In class we derived LU Factorization, or Decomposition, by carrying out Gaussian Elimination using linear algebra. We started with the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}.$$

Now consider the 4×4 identity matrix,

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the columns, we define

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

This gives

$$\begin{aligned} \mathbf{e}_1^T &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{e}_2^T &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \\ \mathbf{e}_3^T &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}, \\ \mathbf{e}_4^T &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

We can use the outer products of these standard basis vectors to proceed with the row operations in Gaussian elimination. Define $\alpha_{21} = A_{21}/A_{11} = 2$. Then, define L_{21} as

$$\begin{aligned} L_{21} &= I - \alpha_{21}\mathbf{e}_2\mathbf{e}_1^T \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \alpha_{21} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{1}$$

Now, compute

$$
\begin{aligned}
L_{21}A &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}. \qquad (2)
\end{aligned}
$$

Note that this is the equivalent of the first row operation.

We can replace 3 and -1 in the first column by computing $\alpha_{31} = A_{31}/A_{11} = 3$, $\alpha_{41} = A_{41}/A_{11} = -1$, and

$$
\begin{aligned}
L_{31} &= I - \alpha_{31}\mathbf{e}_3\mathbf{e}_1^T \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\
L_{41} &= I - \alpha_{41}\mathbf{e}_4\mathbf{e}_1^T \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Computing $L_{41}L_{31}L_{21}A$, we first obtain

$$
\begin{aligned}
L_{41}L_{31}L_{21} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \qquad (3)
\end{aligned}
$$

Then,

$$
L_{41}L_{31}L_{21}A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{bmatrix}.
$$

We move on to column 2 and find L_{32} and L_{42} using $\alpha 32 = -4$ and $\alpha_{42} = 3$.

$$\begin{aligned} L_{32} &= I - \alpha_{32}\mathbf{e}_3\mathbf{e}_2^T \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ L_{42} &= I - \alpha_{42}\mathbf{e}_4\mathbf{e}_2^T \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Using Equation (3), we have

$$\begin{aligned} L_{42}L_{32}L_{41}L_{31}L_{21} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 5 & -4 & 1 & 0 \\ -5 & 3 & 0 & 1 \end{bmatrix} \end{aligned} \tag{4}$$

and

$$L_{42}L_{32}L_{41}L_{31}L_{21}A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix}.$$

Normally, we would still need to evaluate $L_{43} = I - \alpha_{43}\mathbf{e}_4\mathbf{e}_3^T$. But, in this problem $\alpha_{43} = 0$. So, $L_{43} = I$.

What we have obtained is

$$L_{42}L_{32}L_{41}L_{31}L_{21}A = U, \tag{5}$$

where U is an upper triangular matrix and $L_{42}L_{32}L_{41}L_{31}L_{21}$ is a lower triangular matrix. However, for the LU factorization, we want $A = LU$. So, we find $L = (L_{42}L_{32}L_{41}L_{31}L_{21})^{-1}$:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 5 & -4 & 1 & 0 \\ -5 & 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix}. \tag{6}$$