Mocka, doorman, > 1500 knots.
- More ties than we thought, 266,682 distinct tie-knots, 2015.





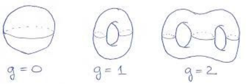
Connection to Surfaces

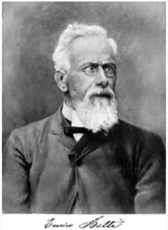


Torus: $V - E + F = 0$
Two-hole Torus: $V - E + F = -2$
Three-hole Torus: $V - E + F = -4$

Bernhard Riemann and Enrico Betti

- Riemann and Betti - connectedness.
- Surfaces with curvature - Manifolds.
- Can we classify surfaces up to a continuous transformation?
- Genus g and Euler Characteristic $\chi = 2 - 2g$.
- Enrico Betti (1823-1892)
- Betti number - maximum number of cuts that can be made without dividing a surface into two separate pieces.





Betti Numbers

- β_0 - number of connected components.
- β_1 - number of handles.
- β_2 - number of voids or. cavities

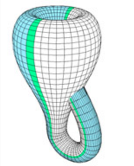
β_0	1	1	1	1
β_1	0	1	0	2
β_2	0	0	1	1

Poincaré Polynomial - Generator of Betti numbers.

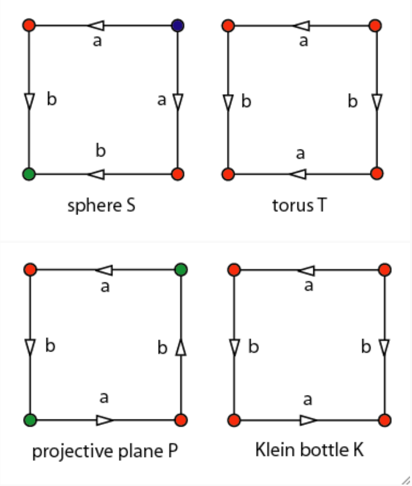
Ex: Torus: $T^2 = S^1 \times S^1$
$p(x) = \beta_0 + \beta_1 x + \beta_2 x^2 = 1 + 2x + x^2 = (1 + x)^2.$

Felix Klein (1849-1925)

- 1872 Felix Klein Public Lecture
  - Erlangen Program, Geometry.
  - Symmetry groups $\rightarrow$ invariants.
  - Euclidean Geometry - invariant under translations, rotations.
  - Topology - invariants under continuous transformations.
  - Klein Bottle.



Equivalence Relations



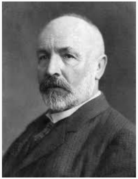
Henri Poincaré (1854-1912)

- 1895 Start of Algebraic Topology.
  - *Analysis Situs*
  - Brings rigor, better Betti Numbers.
- History of Poincaré's Mistakes.
  - 1888 King Oscar II, Sweden, Offered Prize.
  - Judges: Mittag-Leffler, Weierstrass, and Hermite.
  - *Acta Mathematica* - 3 body problem stable.
  - Oops! .... Chaos!
- Topological methods for differential equations.

See Stillwell on Early History of 3-Manifolds

Felix Klein and Georg Cantor

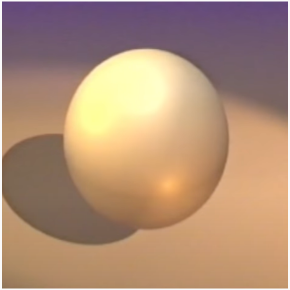
- 1872 Cantor - Open and closed sets.
    - Introduced Set Theory.
    - Infinite and transfinite numbers
    - Cardinality.
- 1902 Hilbert - Neighborhoods.
- 1906 Frechet - Compactness, metric spaces have open and closed sets.
- Riesz 1909 and Hausdorff 1914 - abstract topological spaces.
- 1926 Emmy Noether - Homological groups, corrected Poincaré.



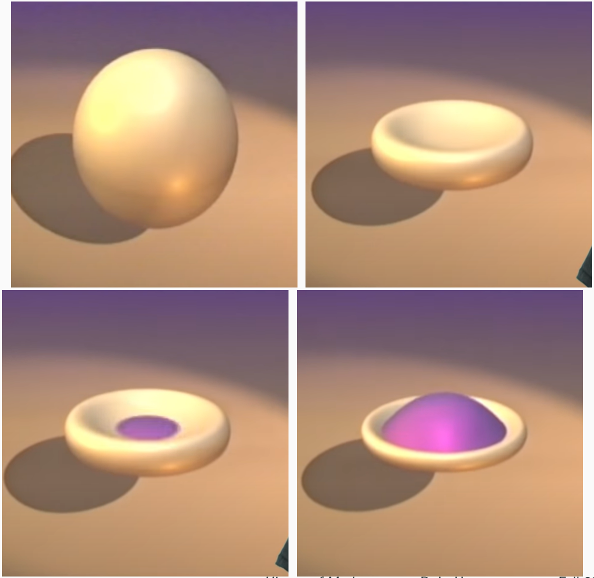

How Do You Turn a Sphere Inside-out?

Sphere Eversion - a continuous deformation, allowing the surface to pass through itself, without puncturing, ripping, creasing, or pinching.

An existence proof for crease-free sphere eversion was first created by Stephen Smale (1957). See Video - *Outside In*



How Do You Turn a Sphere Inside-out?



How Do You Turn a Sphere Inside-out?

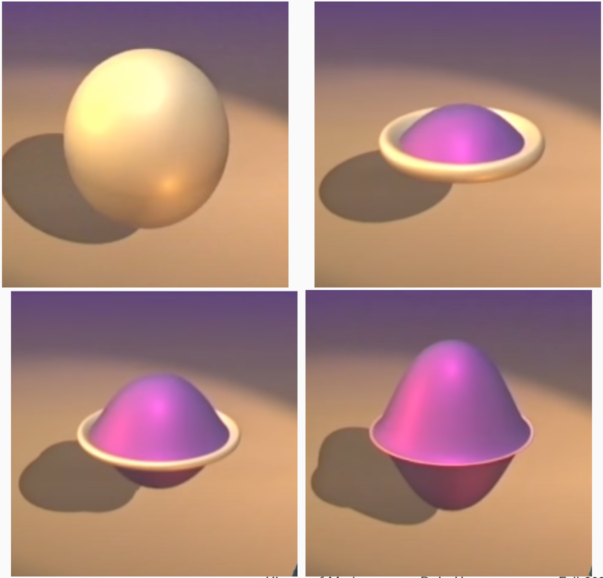
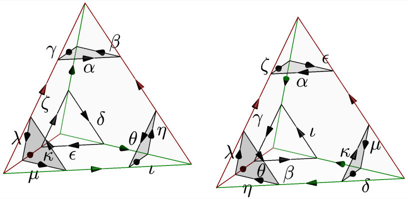


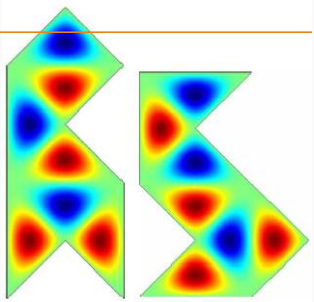
Figure Eight Knot Complement - Honors Thesis

- Start with a Figure Eight Knot.
- Thurston noted knot is the second most commonly occurring knot in garden hoses and vacuum cords.
- What is the space outside the knot?
- 1973 Robert Riley, a graduate student, showed that the figure-eight knot complement had a hyperbolic structure.
- 1978, William Thurston (1936-2012) provides construction.
- Tiled by 2 hyperbolic tetrahedra.



Vibrations and Fourier Analysis

Fall 2023 - R. L. Herman



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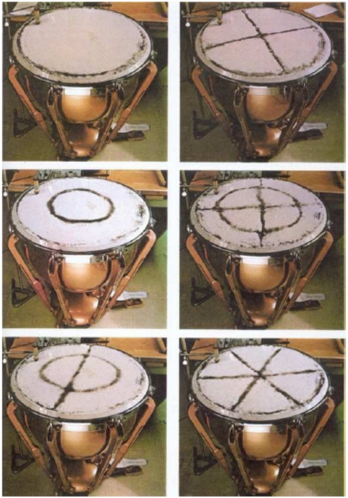
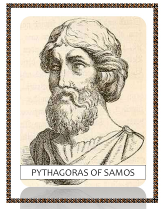


Chladni patterns on a kettledrum
from Risset, *Les instruments de l'orchestre*

Harmonics

- Pythagoras, Ptolemy.
- Galileo and Mersenne, pitch and frequency. Strings produce several tones.
- Joseph Sauveur, 1653-1716, acoustics. Introduced nodes, “harmonic.”
- Johann Bernoulli, 1667-1748.
- Brook Taylor 1685-1731, fundamental.
- Johann Sebastian Bach Bach, 1685-1750.
- Hermann Helmholtz, 1821-1894, acoustics.



The 1700s Debate - Mathematicians vs Physicists

- Jean le Rond d'Alembet, 1717-1783.
- Vibrating string equation and general solution, $y(x,t) = f(x+t) + g(x-t)$. BCs give $g = f$.
- Leonhard Euler's papers, 1748-9. More general equation with c, and $y(x,t) = f(x+ct) + g(x-ct)$.
- Claimed - f from ICs. $y(x,t) = \frac{1}{2}\left(Y(x+ct) + Y(x-ct) + \frac{1}{c}\int_{x+ct}^{x-ct} V(s)\,ds\right)$.
- Y, V are any curves *drawn by hand*.
- Daniel Bernoulli, 1709-1791, solutions are sums of harmonics, 1753:



$$y(x) = A_1 \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + A_2 \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \cdots = f(x+ct) + g(x-ct).$$

The Controversy (from Am. J. of Phys. 55, 33 (1987))

d'Alembert vs Euler



- Euler allowed corners.
- d'Alembert's first response - f must be periodic, odd, differentiable. Introduced separation of variables.
- 1761 - the attack! Use of physical arguments is prohibited.
- If slope discontinuous, then acceleration undefined.
- Euler responded 1762, 1765. For small displacement, the function at corner is infinitesimally close to differentiable.

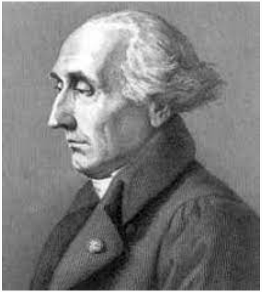
d'Alembert, Euler vs Bernoulli

- d'Alembert did not believe a sum of harmonics.
- Euler sum not general enough - snapped string.
- Bernoulli - "Listen to the string."

They all missed general periodicity.

Joseph-Louis Lagrange

- In enters another math. physicist.
- Born Luigi de la Grange Tournier (1736-1813), in Italy.
- 1759, paper on sound propagation.
- Agreed mostly with Euler, not Bernoulli.
- Avoided wave equation. Used a discrete set of masses.



$$\begin{aligned} y(x,t) &= \frac{2}{L}\int_0^L dX\, Y(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \dots\right] \\ &+ \frac{2}{\pi c}\int_0^L dX\, V(X)\left[\sin\frac{\pi X}{L}\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} + \frac{1}{2}\sin\frac{2\pi X}{L}\sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L} + \dots\right] \end{aligned}$$

He almost discovered Fourier series in 1759. [Fourier was born, 1768.]

Jean-Baptiste Joseph Fourier (1768-1830)

- French Revolution, 1789, several arrests.
- Studied under Lagrange, Laplace, Monge.
- Succeeded Lagrange, chair of analysis and mechanics, 1797.
- Joined Napoleon's invasion of Egypt, scientific adviser with Monge, Malus.
- Organizer of French retreat from Egypt.
- Produced a multi-volume work on Egyptology.
- Studied the heat equation and series solutions.
- Almost forgotten in France, not elsewhere due to P. G. J. Dirichlet who wrote on Fourier series. Open problems led Cantor to set theory.

Siméon-Denis Poisson (1781-1840)

- 1798, entered École Polytechnique.
- Studied under Laplace and Lagrange.
- Degree in mathematics two years.
- Chair of mechanics, Faculty of Sciences, 1809.
- Over 300 papers: definite integrals, Fourier analysis, applied mathematics to physics (mechanics and electrostatics), probability and statistics.
- Poisson brackets, Poisson's constant, Poisson's equation, Poisson's integral, and Poisson's spot.

See D. H. Arnold's Work.

The Heat Equation

- Controversy: Fourier vs Poisson
- Fourier 1805, 1807 - diffusion, series solutions ala D. Bernoulli.
- Examiners: Laplace, Lagrange skeptical.
- Poisson Review 1808.
- 1811 Prize problem. Fourier won, but still critics.
- Third version to be book, 1822. Timing affected by politics.
- 1815, Poisson writes his own paper, then book in 1823.
- Wm. Thomson defense of Fourier in 1845.
- The Age of the Earth and Telegraphy.

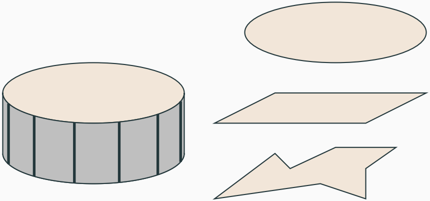
See D. H. Arnold's Work.

On to applications -

- Can you hear the shape of a drum?
- How long does it take to cook a turkey?

"Can One Hear the Shape of a Drum?"

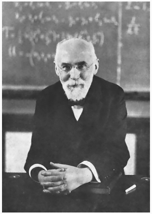
- Kac, Mark (1966). Amer. Math. Monthly. 73, Part II: 1–23.
- Title due to Lipman Bers: "If you had perfect pitch, could you hear the shape of a drum?"
- Can the frequencies (**eigenvalues**) of a resonator (**drum**) determine its shape (**geometry**)?
- Entails features of applied mathematics.
- Historical connections - from radiation theory.



Radiation Theory

- Hendrik Lorentz's (1910) 5 lectures on old/new physics. problems
- 4th - Electromagnetic Radiation Theory.
- Compared vibrations to an organ pipe.
- The number of overtones in frequency range is independent of shape, proportional to volume.
- David Hilbert's prediction
- Hermann Weyl - < 2 yrs

$$N(\lambda) = \sum_{\lambda_n < \lambda} \sim \frac{|\Omega|}{2\pi} \lambda.$$



What Do We Hear? Frequency, $f = \omega/2\pi$,

Seek Harmonic Solutions,
[Recall $e^{i\omega t} = \cos\omega t + i\sin\omega t$.]

$$u(\mathbf{r}, t) = U(\mathbf{r})e^{i\omega t},$$

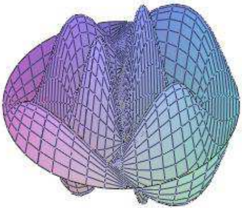
of a Wave Equation, $u(\mathbf{r}, t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$

Helmholtz Equation

$$\nabla^2 U = -k^2 U, \quad k^2 = \frac{\omega}{c}.$$

Eigenvalues $\sim$ frequencies



Vibrations of a String

- Ex: Violin String.
- Harmonics, $u_n(x)$.
- Wavelength, $\lambda = \frac{2L}{n}$.
- Wave Speed, $c = \sqrt{\frac{T}{\mu}}$.
- Frequency, $f = n\frac{c}{2L}$.
- A - $f = 440$ Hz, $L = 32$ cm.
  $c = 2Lf = 280$ m/s.
- Nodes, $u_n(x) = 0$

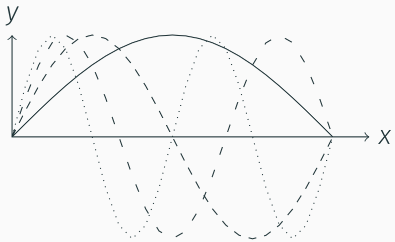


Figure 1: Plot of the eigenfunctions $u_n(x) = \sin \frac{n\pi x}{L}$ for $n = 1, 2, 3, 4$.

Solution of 1D Wave Equation

The one dimensional wave equation, given by

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}, \quad t > 0, \quad 0 \leq x \leq L, \tag{1}$$

subject to the boundary conditions

$$u(0,t) = 0, u(L,t) = 0, \quad t > 0,$$

and the initial conditions

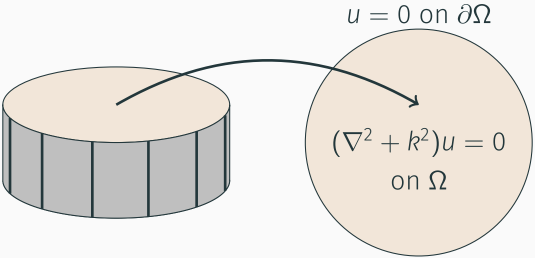
$$u(x,0) = f(x), u_t(x,0) = g(x), \quad 0 < x < L.$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right] \sin \frac{n\pi x}{L}, \tag{2}$$

where $\omega_n = \frac{n\pi c}{L}$.

General 2D Membranes

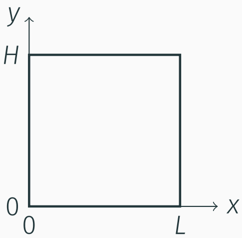
- Membrane Problems.
  - Rectangular
  - Circular
  - Elliptical
  - Irregular
- Solve Helmholtz Equations
  - Normal Modes and Frequencies of Oscillation
  - Eigenvalues of Laplace Operator, $\nabla^2 u = -\lambda u$.



Vibrations of a Rectangular Membrane

- Harmonics
- Frequencies

$$\omega_{mn} = c\sqrt{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2}$$



Boundary-value problem

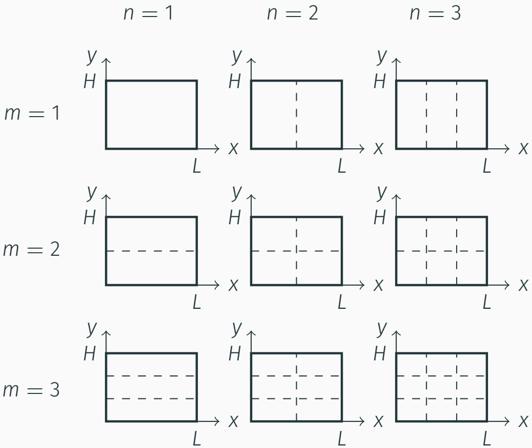
$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad t > 0, 0 < x < L, 0 < y < H, \tag{3}$$

$$u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad t > 0, \quad 0 < y < H,$$
$$u(x, 0, t) = 0, \quad u(x, H, t) = 0, \quad t > 0, \quad 0 < x < L,$$

$$u(x, y, t) = \sum_{n,m} (a_{nm} \cos \omega_{nm} t + b_{nm} \sin \omega_{nm} t) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}.$$

Nodes of a Rectangular Membrane

$$u_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}, \quad f = \frac{c}{2L}\sqrt{n^2+\alpha^2 m^2}, \quad \alpha = \frac{L}{H}.$$

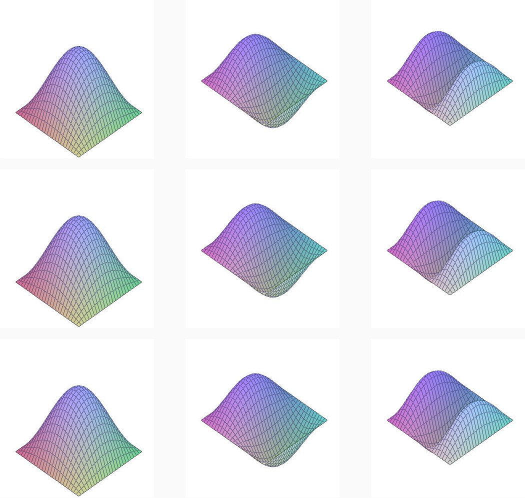


$\alpha = 1$	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

$\alpha = 2$	1	2	3
1	2.236	4.123	6.083
2	2.828	4.472	6.325
3	3.606	5.000	6.708

Vibrations of a Rectangular Membrane

$m = 1$ $m = 2$ $m = 3$



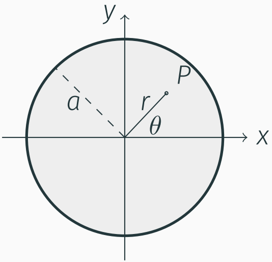
$n = 1$

$n = 2$

$n = 3$

Vibrations of a Circular Membrane

- Circular Symmetry.
- Harmonics
- Frequencies



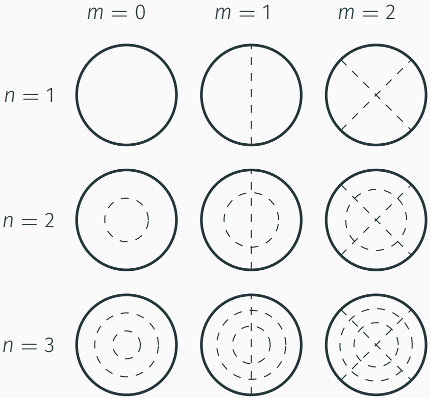
$$\omega_{mn} = \frac{j_{mn}}{a} c.$$

$$u(r,\theta,t) = \left\{ \begin{array}{c} \cos \omega_{mn} t \\ \sin \omega_{mn} t \end{array} \right\} \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} J_m(\frac{j_{mn}}{a} r). \qquad (4)$$

$$J_m(j_{mn}) = 0 \quad m = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

Nodes of a Circular Membrane

$$u_{mn}(r,\theta) = J_m\left(\frac{j_{mn}}{a}r\right)\cos m\theta, \quad f_{mn} = \frac{j_{mn}c}{2\pi a}.$$

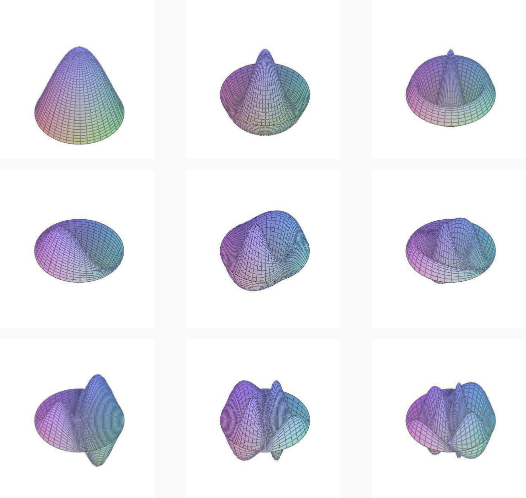


j_{mn}	0	1	2
1	2.405	3.832	5.136
2	5.520	7.016	8.417
3	8.654	10.173	11.62

f_{mn}	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Vibrations of a Circular Membrane

$n = 1$ $n = 2$ $n = 3$

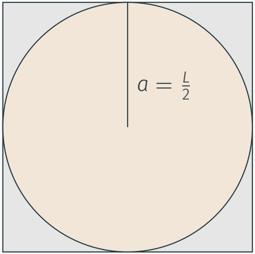


$m = 0$

$m = 1$

$m = 2$

Rectangular and Circular Membrane Frequencies



Rectangular

	1	2	3
1	1.414	2.236	3.162
2	2.236	2.828	3.606
3	3.162	3.606	4.243

Circular $a = \frac{L}{2}$

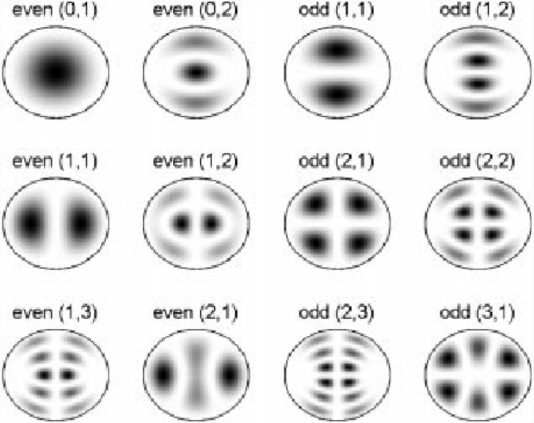
	0	1	2
1	1.531	2.440	3.270
2	3.514	4.467	5.358
3	5.509	6.476	7.398

Circular $\pi a^2 = L^2$

	0	1	2
1	1.357	2.162	2.898
2	3.114	3.958	4.749
3	4.882	5.740	6.556

Vibrations of an Elliptical Membrane

$$\left[\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + (kh)^2(\cosh^2\xi - \cos^2\eta)\right] u(\xi,\eta) = 0.$$



Vibrations of a Balloon

The wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} L u, \quad \text{where} \quad L Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$, The general solution is found as

$$u(\theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell m} \cos \omega_\ell t + B_{\ell m} \sin \omega_\ell t \right] Y_{\ell m}(\theta, \phi),$$

where $\omega_\ell = \sqrt{\ell(\ell+1)} \frac{c}{R}$.

Modes for a Vibrating Spherical Membrane (Balloon?)

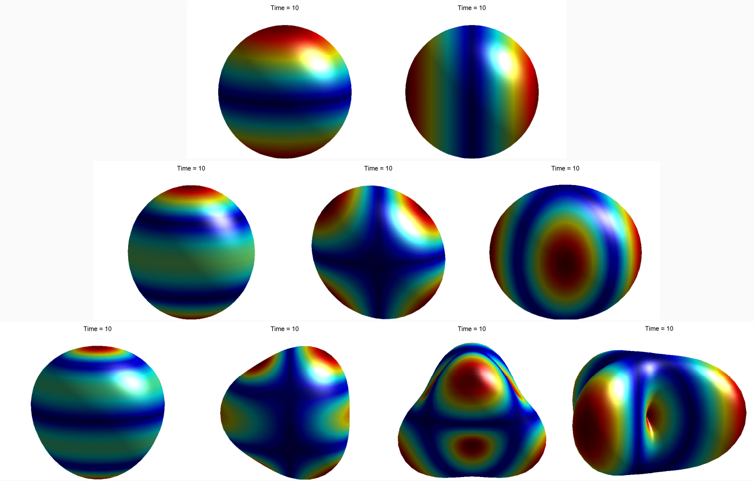
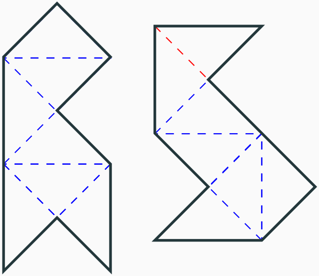
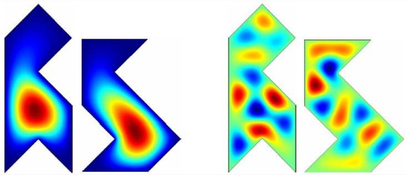


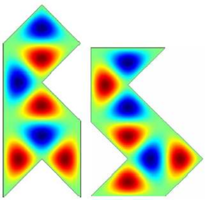
Figure 2: *http://people.uncw.edu/hermanr/pde1/sphmem/*
Row 1: $(1,0),(1,1)$; Row 2: $(2,0),(2,1),(2,2)$;
Row 3 $(3,0),(3,1),(3,2),(3,3)$.

Vibrations of a Irregular Membranes

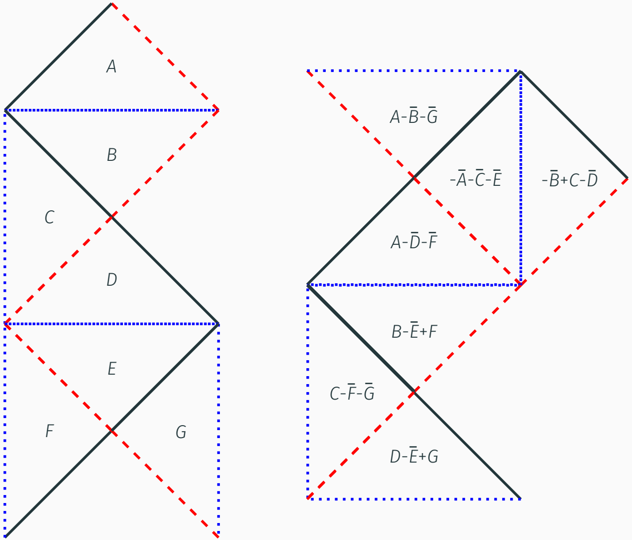
- Gordon, C., Webb, D., and Wolpert, S.(1992) - *You Cannot Hear the Shape of a Drum*
- Shapes on right have same set of frequencies - **isospectral drums.**





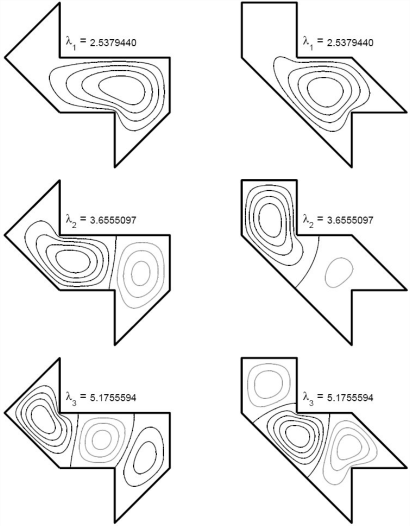


Isospectral Drums



Spectra of Isospectral Drums

$\lambda = 2.5379440, 3.6555097, 5.1755594.$



Other Isospectral Drums

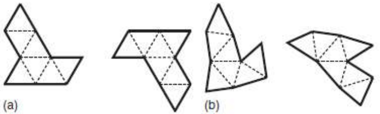


FIG. 25. Pair 7_2. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 1)(2\ 5)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 4)(2\ 3)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

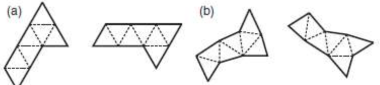


FIG. 26. Pair 7_3. Sunada triple $G=\mathbf{PSL}(3,2)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(2\ 5)(4\ 6)$, $b_1=(1\ 5)(3\ 4)$, $c_1=(0\ 4)(1\ 6)$, $a_2=(0\ 3)(2\ 4)$, $b_2=(0\ 6)(1\ 4)$, and $c_2=(0\ 2)(1\ 5)$.

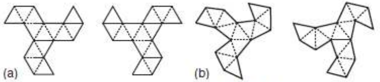


FIG. 27. Pair 13_1. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 12)(1\ 10)(3\ 5)(6\ 7)$, $b_1=(0\ 10)(2\ 9)(3\ 4)(5\ 8)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $b_2=(0\ 1\ 2)(1\ 4)(5\ 11)(6\ 9)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

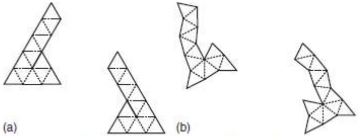


FIG. 31. Pair 13_5. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(1\ 7)(3\ 5)(4\ 9)(6\ 10)$, $b_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $c_1=(0\ 4)(1\ 6)(2\ 11)(9\ 12)$, $a_2=(0\ 9)(4\ 10)(6\ 8)(7\ 12)$, $b_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$, and $c_2=(0\ 10)(1\ 5)(2\ 7)(3\ 12)$.

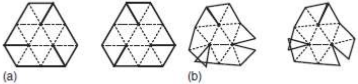


FIG. 32. Pair 13_6. Sunada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 6)(2\ 4)(3\ 8)(5\ 9)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 8)(1\ 10)(5\ 11)(7\ 9)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

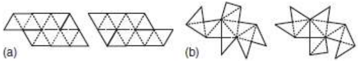


FIG. 33. Pair 13_7. Sun+ada triple $G=\mathbf{PSL}(3,3)$, $G_i=\langle a_i,b_i,c_i\rangle$, $i=1,2$, with $a_1=(0\ 2)(1\ 7)(3\ 6)(5\ 10)$, $b_1=(0\ 4)(2\ 3)(6\ 8)(9\ 10)$, $c_1=(0\ 5)(1\ 2)(6\ 12)(9\ 11)$, $a_2=(0\ 7)(3\ 11)(6\ 8)(9\ 12)$, $b_2=(0\ 12)(1\ 1\ 0)(3\ 5)(6\ 7)$, and $c_2=(0\ 11)(1\ 8)(2\ 7)(3\ 4)$.

Can one hear the shape of a drum? -

No!

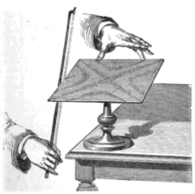
Membranes - Rectangular, circular, elliptical, irregular

Never look at MATLAB logo the same way again - Why?

MATLAB

Chladni Plates

- Recall Sophie Germain.
- Ernst Chladni, 1756-1827, physicist and musician.
- In 1808, Chladni demonstrated vibrating plates at the Academy of Science in Paris.
- Napoleon, who attended, proposed a prize.
- Lagrange, Laplace and others – felt that it was beyond reach.
- Germain only one to try.
- 1816, two more tries, first woman awarded Grand Prize in Mathematics of the Paris Academy of Sciences.



Heat Equation vs Wave Equation

1D Wave Equation

$$u_{tt} = c^2 u_{xx}$$

1D Heat Equation

$$u_t = k u_{xx}$$

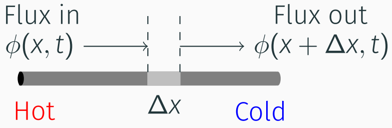
History of Heat Equation

Developed by Joseph Fourier (1768-1830)

- Discovered in early 1807 and published later in 1822

  Afterwards, diffusion processes studied outside of France.
  Lead to research in partial differential equations.
- Describes conduction and storage of heat (energy) in a body.
- Involves heat exchange with surroundings, conservation of energy.
- Leads to temperature changes inside body (diffusion).
- Uses the relation of heat energy to temperature (gradient), Fourier Law of Heat Conduction.

Heat Equation - Mathematics



Rate of change of heat energy = Flux in - Flux out

$$\frac{dQ}{dt} = \phi(x,t) - \phi(x+\Delta x,t).$$

Flux density = conductivity $\times$ temperature gradient

$$\phi = k\frac{dT}{dx}.$$

Heat energy is proportional to temperature

$$Q = mcT.$$

$q =$ Heat energy per vol, $u =$ temperature per vol

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \quad D = \frac{k}{mc}.$$

Thanksgiving Turkey!

- Native to North America.
- Introduced in Spain in 1500's.
- Benjamin Franklin - national bird.
- Holiday bird in Europe in 1800's
  - replacing goose.
- Turkeys mostly walk.
- Harold McGee: Breast 155-160 F, Legs 180 F.
- Cooking times

  Constant oven temp, diffusivity constant, Turkey plump
  Small - 20 min/lb + 20.
  Large - 15 min/lb + 15.
  $t \sim M^{2/3}$.

How long does it take to cook a turkey?

Example 1
If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

Figure 3: A Thanksgiving turkey - From 2015.

Panofsky Equation

- Pief Panofsky [SLAC Director Emeritus] *SLAC Today*, Nov 26, 2008 *http://today.slac.stanford.edu/a/2008/11-26.htm*
  For a stuffed turkey at 325° F

$$t = \frac{W^{2/3}}{1.5}$$

  vs. 30 minutes/lb.
- Also, check out WolframAlpha *http://www.wolframalpha.com/input/?i=how+long+should+you+cook+a+turkey*
- Musings of an Energy Nerd *http://www.greenbuildingadvisor.com/blogs/dept/musings/heat-transfer-when-roasting-turkey*

Consider a Spherical Turkey

Figure 4: The depiction of a spherical turkey.

Scaling a Spherically Symmetric Turkey

The baking follows the heat equation.

Rescale the coordinates (r, t) to (ρ, τ) :

$$r = \beta\rho \text{ and } t = \alpha\tau.$$

Then, the heat equation rescales as

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right).$$

- Invariance of heat equation implies $\alpha = \beta^2$.
- So, if the radius increases by a factor of β, then the time to cook the turkey increases by β^2.

Problem Solution

Example 1

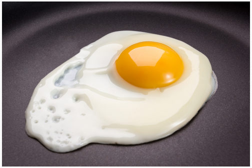
If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

- The weight doubles $\Rightarrow$ the volume doubles.
  (if density $=$ constant).
- $V \propto r^3 \Rightarrow r$ increases by factor: $2^{1/3}$.
- Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$.
- If 4 lb turkey takes 4 hrs, then a 20 lb turkey takes

$$t = 4(2^{2/3}) = 2^{8/3} \approx 6.35 \text{ hours.}$$

- In general, if the weight increases by a factor of x, then the time increases by $x^{2/3}$.

Eggs


Omelettes

Egg Protein

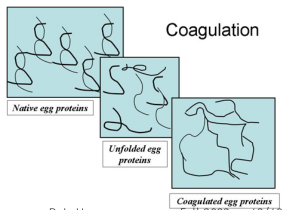
Proteins in eggs can be used

- to help food set (e.g. egg custards),
- as a foam to add air and volume (e.g. sponge cakes),
- as an emulsifier (e.g. mayonnaise).

Two different major proteins, egg white (albumin) and egg yolk,

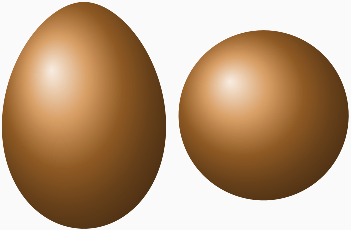
- Albumin starts coagulating at 63^oC
- Yolks start at 70^oC

Coagulation - protein unfolds, denaturation.
As heat increases the proteins rearrange and coagulate.
Egg albumin turns from clear to cloudy white.



Egg Cooking Time

Peter Barnham, *The Science of Cooking* & Dr. Charles Williams of Exeter:

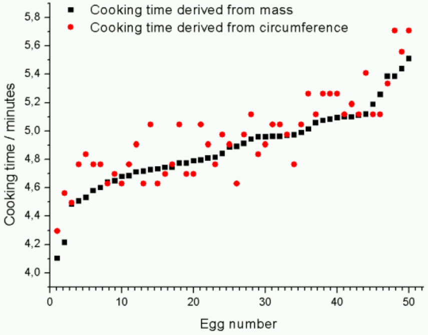


$$t = 0.0152 d^2 \log\left[2 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}}\right],$$
$$t = 0.451 M^{2/3} \log\left[0.76 \times \frac{(T_{\text{water}} - T_0)}{T_{\text{water}} - T_{\text{yolk}}}\right],$$

for t min, diameter d cm, M g, and temperatures in ^{o}C.

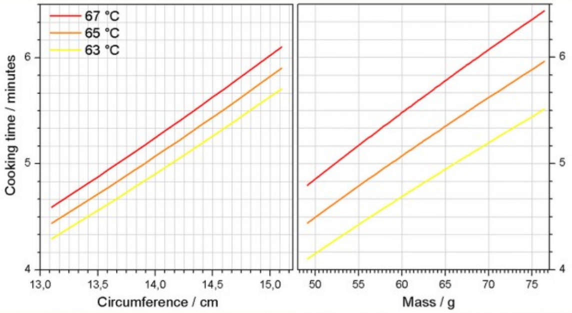
Egg Cooking Time - Data

From Khymos *Towards the perfect soft boiled egg* by Martin Lersch, April 9th, 2009. See also University of Oslo Applet



50 eggs with $T_{yolk} = 63^oC$, $T_{water} = 100^oC$ and $T_{egg} = 4^oC$.

Egg Cooking Time - Formula



Given circumference or mass to reach to reach 63, 65 and 67^o C, respectively, at the yolk-white boundary with $T_{water} = 100^o$ C and $T_{egg} = 4^o$ C.

Egg Consistency

Temp	White	Yolk
62	Begins to set, runny	Liquid
64	Partly set, runny	Begins to set
66	Largely set, still runny	Soft solid
70	Tender solid	Soft solid, waxy
80	Firm	Firm
90	Rubbery solid	Crumbly texture

At sea level, boiling water is 100^o C. At higher altitudes, the boiling temperature of water is lowered 0.3^o C for each additional 100 m above sea level.

Fast Fourier Transform - FFT

- One of top algorithms of 20th Century.
- Developed by Cooley and Tukey, 1965, to compute DFT (Discrete fourier transform)
- Some traced the ideas back to Gauss.
- Limit of Fourier series = Fourier Transform.
- Related to Laplace transform.

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk. \\ F(s) &= \int_{0}^{\infty} f(t) e^{-st}\, dt. \end{aligned} \tag{5}$$

Left for another course!

References for Drums

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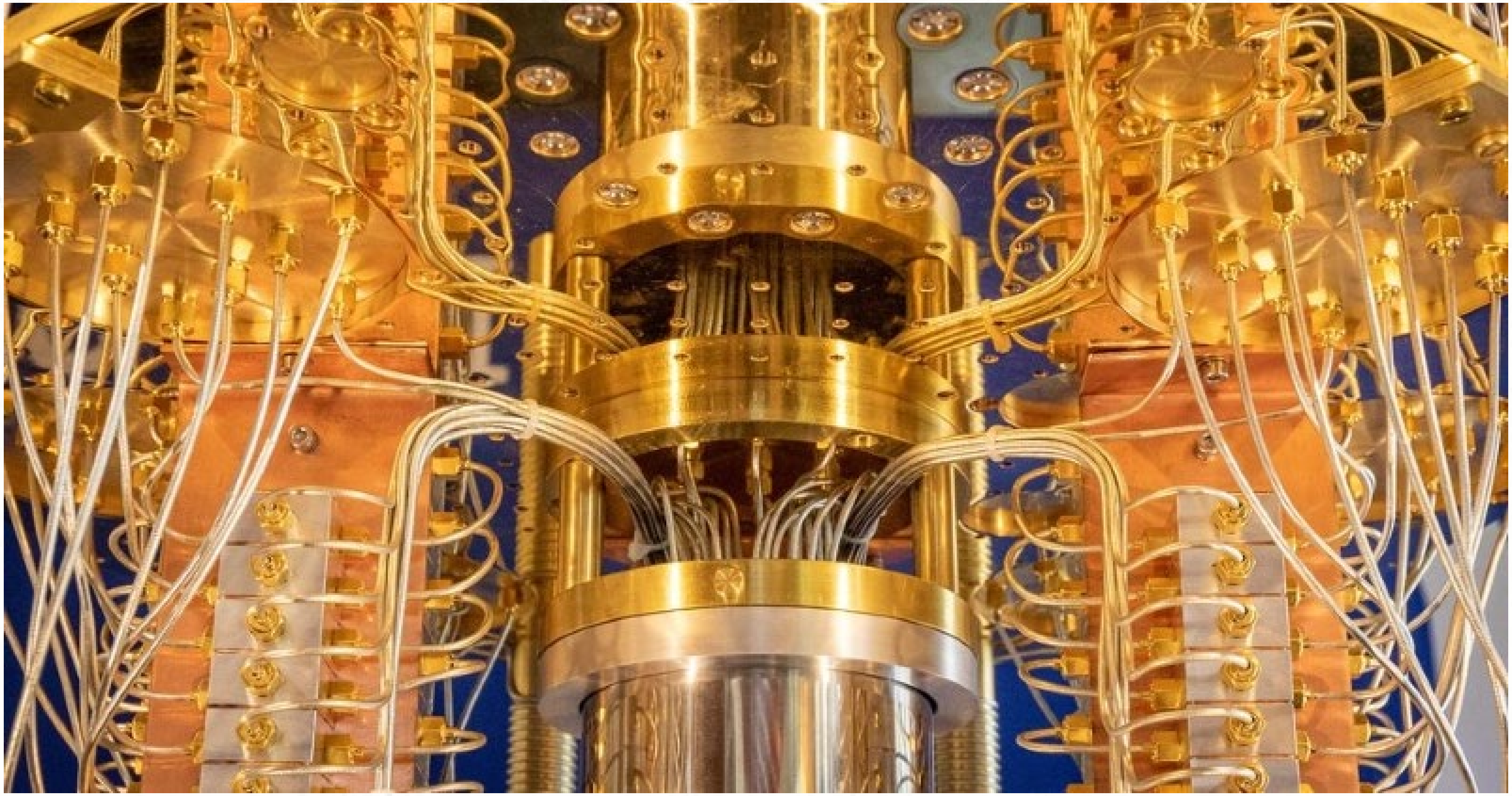
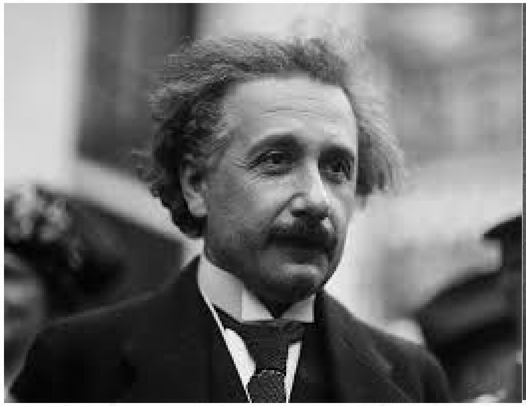
- Tobin Driscoll, Eigenmodes of isospectral drums, SIAM Review 39 (1997), 1-17.
- Carolyn Gordon, David Webb, Scott Wolpert, One cannot hear the shape of a drum, *Bull. Amer. Math. Soc.* 27 (1992), 134-138.
- Marc Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* 73 (1966), 1-23.
- Cleve Moler, The MathWorks logo is an eigenfunction of the wave equation (2003).
- Lloyd N. Trefethen and Timo Betcke, Computed eigenmodes of planar regions (2005).

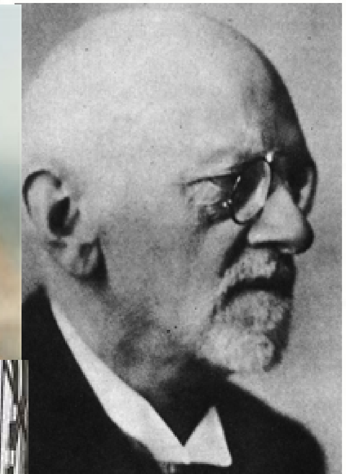
History of Mathematics - 1900s

Dr. R. L. Herman, UNCW

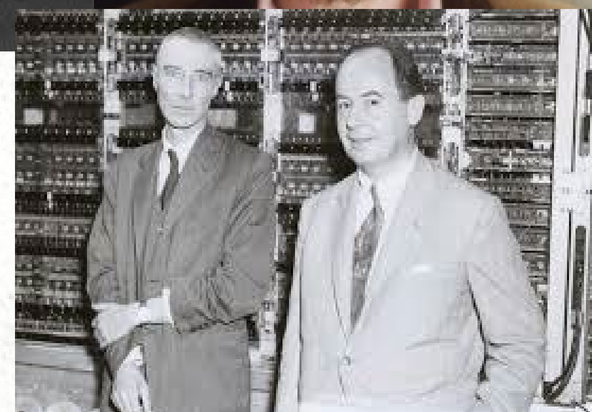
MAT 346, Fall 2023



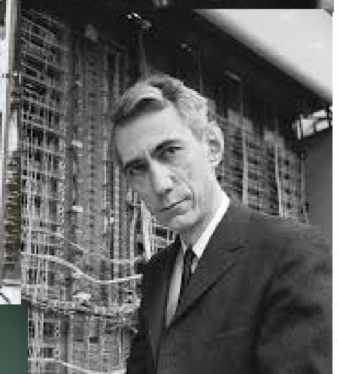


Outline

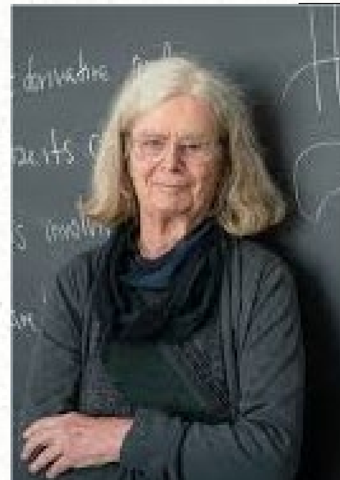
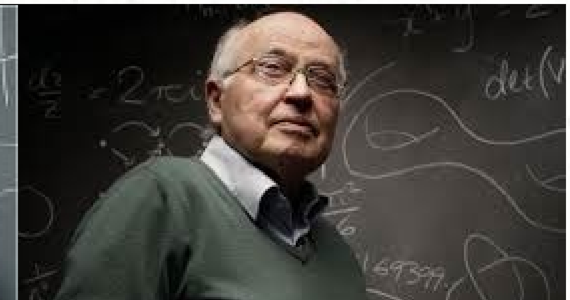


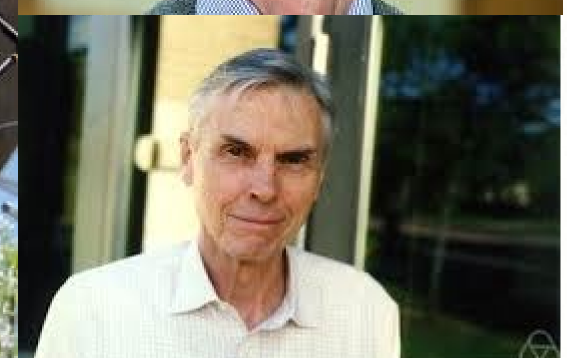
- The Rise of Rigor
- Set Theory
- N. Bourbaki
- Physics Revolutions
- Hilbert's Problems
- Mathematics Prizes

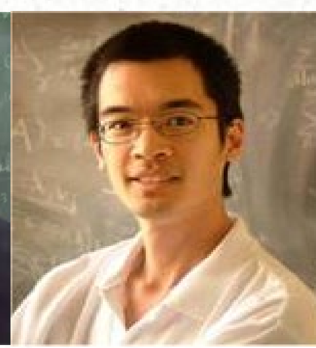



Mathematics Prizes

- Fields Medal
- Abel Prize
- Wolf Prize
- Millenium Prize

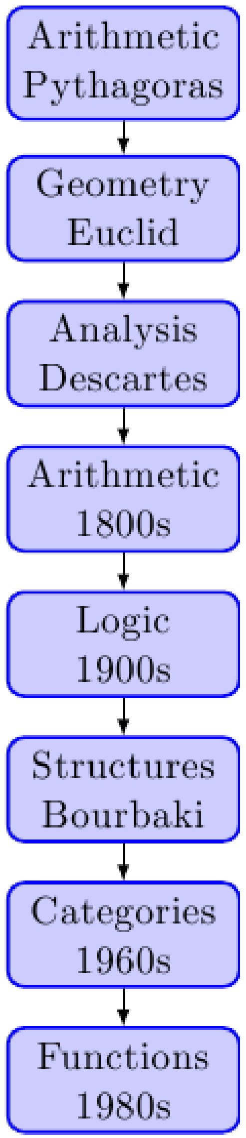



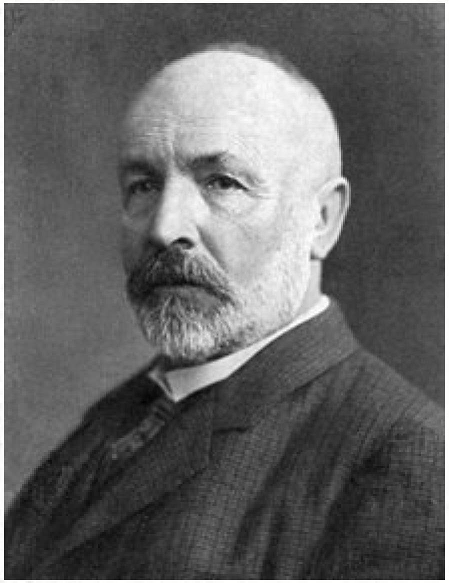



Evolution of Mathematics

- Pythagoreans - Arithmetic
- Euclid - Geometry
- Descartes – Analytic Geometry
- Newton – Calculus, Mechanics
- Euler – Numbers, Applications
- Gauss – Noneuclidean geometry
- Cantor – Set Theory, Infinity
- Frege – Predicate Logic
- Russell – *Principia Mathematica*
- Gödel – Incompleteness Theorems



Georg Cantor (1845-1918)



- Created set theory.

  How did Cantor Discover Set Theory and Topology?

  – Connection to Fourier series.

- Cardinality of sets.

- Not all infinities have same cardinality.

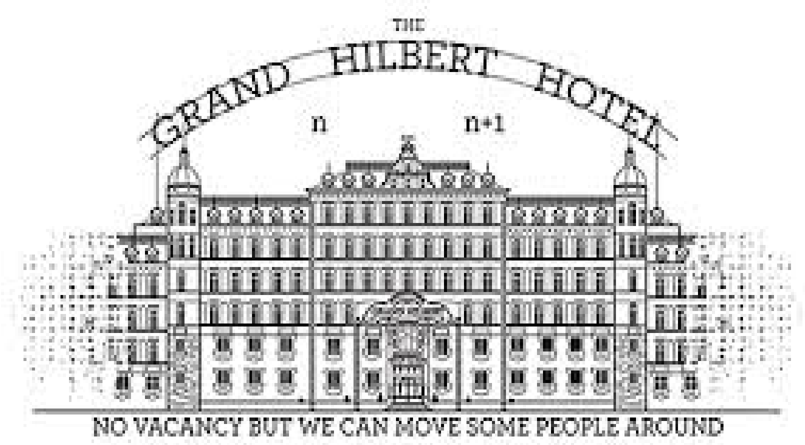
– Natural Numbers, Rationals, Reals

- Transfinite numbers and the Continuum hypothes $\aleph_0$

  There are no intermediate cardi $\aleph_1$ numbers between (aleph-null) and the cardinality of the continuum (set of real numbers).

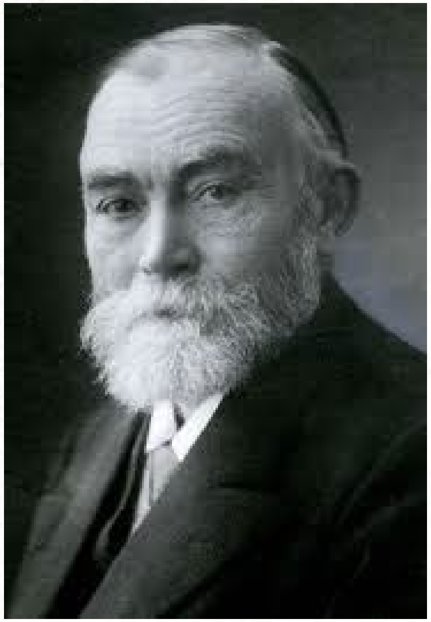
Hilbert's Grand Hotel - 1924

- David Hilbert (1862-1923)
- Infinite # rooms fully occupied with infinite # people.
- First add one person.
- Then, a bus with an infinite number of guests.
- What about an infinite number of buses filled with an infinite number of people?



Gottlob Frege (1848-1925)

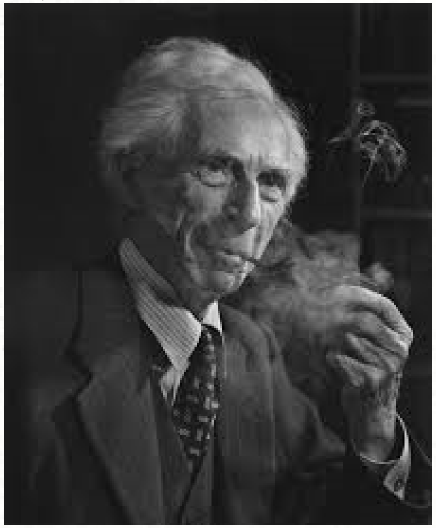
Invented axiomatic predicate logic,
Essential to



- Principia Mathematica (1910–13) Bertrand Russell, (1872–1970), and Alfred North Whitehead, (1861–1947).
- Kurt Gödel's (1906–78) incompleteness theorems.
- Alfred Tarski's (1901–83) theory of truth.
- Development of set theory.

Bertrand Russell, (1872–1970)

- Philosopher, logician, mathematician, historian, writer, essayist, social critic, political activist, and Nobel laureate



- Liar's Paradox

  "This sentence is a lie."

- Russell's Paradox

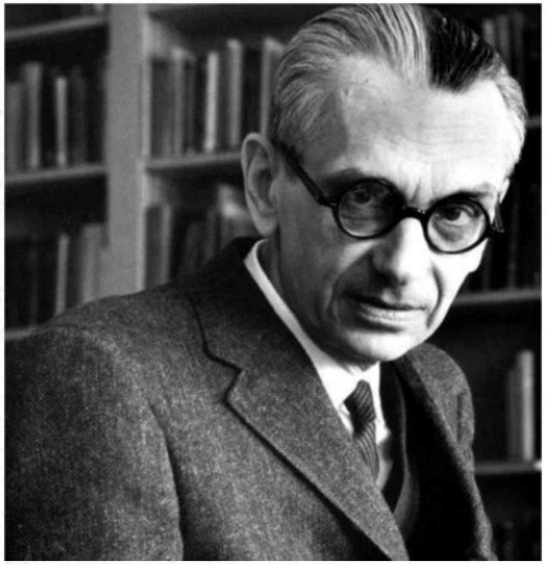
  Consider the set of all sets (power set) that are not members of themselves.

$$\text{Let } R = \{x \mid x \notin x\}, \text{ then } R \in R \iff R \notin R$$

- Avoid paradox - Zermelo–Fraenkel set theory

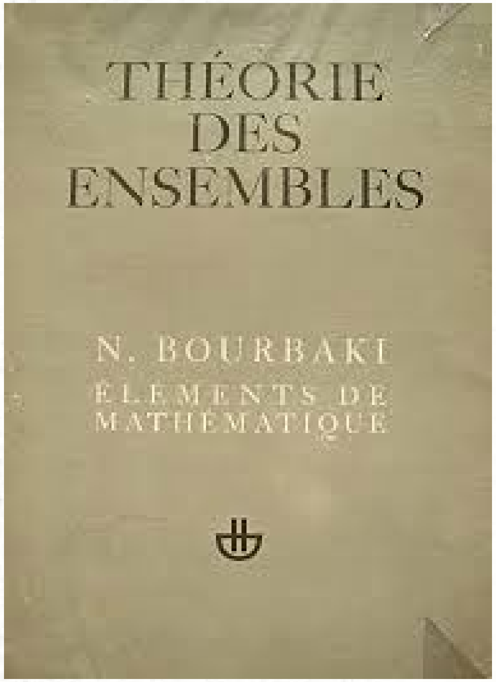
Kurt Gödel (1906–78)

- 1930 Incompleteness Theorems
- 1. If a logical, or axiomatic formal, system is consistent, it cannot be complete.
- 2. The consistency of axioms cannot be proved within their own system.
- Met Einstein 1933.
- Moved to Princeton 1940.
- 1949 Rotating universes and time travel.



Nicolas Bourbaki (1935 -)

École Normale Supérieure, Paris



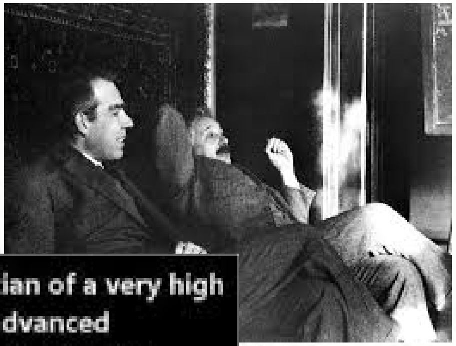
- Founders
  - Henri Cartan,
  - Claude Chevalley,
  - Jean Coulomb,
  - Jean Delsarte,
  - Jean Dieudonné,
  - Charles Ehresmann,
  - René de Possel, Szolem Mandelbrojt,
  - André Weil.
- Notable participants in later days:
  - Schwartz, Serre, Grothendieck, Eilenberg, and Lang.

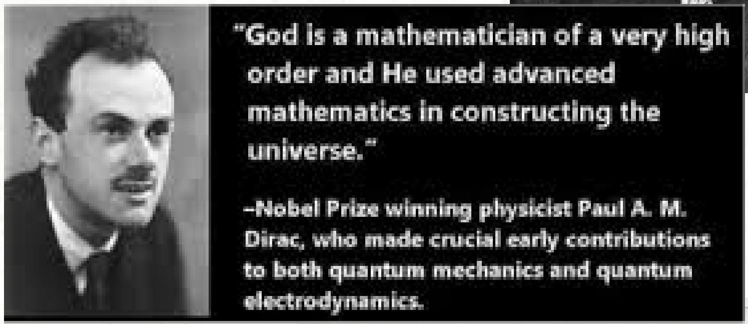
Bourbaki (1935 -)

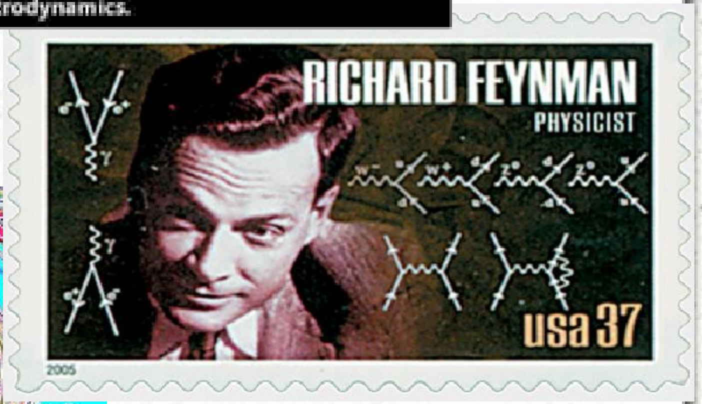
École Normale Supérieure, Paris

- Sought to write better analysis texts (Henri Cartan complained to Weil).
- Fall of French mathematics –due to WWI loss of a generation of French mathematicians.
- The rise of German Mathematics (and physicists).
- Bourbaki – a secret society, name based on past French general.
- Original plan -one volume of 1000 pages, collectively written, – a treatise on analysis. plan and became *Éléments de Mathématique*
- *By 1967:* 10 books in several volumes, over 60 chapters, publications starting in 1939. set Theory, Algebra, General Topology, Real Analysis, Topological Vector Spaces, Integration, Commutative Algebra, Varieties, Lie Groups and Algebras, Spectral Theory.
- In 2016 – Algebraic Topology; 2019 Revised Spectral Theory.

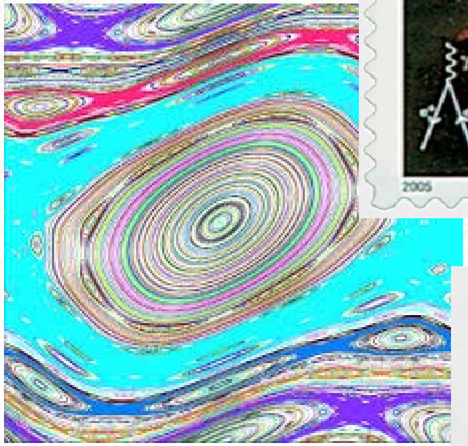
Revolutions – Paradigm Shifts

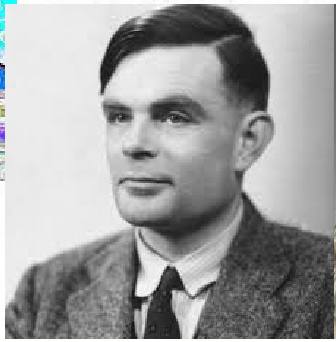






- Quantum Theory
- Special Relativity
- General Relativity
- String Theory Revolution
- Nonlinear Dynamics and Chaos
- Information Age
- *Focus on higher dimensions …*



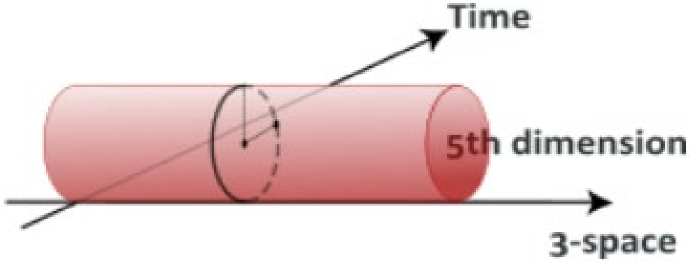


Kaluza-Klein to Calabi-Yau

The Search for Higher Dimensions

- The 4th Dimension
  - Special Relativity 1905;
  - Minkowski – 4D spacetime.
  - General Relativity 1915.
- Theodor Kaluza (1885-1954) In 1921
  - Solved Einstein Equations in 5D.
  - Unified General Relativity with E&M.
- Oskar Klein (1894-1977) – QM Interpretation
  - Fifth dimension - curled up, microscopic
- Led to gauge theories on fiber bundles.

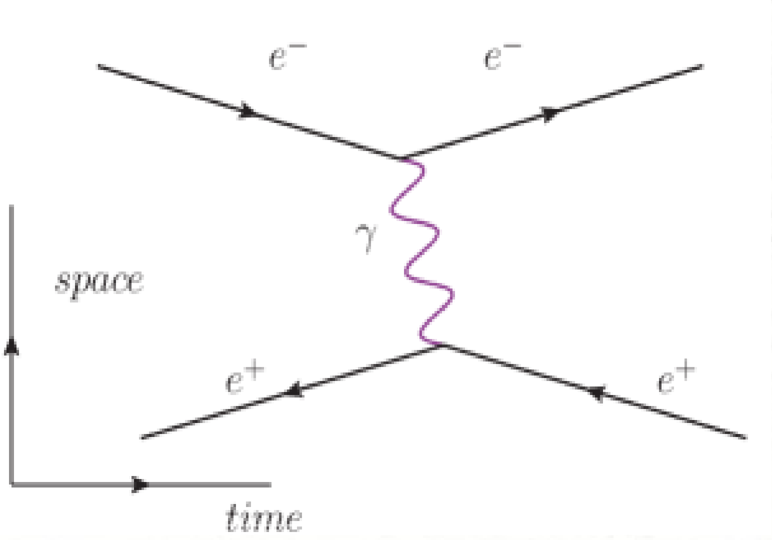

Theodor Kaluza




Oskar Klein

Feynman Diagrams and QED

- Richard Feynman (1918-1988)
- Quantum Electrodynamics (QED) – 1948-9
  - Interaction of light and matter
  - Nobel Prize with Schwinger, Tomonaga 1965

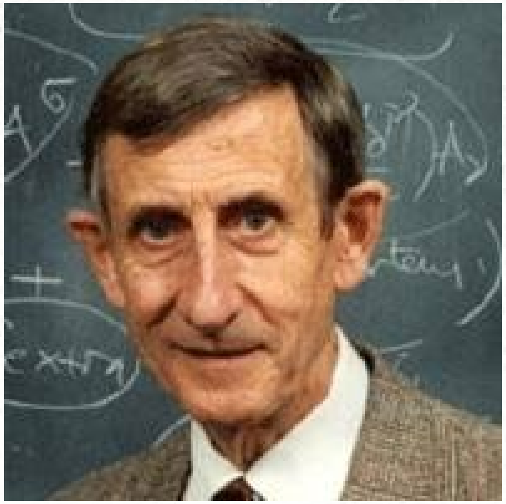


Not to F. Dyson

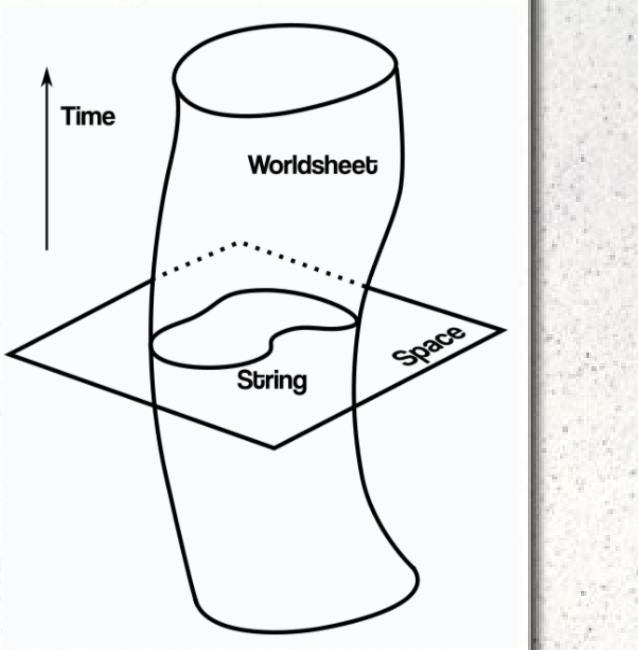
Freeman Dyson (1923-2020)

1972 Gibbs Lecture

I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has ended in divorce.



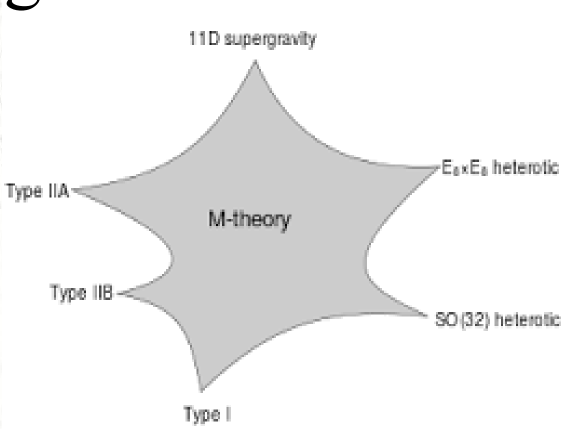
String Theory



- Gabriele Veneziano (1942-)
- In 1968 "Thumbing through old math books, they [Mahiko Suzuki] stumbled by chance on the **[Euler] Beta function** …," Michio Kaku

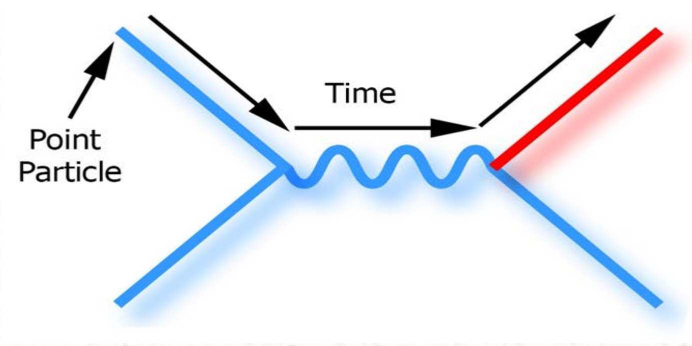
$$\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

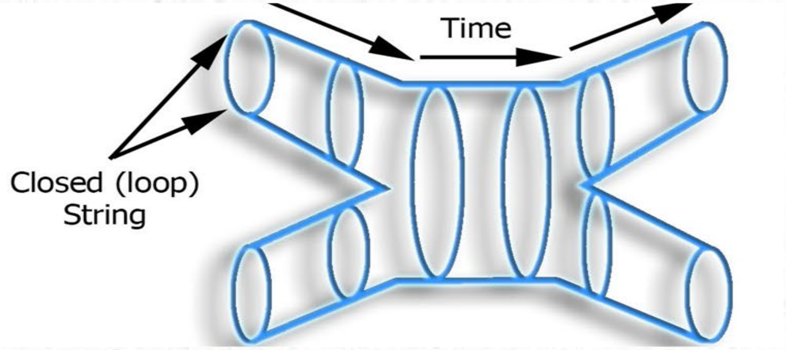
- Point-like particles modeled as 1D strings.
- Extra Dimensions 70s-90s
  - Bosonic string theory, 26-dimensional.
  - Superstring theory, 10-dimensional.
  - M-theory, 11-dimensional.



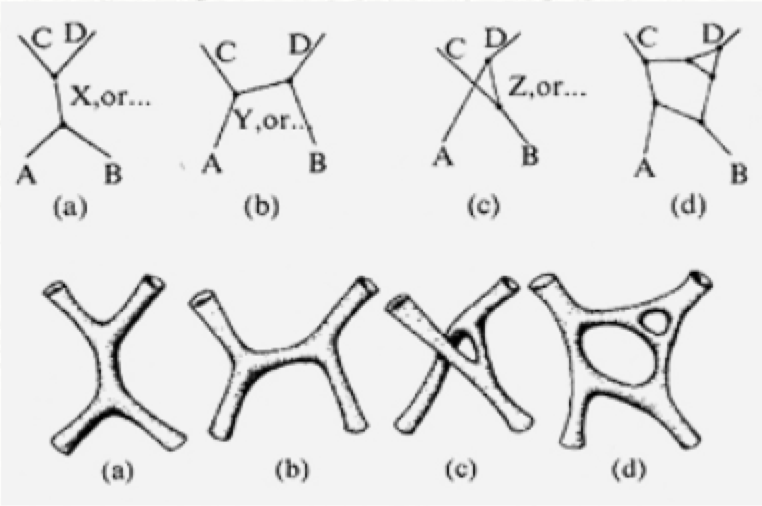
Feynman Diagrams to Strings

- Extend Feynman diagrams





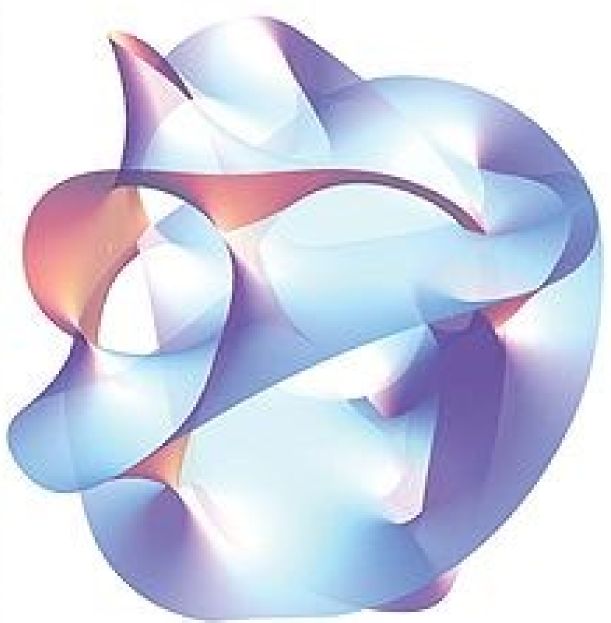
- Differential Geometry
- Topology
- Knot Theory

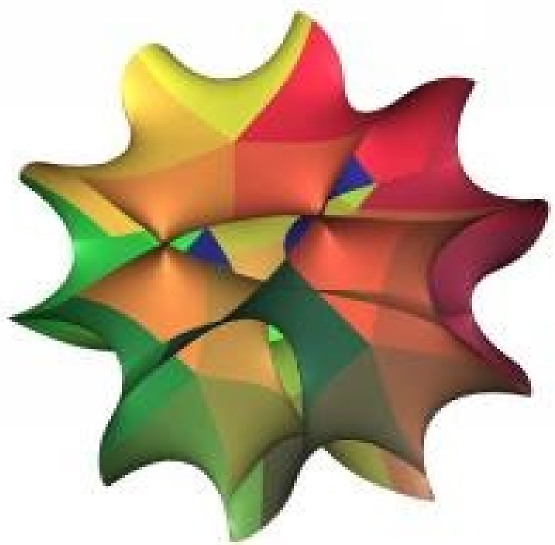


Calabi-Yau Manifolds

- Compactification (curl up extra 6 dimensions).
- Eugenio Calabi and Shing-Tung Yau, mathematicians

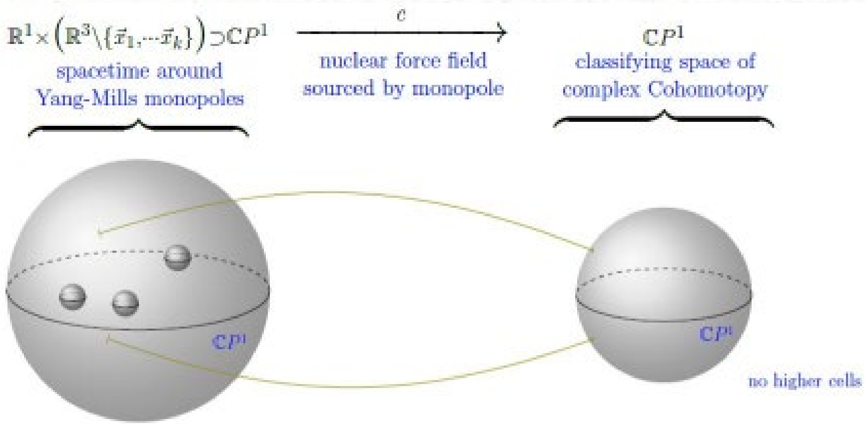
$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + \psi x_1 x_2 x_3 x_4 = 1$$





Mathematics and Physics

- Not Divorced

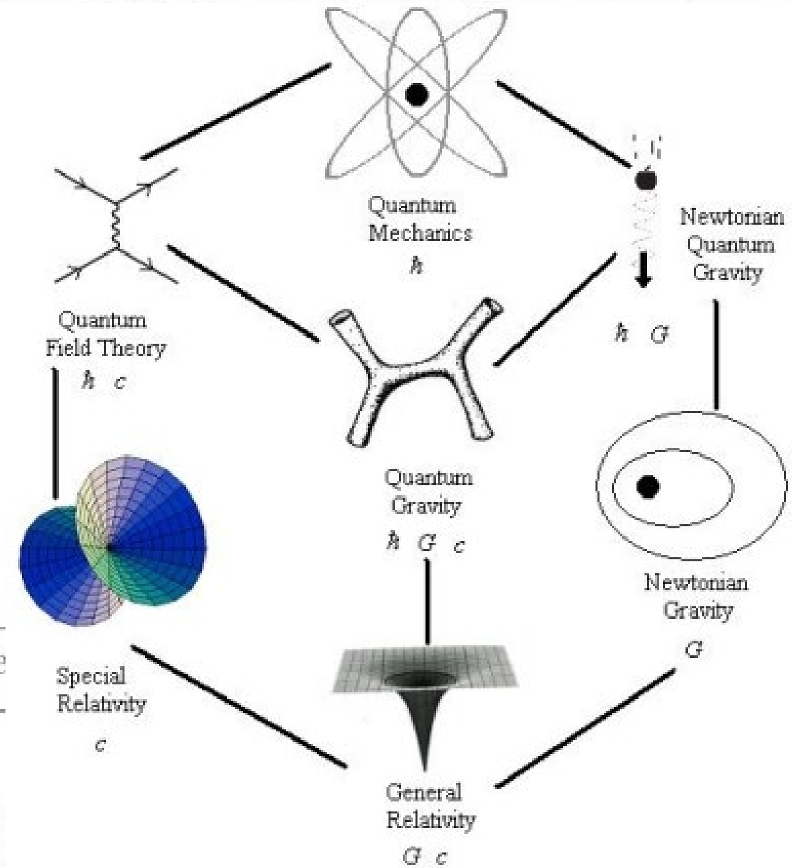


$$[c] \in \left\{\mathbb{C}P^1 \longrightarrow \mathbb{C}P^1\right\}/_{\sim \text{homotopy}} \simeq \mathbb{Z}$$

charge = homotopy class

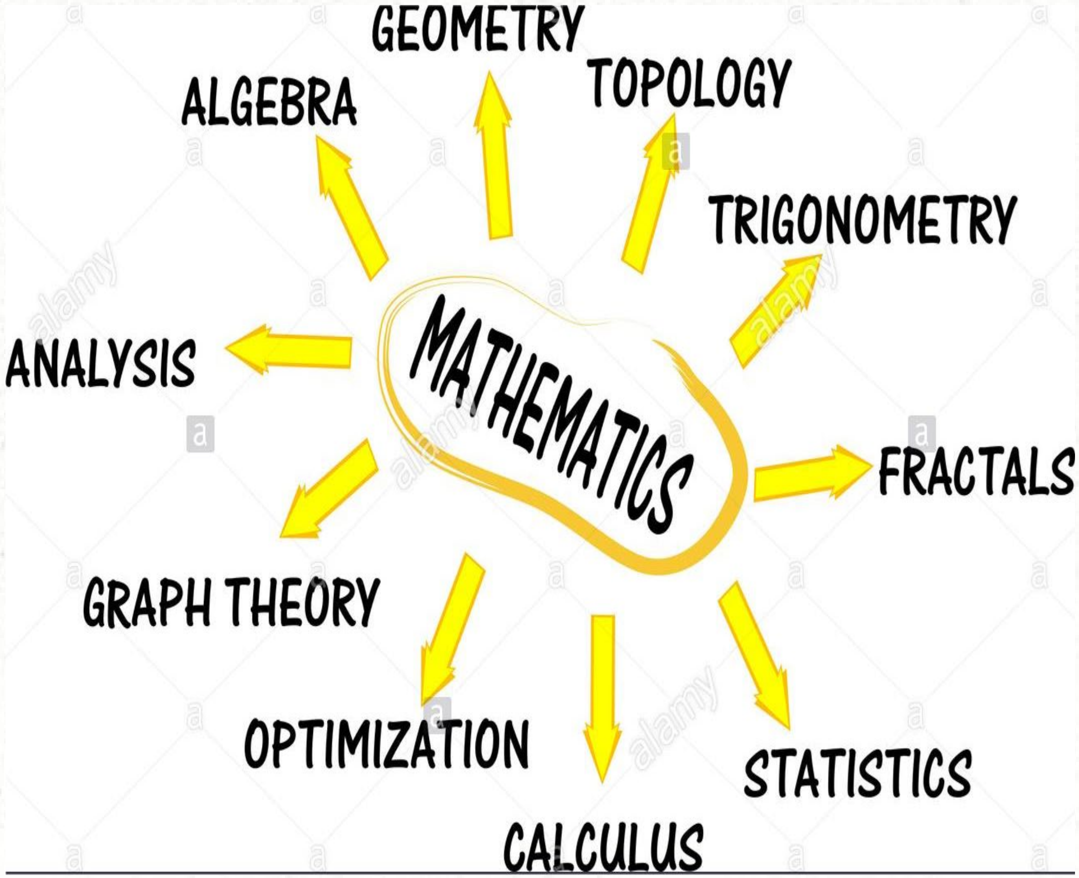
charge lattice

Atiyah-Hitchin charge quantization – The moduli space of SU(2) Yang-Mills monopoles is the cocycle space of complex-rational Cohomotopy of any sphere enclosing them.



Michael Atiyah (1929-2019)

The Greatest Problems and Prizes



David Hilbert (1862-1943)

- Opened the International Congress of Mathematicians in Paris in the year 1900.
- Outlined 23 major mathematical problems to provide solutions for in the coming new century.

Hilbert's Problems 1-6

1. Cantor's problem of the cardinal number of the continuum. (partially resolved)
2. The compatibility of the arithmetic axioms.
3. The equality of two volumes of two tetrahedra of equal bases and equal altitudes. - (resolved)
4. Problem of the straight line as the shortest distance between two points. (vague)
5. Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group. (i.e., are continuous groups automatically differential groups?)
6. Mathematical treatment of the axioms of physics.

Hilbert's Problems 7-14

7. Irrationality and transcendence of certain numbers.
8. Problems (with the distribution) of prime numbers.
9. Proof of the most general law of reciprocity in any number field.

Red - Unresolved

10. Determination of the solvability of a diophantine equation.
11. Quadratic forms with any algebraic numerical coefficients.
12. Extension of Kronecker's theorem on abelian fields.
13. Impossibility of the solution of the general equation of the 7th degree.
14. Proof of the finiteness of certain complete systems of functions.

Hilbert's Problems 15-23

15. Rigorous foundation of Schubert's calculus.
16. Problem of the topology of algebraic curves and surfaces.
17. Expression of definite forms by squares.
18. Building space from congruent polyhedra.
19. Are the solutions of regular problems in the calculus of variations always necessarily analytic?
20. The general problem of boundary curves.
21. Proof of the existence of linear differential equations having a prescribed monodromic group.
22. Uniformization of analytic relations by means of automorphic functions.
23. Further development of the methods of the calculus of variations.

Fields Medal

Prize (John Charles Fields)

- 2-4 mathematicians under 40 yrs.
- The International Congress of the International Mathematical Union
- Every four years.

Transire suum pectus mundoque potiri.
Rise above oneself and grasp the world.

See Fields Medalists

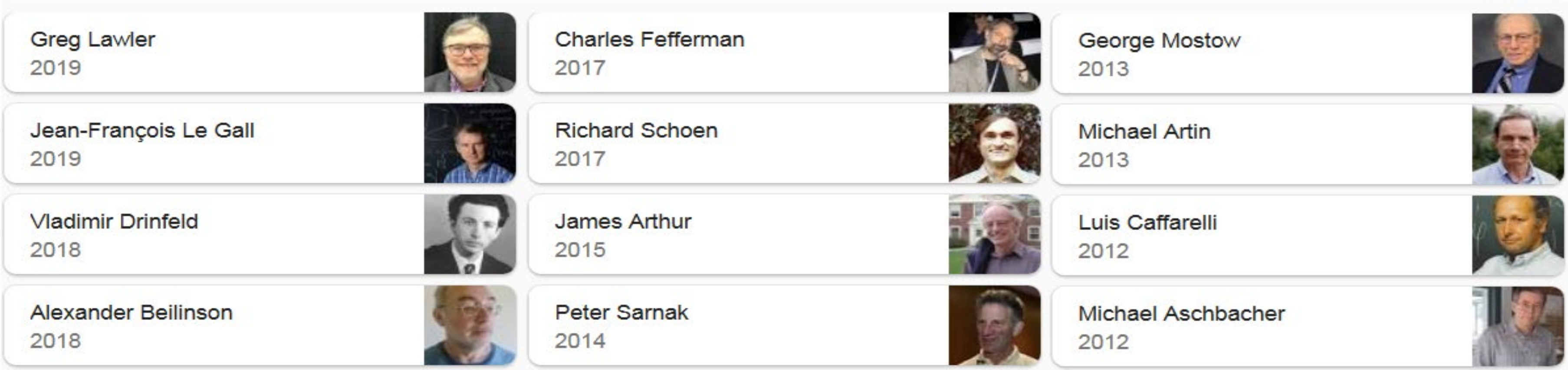
Abel Prize

- 1899 Proposed by the Norwegian mathematician Sophus Lie (1842-1899).
- He learned of Alfred Nobel's plans.
- First Awarded 2003
- See Laureates

Niels Henrik Abel (1802-1829)

Wolf Prize in Mathematics

- Awarded almost annually by the Wolf Foundation, in Israel.
- One of the 6 Wolf Prizes since 1978;
  - Agriculture, Chemistry, Medicine, Physics and Arts.
- See [Winners]

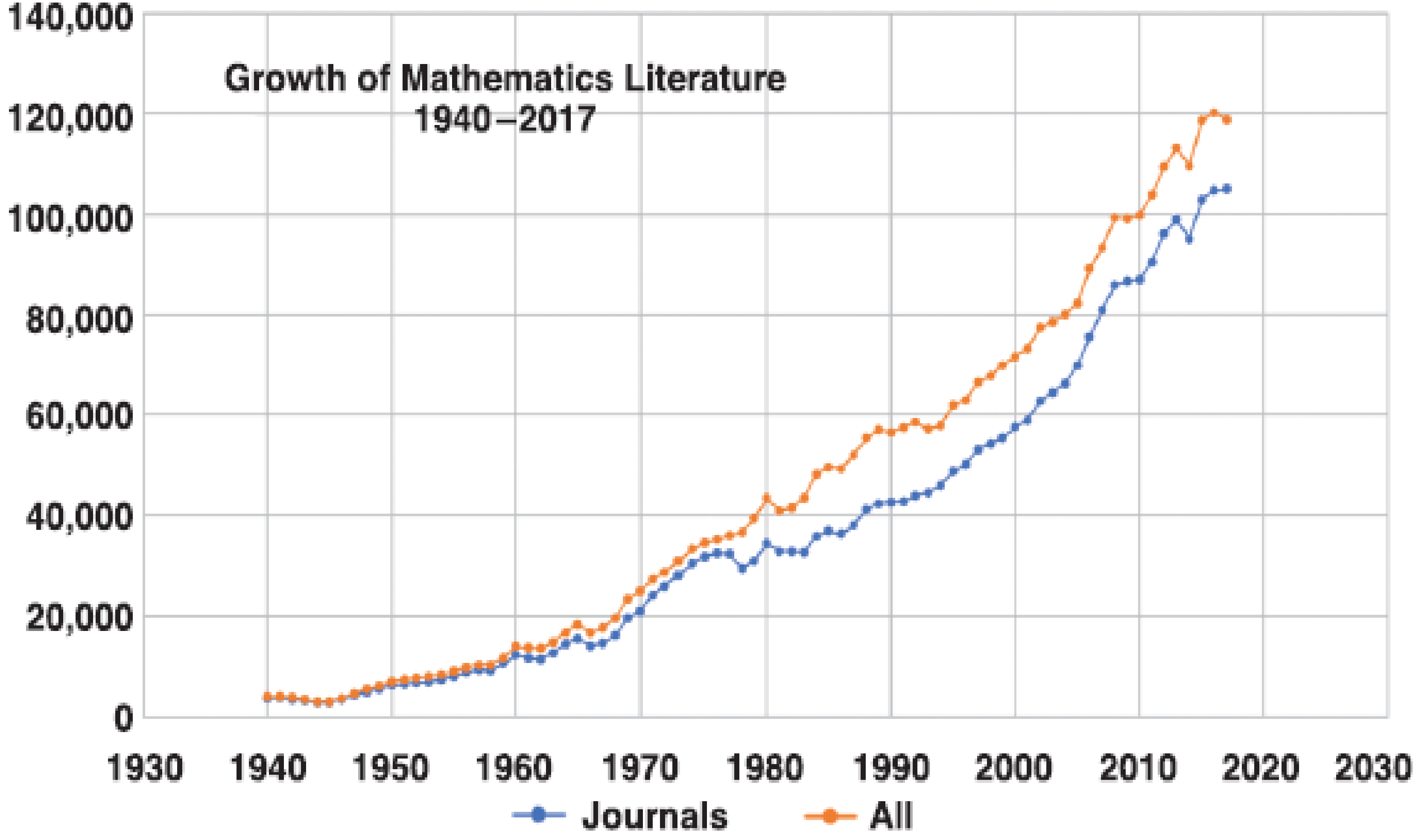


Millennium Prize Problems

Stated by the Clay Mathematics Institute on May 24, 2000 – for One Million Dollars

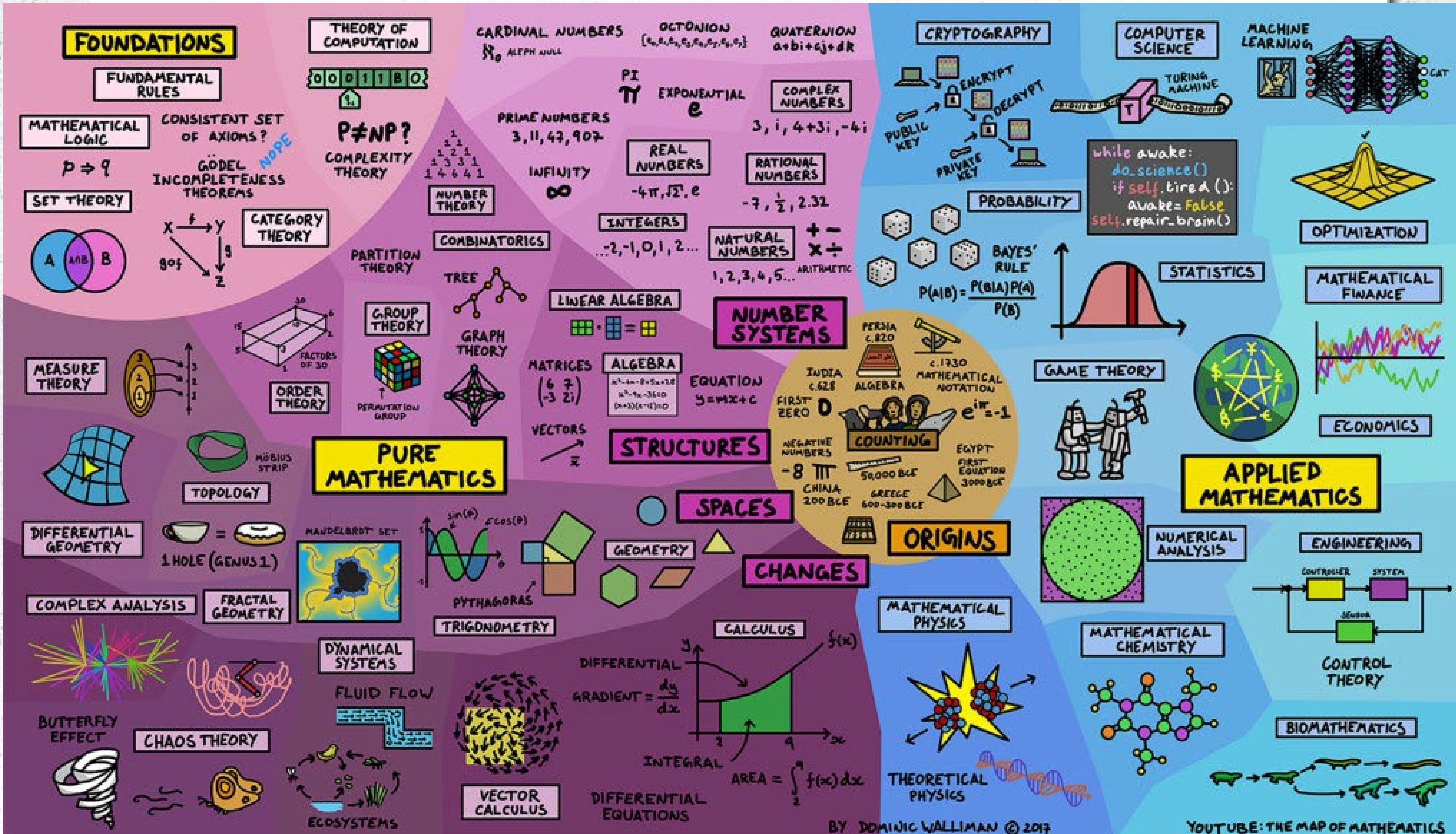
- Birch and Swinnerton-Dyer conjecture,
- Hodge conjecture,
- Navier–Stokes existence and smoothness,
- P versus NP problem,
- Poincaré conjecture (Solved! - Perelman),
- Riemann hypothesis, and
- Yang–Mills existence and mass gap

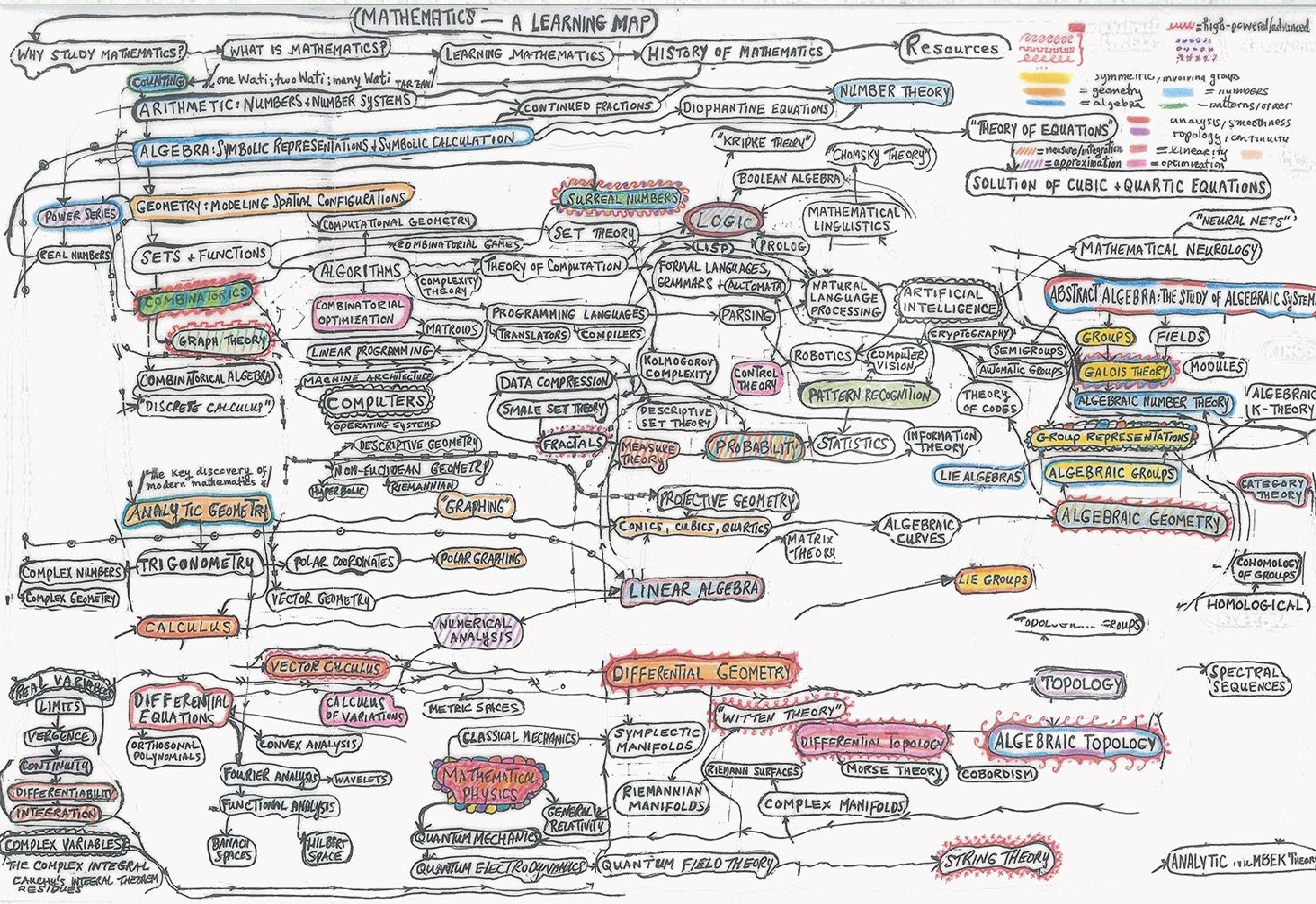
Mathematics Publications



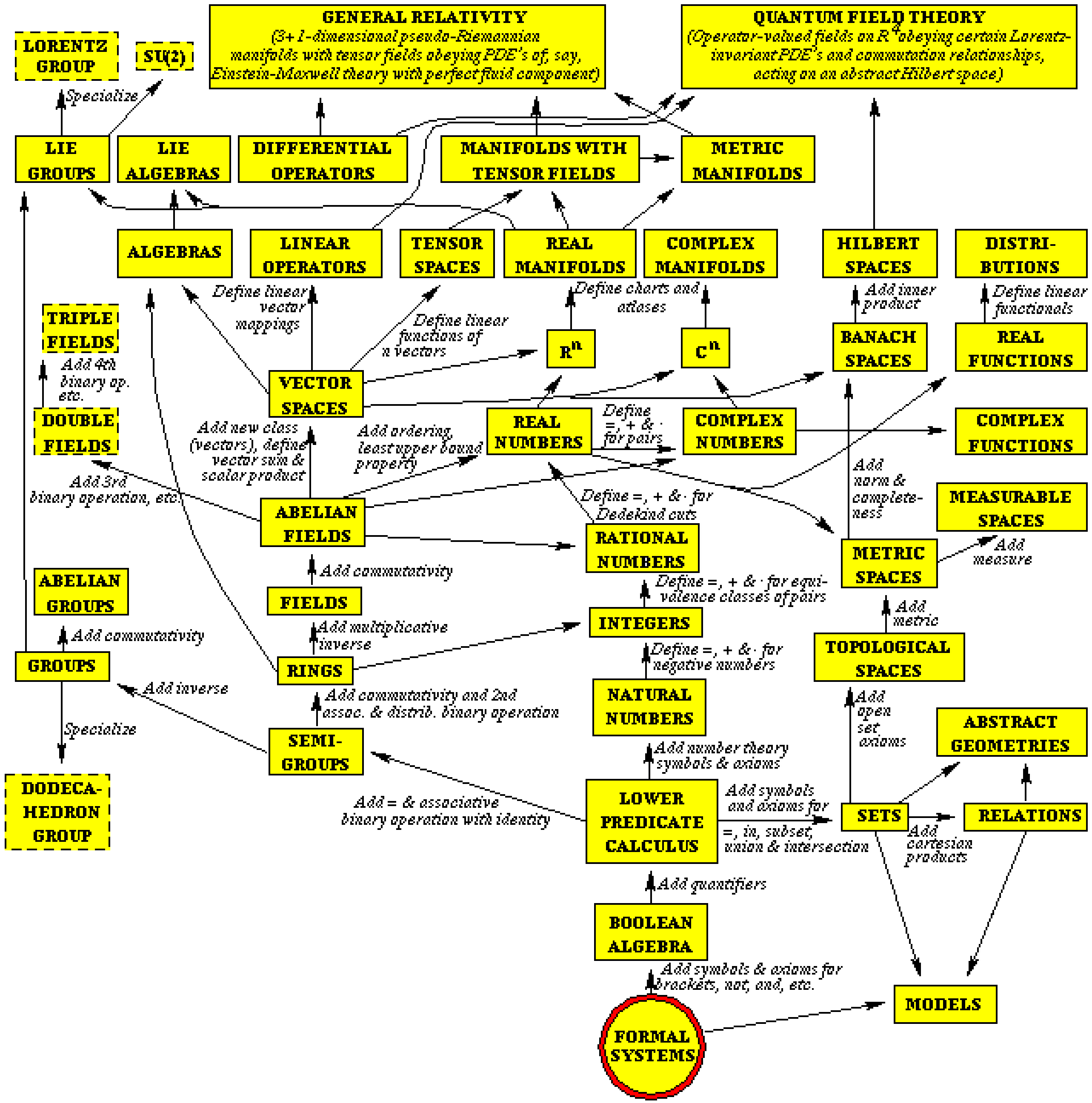
Mind Maps of Mathematics

https://www.sciencealert.com/this-mind-boggling-map-explains-how-everything-in-mathematics-is-connected-3





https://math.stackexchange.com/questions/166862/detailed-diagram-with-mathematical-fields-of-study



GENERAL RELATIVITY
(3+1-dimensional pseudo-Riemannian manifolds with tensor fields obeying PDE's of, say, Einstein-Maxwell theory with perfect fluid component)
QUANTUM FIELD THEORY
(Operator-valued fields on R^4 obeying certain Lorentz-invariant PDE's and commutation relationships, acting on an abstract Hilbert space)
LORENTZ GROUP
SU(2)
Specialize
LIE GROUPS
LIE ALGEBRAS
DIFFERENTIAL OPERATORS
MANIFOLDS WITH TENSOR FIELDS
METRIC MANIFOLDS
ALGEBRAS
LINEAR OPERATORS
TENSOR SPACES
REAL MANIFOLDS
COMPLEX MANIFOLDS
HILBERT SPACES
DISTRI-BUTIONS
Define linear vector mappings
Define linear functions of n vectors
Define charts and atlases
Add inner product
Define linear functionals
TRIPLE FIELDS
R^n
C^n
BANACH SPACES
REAL FUNCTIONS
Add 4th binary op. etc.
VECTOR SPACES
DOUBLE FIELDS
Add new class (vectors), define vector sum & scalar product
Add ordering, least upper bound property
REAL NUMBERS
Define =, + & · for pairs
COMPLEX NUMBERS
COMPLEX FUNCTIONS
Add norm & complete-ness
Add 3rd binary operation, etc.
ABELIAN FIELDS
Define =, + & · for Dedekind cuts
MEASURABLE SPACES
RATIONAL NUMBERS
METRIC SPACES
Add measure
Add commutativity
Define =, + & · for equi-valence classes of pairs
ABELIAN GROUPS
FIELDS
Add metric
INTEGERS
Add multiplicative inverse
Add commutativity
TOPOLOGICAL SPACES
Define =, + & · for negative numbers
GROUPS
RINGS
Add inverse
Add commutativity and 2nd assoc. & distrib. binary operation
NATURAL NUMBERS
Add open set axioms
ABSTRACT GEOMETRIES
Specialize
SEMI-GROUPS
Add number theory symbols & axioms
DODECA-HEDRON GROUP
Add = & associative binary operation with identity
LOWER PREDICATE CALCULUS
Add symbols and axioms for =, in, subset, union & intersection
SETS
RELATIONS
Add cartesian products
Add quantifiers
BOOLEAN ALGEBRA
Add symbols & axioms for brackets, not, and, etc.
MODELS
FORMAL SYSTEMS

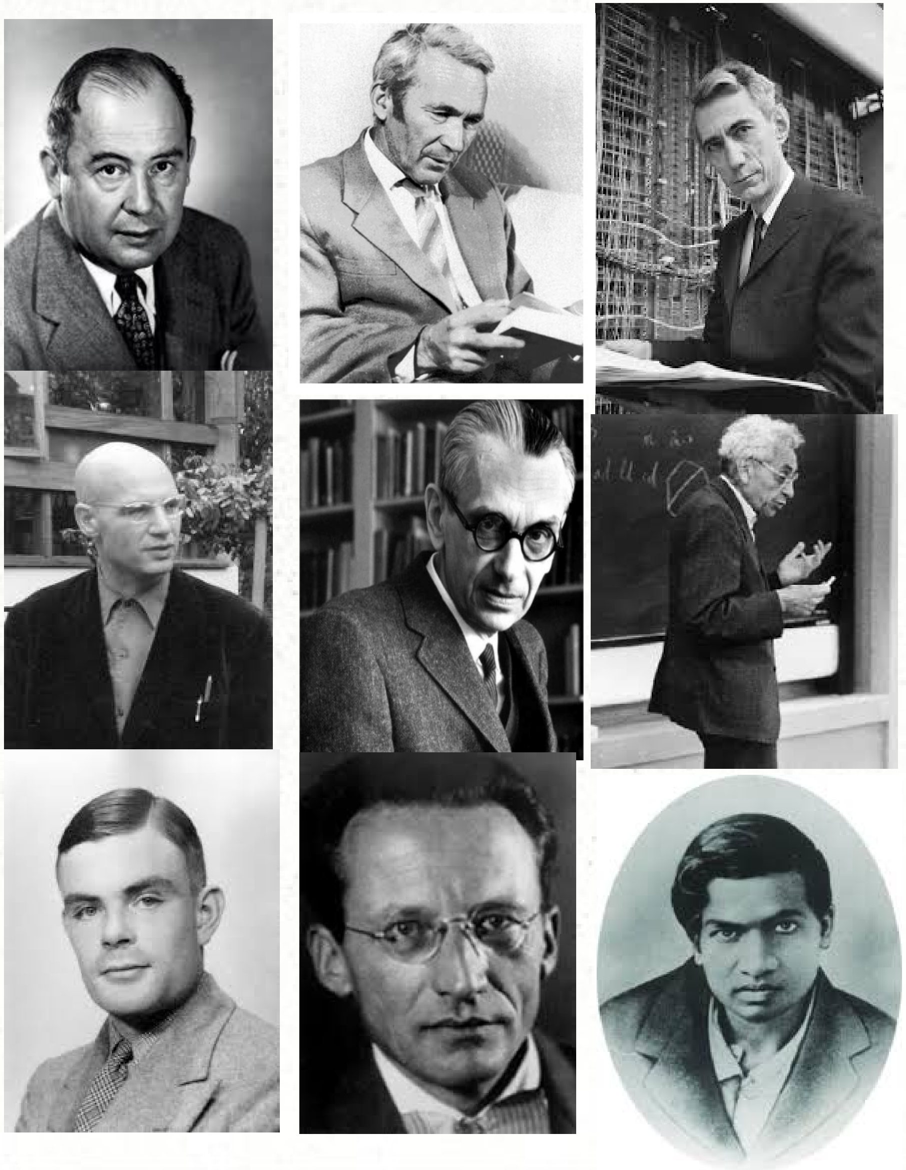
Top Numerical Algorithms (2000)

Another list (2015)

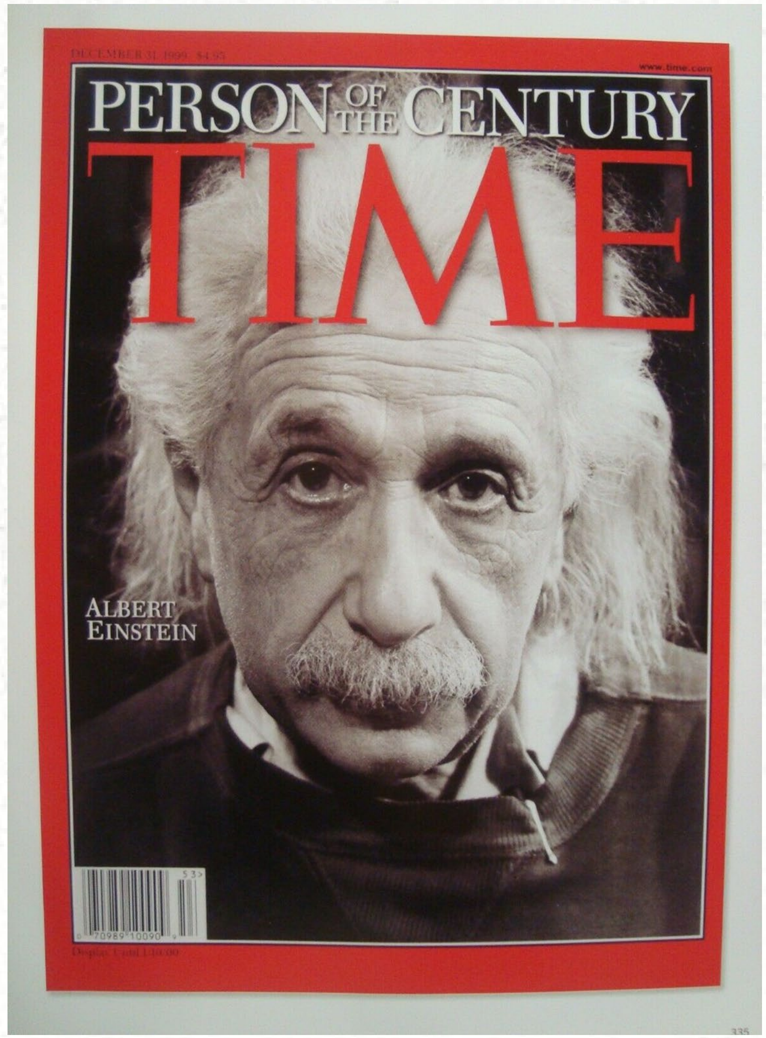
1. Newton and quasi-Newton methods
2. Matrix factorizations (LU, Cholesky, QR)
3. Singular value decomposition, QR and QZ algorithms
4. Monte-Carlo methods
5. Fast Fourier transform
6. Krylov subspace methods (conjugate gradients, Lanczos, GMRES, minres)
7. JPEG
8. PageRank
9. Simplex algorithm
10. Kalman filter

Greatest Mathematicians of 20th Century?

- John von Neumann
- Andrey Kolmogorov
- Claude Shannon
- Alexander Grothendieck
- Kurt Godel
- Paul Erdos
- Alan Turing
- Hermann Weyl
- Srinivasa Ramanujan

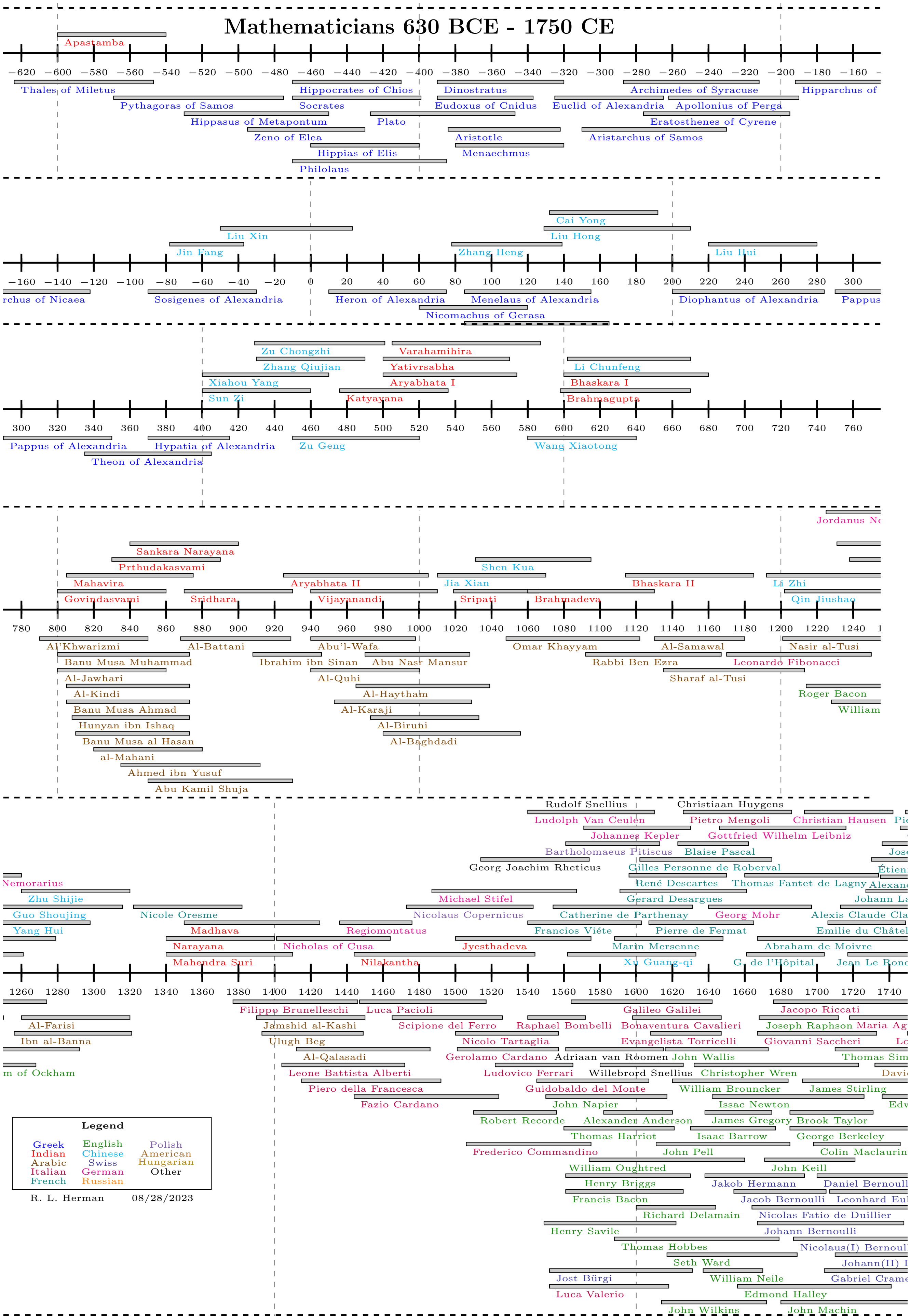


Person of the Century

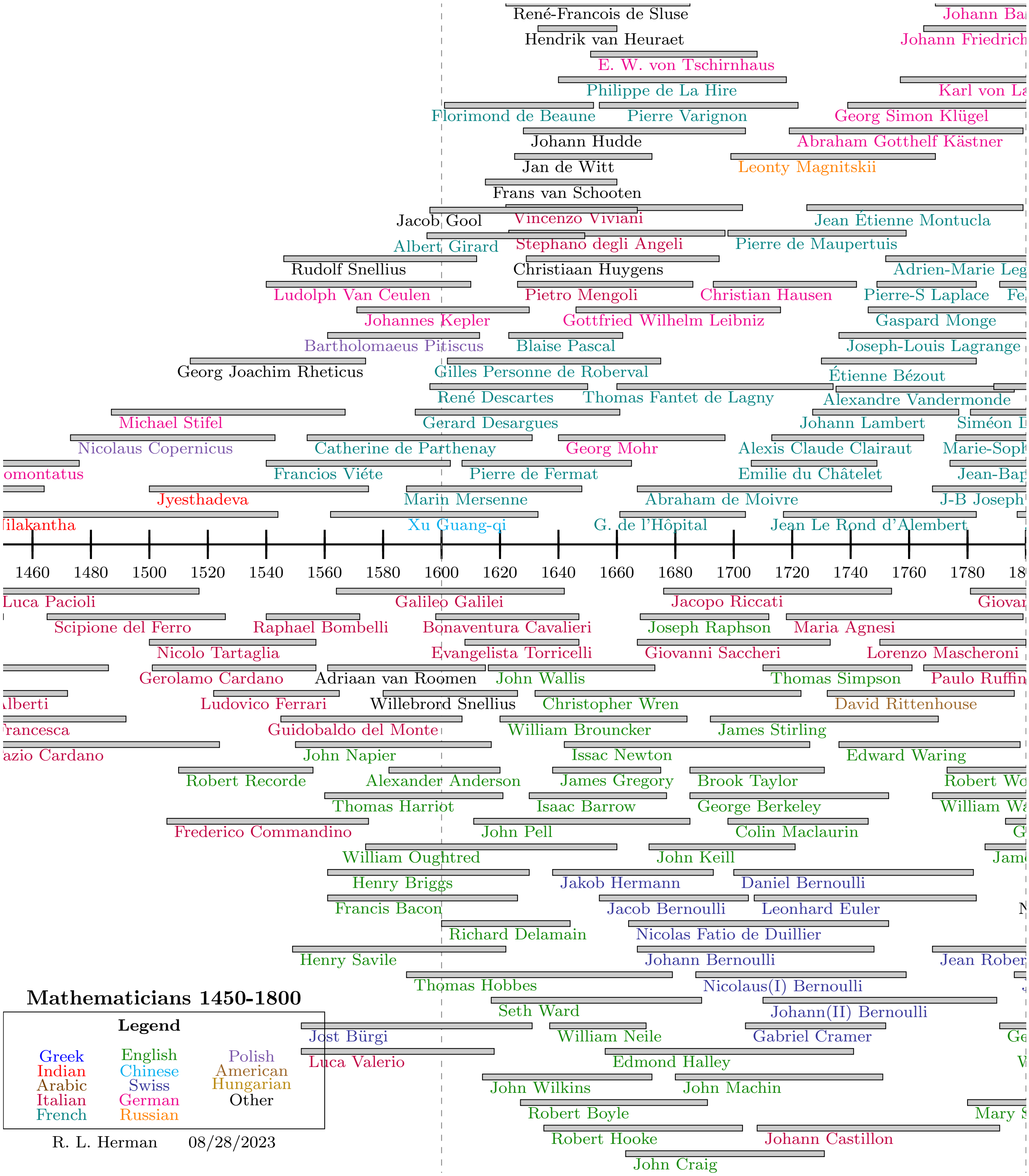


Good Luck on Your Finals!

Mathematicians 630 BCE - 1750 CE



Mathematicians 1590-2000


René-Francois de Sluse
Hendrik van Heuraet
E. W. von Tschirnhaus
Philippe de La Hire
Florimond de Beaune
Pierre Varignon
Johann Hudde
Jan de Witt
Frans van Schooten
Jacob Gool
Vincenzo Viviani
Albert Girard
Stephano degli Angeli
Rudolf Snellius
Christiaan Huygens
Ludolph Van Ceulen
Pietro Mengoli
Christian Hausen
Johannes Kepler
Gottfried Wilhelm Leibniz
Bartholomaeus Pitiscus
Blaise Pascal
Georg Joachim Rheticus
Gilles Personne de Roberval
René Descartes
Thomas Fantet de Lagny
Michael Stifel
Gerard Desargues
Nicolaus Copernicus
Catherine de Parthenay
Georg Mohr
Francios Viéte
Pierre de Fermat
Jyesthadeva
Marin Mersenne
Xu Guang-qi
G. de l'Hôpital
Georg Simon Klügel
Abraham Gotthelf Kästner
Leonty Magnitskii
Jean Étienne Montucla
Pierre de Maupertuis
Pierre-S Laplace
Gaspard Monge
Joseph-Louis Lagrange
Étienne Bézout
Alexandre Vandermonde
Johann Lambert
Alexis Claude Clairaut
Emilie du Châtelet
Abraham de Moivre
Jean Le Rond d'Alembert
1460 1480 1500 1520 1540 1560 1580 1600 1620 1640 1660 1680 1700 1720 1740 1760 1780
Luca Pacioli
Galileo Galilei
Jacopo Riccati
Scipione del Ferro
Raphael Bombelli
Bonaventura Cavalieri
Joseph Raphson
Maria Agnesi
Nicolo Tartaglia
Evangelista Torricelli
Giovanni Saccheri
Lorenzo Mascheroni
Gerolamo Cardano
Adriaan van Roomen
John Wallis
Thomas Simpson
Ludovico Ferrari
Willebrord Snellius
Christopher Wren
David Rittenhouse
Guidobaldo del Monte
William Brouncker
James Stirling
John Napier
Issac Newton
Edward Waring
Robert Recorde
Alexander Anderson
James Gregory
Brook Taylor
Thomas Harriot
Isaac Barrow
George Berkeley
Frederico Commandino
John Pell
Colin Maclaurin
William Oughtred
John Keill
Henry Briggs
Jakob Hermann
Daniel Bernoulli
Francis Bacon
Jacob Bernoulli
Leonhard Euler
Richard Delamain
Nicolas Fatio de Duillier
Henry Savile
Johann Bernoulli
Thomas Hobbes
Nicolaus(I) Bernoulli
Mathematicians 1450-1800
Seth Ward
Johann(II) Bernoulli
Jost Bürgi
William Neile
Gabriel Cramer
Luca Valerio
Edmond Halley
John Wilkins
John Machin
Robert Boyle
Robert Hooke
Johann Castillon
John Craig
Legend
Greek
Indian
Arabic
Italian
French
English
Chinese
Swiss
German
Russian
Polish
American
Hungarian
Other
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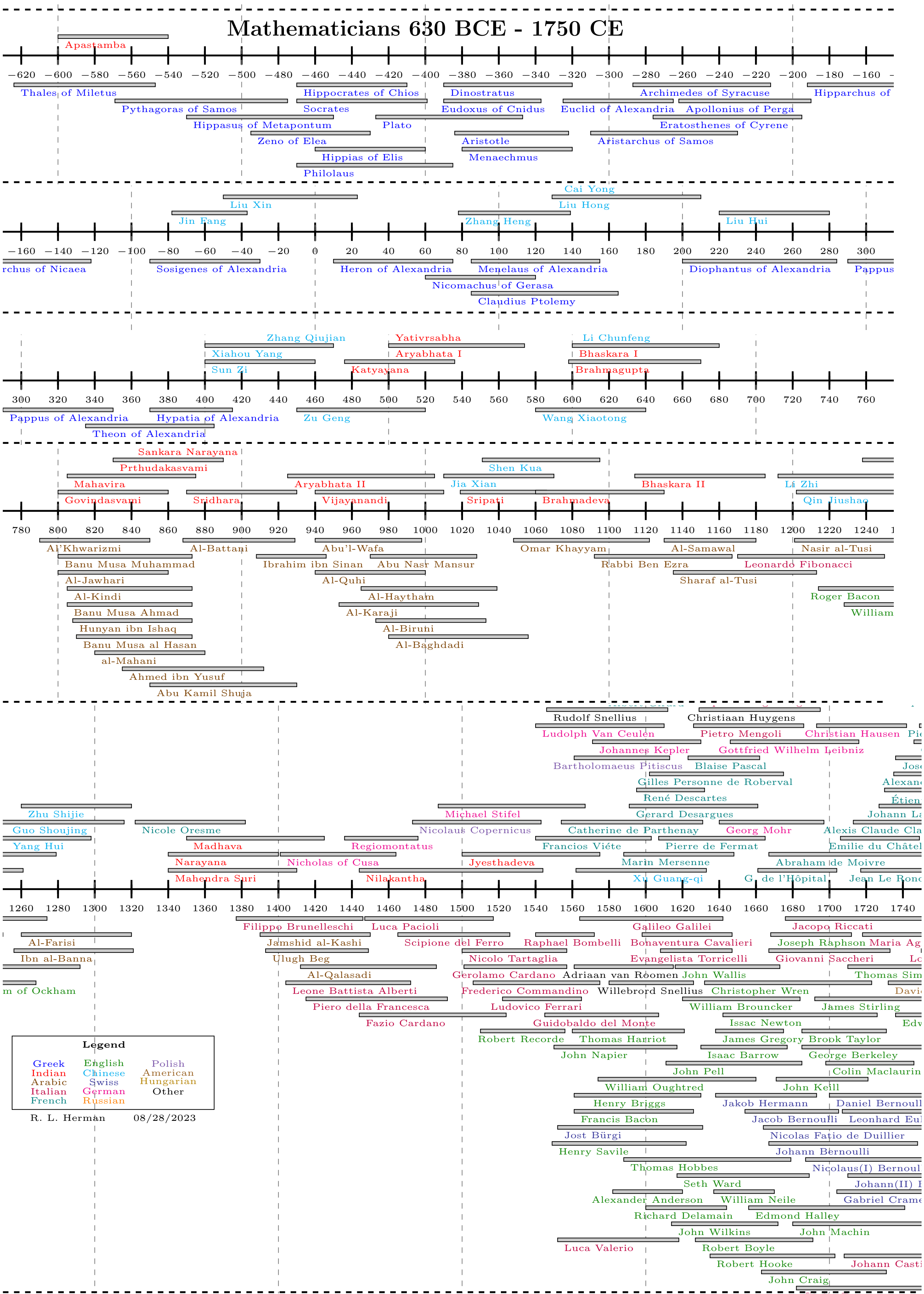
Mathematicians 630 BCE - 1750 CE
Apastamba
Thales of Miletus
Pythagoras of Samos
Hippasus of Metapontum
Zeno of Elea
Hippias of Elis
Philolaus
Hippocrates of Chios
Socrates
Plato
Dinostratus
Eudoxus of Cnidus
Aristotle
Menaechmus
Euclid of Alexandria
Aristarchus of Samos
Archimedes of Syracuse
Apollonius of Perga
Eratosthenes of Cyrene
Hipparchus of
Liu Xin
Jin Fang
Zhang Heng
Cai Yong
Liu Hong
Liu Hui
rchus of Nicaea
Sosigenes of Alexandria
Heron of Alexandria
Nicomachus of Gerasa
Menelaus of Alexandria
Claudius Ptolemy
Diophantus of Alexandria
Pappus
Zhang Qiujian
Xiahou Yang
Sun Zi
Yativrsabha
Aryabhata I
Katyayana
Li Chunfeng
Bhaskara I
Brahmagupta
Pappus of Alexandria
Hypatia of Alexandria
Theon of Alexandria
Zu Geng
Wang Xiaotong
Sankara Narayana
Prthudakasvami
Mahavira
Govindasvami
Sridhara
Aryabhata II
Vijayanandi
Shen Kua
Jia Xian
Sripati
Brahmadeva
Bhaskara II
Li Zhi
Qin Jiushao
Al'Khwarizmi
Banu Musa Muhammad
Al-Jawhari
Al-Kindi
Banu Musa Ahmad
Hunyan ibn Ishaq
Banu Musa al Hasan
al-Mahani
Ahmed ibn Yusuf
Abu Kamil Shuja
Al-Battani
Ibrahim ibn Sinan
Abu'l-Wafa
Al-Quhi
Abu Nasr Mansur
Al-Haytham
Al-Karaji
Al-Biruni
Al-Baghdadi
Omar Khayyam
Rabbi Ben Ezra
Al-Samawal
Sharaf al-Tusi
Nasir al-Tusi
Leonardo Fibonacci
Roger Bacon
William
Zhu Shijie
Guo Shoujing
Yang Hui
Nicole Oresme
Madhava
Narayana
Mahendra Suri
Nicholas of Cusa
Regiomontatus
Nilakantha
Michael Stifel
Nicolaus Copernicus
Jyesthadeva
Rudolf Snellius
Ludolph Van Ceulen
Johannes Kepler
Bartholomaeus Pitiscus
Christiaan Huygens
Pietro Mengoli
Christian Hausen
Gottfried Wilhelm Leibniz
Blaise Pascal
Gilles Personne de Roberval
René Descartes
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Abraham de Moivre
Alexis Claude Cla
Emilie du Châtel
Jean Le Ronc
Johann La
Étien
Alexan
Jose
Al-Farisi
Ibn al-Banna
m of Ockham
Filippo Brunelleschi
Jamshid al-Kashi
Ulugh Beg
Al-Qalasadi
Leone Battista Alberti
Piero della Francesca
Fazio Cardano
Luca Pacioli
Scipione del Ferro
Nicolo Tartaglia
Gerolamo Cardano
Frederico Commandino
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Robert Recorde
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John Napier
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James Gregory
Isaac Barrow
John Pell
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Jost Bürgi
Henry Savile
Thomas Hobbes
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William Neile
Richard Delamain
Edmond Halley
John Wilkins
Luca Valerio
Robert Boyle
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Maria Ag
Giovanni Saccheri
Lo
Thomas Sim
Davi
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George Berkeley
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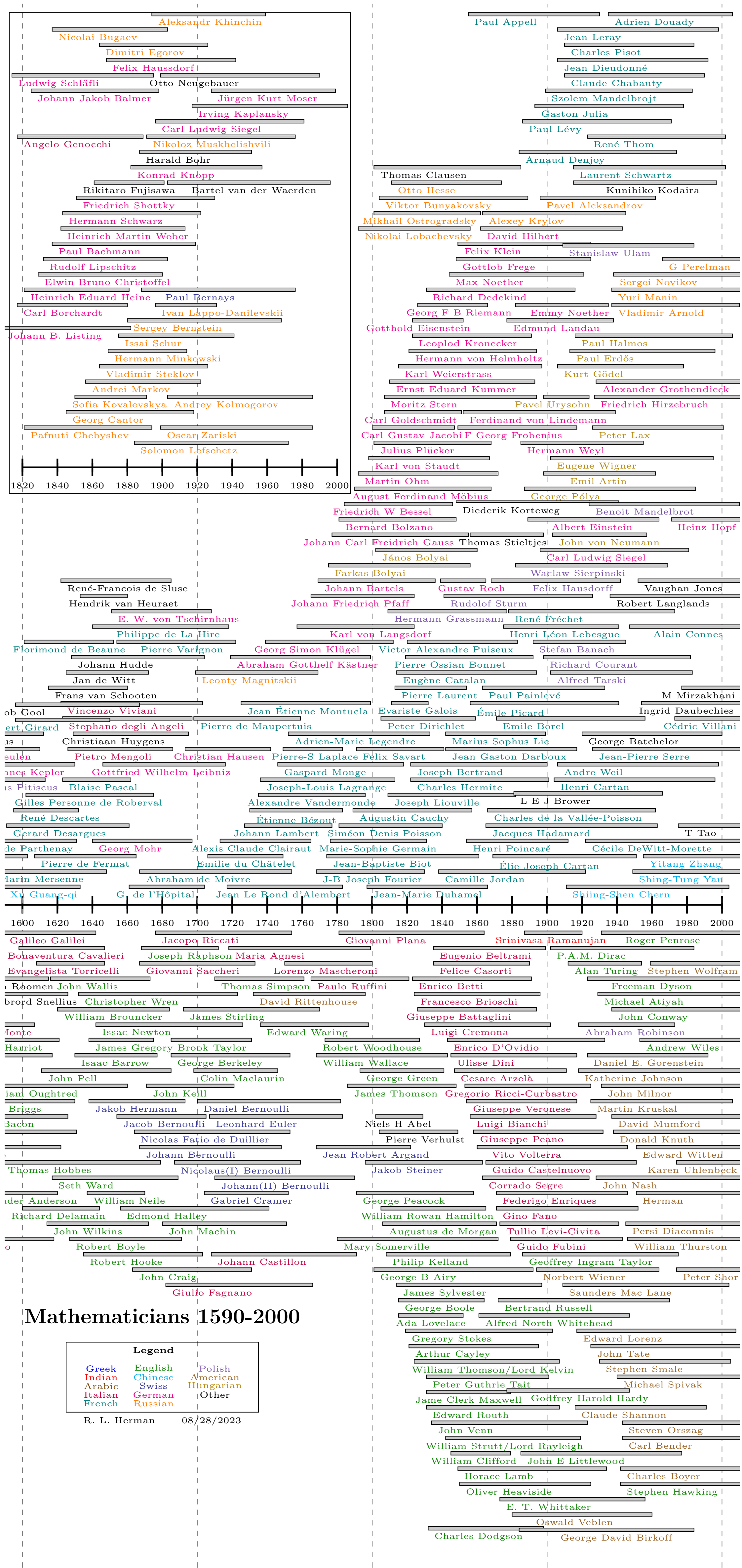
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Aleksandr Khinchin
Nicolai Bugaev
Dimitri Egorov
Felix Haussdorf
Ludwig Schläfli
Otto Neugebauer
Johann Jakob Balmer
Jürgen Kurt Moser
Irving Kaplansky
Carl Ludwig Siegel
Angelo Genocchi
Nikoloz Muskhelishvili
Harald Bohr
Konrad Knopp
Rikitarō Fujisawa
Bartel van der Waerden
Friedrich Shottky
Hermann Schwarz
Heinrich Martin Weber
Paul Bachmann
Rudolf Lipschitz
Elwin Bruno Christoffel
Heinrich Eduard Heine
Paul Bernays
Carl Borchardt
Ivan Lappo-Danilevskii
Johann B. Listing
Sergey Bernstein
Issai Schur
Hermann Minkowski
Vladimir Steklov
Andrei Markov
Sofia Kovalevskaya
Andrey Kolmogorov
Georg Cantor
Pafnuti Chebyshev
Oscar Zariski
Solomon Lefschetz
1820
1840
1860
1880
1900
1920
1940
1960
1980
2000
Paul Appell
Adrien Douady
Jean Leray
Charles Pisot
Jean Dieudonné
Claude Chabauty
Szolem Mandelbrojt
Gaston Julia
Paul Lévy
René Thom
Arnaud Denjoy
Thomas Clausen
Laurent Schwartz
Otto Hesse
Kunihiko Kodaira
Viktor Bunyakovsky
Pavel Aleksandrov
Mikhail Ostrogradsky
Alexey Krylov
Nikolai Lobachevsky
David Hilbert
Felix Klein
Stanislaw Ulam
Gottlob Frege
G Perelman
Max Noether
Sergei Novikov
Richard Dedekind
Yuri Manin
Georg F B Riemann
Emmy Noether
Vladimir Arnold
Gotthold Eisenstein
Edmund Landau
Leopold Kronecker
Paul Halmos
Hermann von Helmholtz
Paul Erdős
Karl Weierstrass
Kurt Gödel
Ernst Eduard Kummer
Alexander Grothendieck
Moritz Stern
Pavel Urysohn
Friedrich Hirzebruch
Carl Goldschmidt
Ferdinand von Lindemann
Carl Gustav Jacobi
F Georg Frobenius
Peter Lax
Julius Plücker
Hermann Weyl
Karl von Staudt
Eugene Wigner
Martin Ohm
Emil Artin
August Ferdinand Möbius
George Pólya
Friedrich W Bessel
Diederik Korteweg
Benoit Mandelbrot
Bernard Bolzano
Albert Einstein
Heinz Hopf
Johann Carl Freidrich Gauss
Thomas Stieltjes
John von Neumann
János Bolyai
Carl Ludwig Siegel
Farkas Bolyai
Waclaw Sierpinski
René-Francois de Sluse
Johann Bartels
Gustav Roch
Felix Hausdorff
Vaughan Jones
Hendrik van Heuraet
Johann Friedrich Pfaff
Rudolof Sturm
Robert Langlands
E. W. von Tschirnhaus
Hermann Grassmann
René Fréchet
Philippe de La Hire
Karl von Langsdorf
Henri Léon Lebesgue
Alain Connes
Florimond de Beaune
Pierre Varignon
Georg Simon Klügel
Victor Alexandre Puiseux
Stefan Banach
Johann Hudde
Abraham Gotthelf Kästner
Pierre Ossian Bonnet
Richard Courant
Jan de Witt
Leonty Magnitskii
Eugène Catalan
Alfred Tarski
Frans van Schooten
Pierre Laurent
Paul Painlevé
M Mirzakhani
ob Gool
Vincenzo Viviani
Jean Étienne Montucla
Evariste Galois
Émile Picard
Ingrid Daubechies
ert Girard
Stephano degli Angeli
Pierre de Maupertuis
Peter Dirichlet
Emile Borel
Cédric Villani
Christiaan Huygens
Adrien-Marie Legendre
Marius Sophus Lie
George Batchelor
Pietro Mengoli
Christian Hausen
Pierre-S Laplace
Félix Savart
Jean Gaston Darboux
Jean-Pierre Serre
Gottfried Wilhelm Leibniz
Gaspard Monge
Joseph Bertrand
Andre Weil
Blaise Pascal
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Charles Hermite
Henri Cartan
L E J Brower
Gilles Personne de Roberval
Alexandre Vandermonde
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René Descartes
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Augustin Cauchy
Charles de la Vallée-Poisson
Gerard Desargues
Johann Lambert
Siméon Denis Poisson
Jacques Hadamard
T Tao
Georg Mohr
Alexis Claude Clairaut
Marie-Sophie Germain
Henri Poincaré
Cécile DeWitt-Morette
Pierre de Fermat
Emilie du Châtelet
Jean-Baptiste Biot
Élie Joseph Cartan
Yitang Zhang
Abraham de Moivre
J-B Joseph Fourier
Camille Jordan
Shing-Tung Yau
Xu Guang-qi
G. de l'Hôpital
Jean Le Rond d'Alembert
Jean-Marie Duhamel
Shiing-Shen Chern
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Galileo Galilei
Jacopo Riccati
Giovanni Plana
Srinivasa Ramanujan
Roger Penrose
Bonaventura Cavalieri
Joseph Raphson
Maria Agnesi
Eugenio Beltrami
P.A.M. Dirac
Evangelista Torricelli
Giovanni Saccheri
Lorenzo Mascheroni
Felice Casorti
Alan Turing
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Paulo Ruffini
Enrico Betti
Freeman Dyson
Christopher Wren
David Rittenhouse
Francesco Brioschi
Michael Atiyah
William Brouncker
James Stirling
Giuseppe Battaglini
John Conway
Issac Newton
Edward Waring
Luigi Cremona
Abraham Robinson
James Gregory
Brook Taylor
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Andrew Wiles
Isaac Barrow
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John Kell
James Thomson
Gregorio Ricci-Curbastro
John Milnor
Jakob Hermann
Daniel Bernoulli
Giuseppe Veronese
Martin Kruskal
Jacob Bernoulli
Leonhard Euler
Niels H Abel
Luigi Bianchi
David Mumford
Nicolas Fatio de Duillier
Pierre Verhulst
Giuseppe Peano
Donald Knuth
Johann Bernoulli
Jean Robert Argand
Vito Volterra
Edward Witten
Thomas Hobbes
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Jakob Steiner
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Karen Uhlenbeck
Seth Ward
Johann(II) Bernoulli
Corrado Segre
John Nash
William Neile
Gabriel Cramer
George Peacock
Federigo Enriques
Herman
Richard Delamain
Edmond Halley
William Rowan Hamilton
Gino Fano
John Wilkins
John Machin
Augustus de Morgan
Tullio Levi-Civita
Persi Diaconnis
Robert Boyle
Mary Somerville
Guido Fubini
William Thurston
Robert Hooke
Johann Castillon
Philip Kelland
Geoffrey Ingram Taylor
John Craig
George B Airy
Norbert Wiener
Peter Shor
Giulio Fagnano
James Sylvester
Saunders Mac Lane
George Boole
Bertrand Russell
Ada Lovelace
Alfred North Whitehead
Gregory Stokes
Edward Lorenz
Arthur Cayley
John Tate
William Thomson/Lord Kelvin
Stephen Smale
Peter Guthrie Tait
Michael Spivak
James Clerk Maxwell
Godfrey Harold Hardy
Edward Routh
Claude Shannon
John Venn
Steven Orszag
William Strutt/Lord Rayleigh
Carl Bender
William Clifford
John E Littlewood
Horace Lamb
Charles Boyer
Oliver Heaviside
Stephen Hawking
E. T. Whittaker
Oswald Veblen
Charles Dodgson
George David Birkoff