Project: Getting nervous?
This problem from the FNC authors was modified.

There are many important *activator–inhibitor* phenomena, in which two components of a system essentially compete to pull it in different directions, leading to oscillatory behavior. A classic example is the *FitzHugh–Nagumo* model of a neuron:

$$\begin{aligned} \frac{dv}{dt} &= v(a-v)(v-1) - w + \beta, \\ \frac{dw}{dt} &= bv - cw, \end{aligned} \tag{1}$$

where $0 < a < 1$, and b, c, and β are positive constants. In physically realistic models, v stands for an activation potential, and the peaks in v correspond to "firing" a signal. The parameter β is analogous to an external current enhancing the growth of v, and it affects the period of the oscillations.

A steady state of the ODE occurs when $v'(t) = w'(t) = 0$. These conditions imply a nonlinear system of two equations for the two variables v and w.

You can see solutions of the model as an initial-value problem for two different values of β in Figure 1, for which

$$a = 0.25, \quad b = c = 0.02, \quad v(0) = 0.1, \quad w(0) = 0. \tag{2}$$

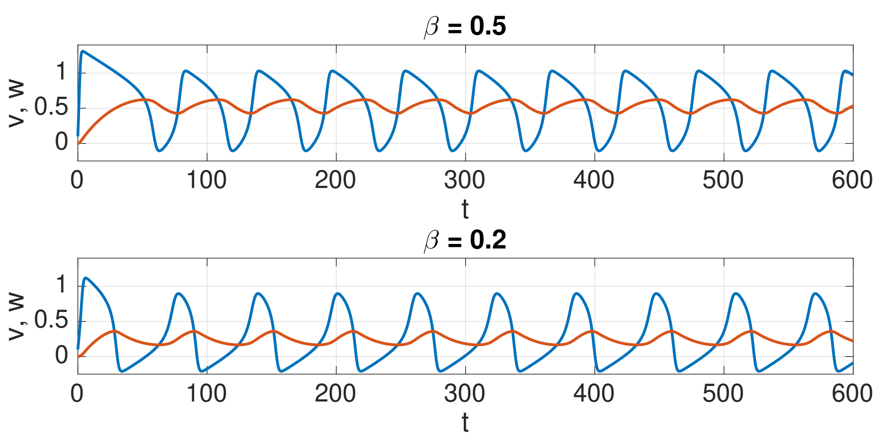

Figure 1: Solutions of the system (1) using the values in (2) and two different values of β.

Objective 1. Recreate Figure 1 using the single-neuron model (1) and the parameters in (2). Then, on another graph, plot the two solutions in the (v, w) phase plane.

From Figure 1 (top half) we can estimate the period of oscillation when $\beta = 0.5$. Imagine drawing a horizontal line in Figure 1 with $v = 1/2$. For each oscillation, this line makes two intersections with the graph of $v(t)$ in blue: one when v is increasing, and another when v is decreasing. Hence we can define one period of the oscillation as the time between two similar kinds of intersections (i.e., both increasing or both decreasing).

We can extend the idea to get much more precision than a simple graphical estimate allows. Note that whichever kind of intersection we pick (increasing or decreasing), there is a particular value of w associated with it. So we can define the period T via the conditions $v(0) = v(T) = 1/2$, $w(0) = w(T)$. If we let the initial conditions be $v(0) = 1/2$, $w(0) = \gamma$, then we can solve the model over $[0, T]$ and check the equations $v(T) = 1/2$, $w(T) = \gamma$, which will be valid only when γ and T are chosen correctly. In other words, we have defined a system of two equations $\mathbf{f}(\mathbf{x}) = \mathbf{zero}$ for the unknowns $x_1 = \gamma$ and $x_2 = T$.

Objective 2. Find the oscillation period with $a = 0.25$, $b = c = 0.02$, and $\beta = 0.5$ by using `levenberg` from the text on the system of equations described above. As always, you must provide `levenberg` with the function $\mathbf{f}(\mathbf{x})$. You can find a good starting value for `levenberg` using the graph you made for Objective 1.

Some oscillators exhibit an interesting phenomenon of synchronization. If the oscillators are coupled together, even weakly, they can adjust their individual natural frequencies to oscillate together. This can occur with pendulums, fireflies, and neurons. To extend our equations to two neurons, we use

$$\begin{aligned}
\frac{dv_1}{dt} &= v_1(a - v_1)(v_1 - 1) - w_1 + \beta_1 + \alpha v_2, \\
\frac{dv_2}{dt} &= v_2(a - v_2)(v_2 - 1) - w_2 + \beta_2 + \alpha v_1, \\
\frac{dw_1}{dt} &= b v_1 - c w_1, \\
\frac{dw_2}{dt} &= b v_2 - c w_2,
\end{aligned} \tag{3}$$

Here, the excitation of v_i occurs in response to the firing of the other neuron, with a strength expressed as $\alpha > 0$. Larger values of α mean stronger coupling.

Objective 3. Implement the model (3), using $\beta_1 = 0.5$, $\beta_2 = 0.2$, $v_1(0) = v_2(0) = 0.25$, $w_1(0) = w_2(0) = 0$, and a, b, and c as given in (2). Make a 2×1 subplot grid. In the top axes plot $v_1(t)$ and $v_2(t)$ over $0 \le t \le 1000$ for $\alpha = 0.002$, and in the bottom axes make the same plot for the solution with $\alpha = 0.05$. It should be clear that the top pair never synchronize while the bottom pair quickly reach the same frequency.