MAT 325. Computational Mathematics
UNCW Mathematics and Statistics

Homework 11
Spring 2022

Eigenvalue Problems

Set I.

***Problem* 1** (Adjacency)**:** Consider the graph adjaceny matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. How many vertices are adjacent to vertex 5?

b. How many edges are in the graph?

c. Draw the graph.

***Problem* 2** (Image Manipulation)**:** In MATLAB you can read an image and manipulate it. Load **peppers.png**. Display it a. upside-down, b. mirror-reversed, and c. crop it to only show the garlic bulb.

***Problem* 3** (Triangular Matrix)**:** Prove that the eigenvalues of a 3×3 triangular matrix are the diagonal entries.

$$T = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

***Problem* 4** (EVD)**:** Find the EVD of each matrix. for each eigenvalue verify that $\lambda I - A$ is singular.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

***Problem* 5** (Rotation)**:** Recall the 2D rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

- Show that R_θ is unitary.
- R_θ is a passive rotation since it rotates the basis, or coordinate axes. What rotation matrix $\tilde{R}_\theta$ would represent an active rotation, one that rotates vectors?
- If one actively rotates a vector $\mathbf{x}$ by θ to obtain $\mathbf{x}'$ and then rotates $\mathbf{x}'$ by ϕ to obtain $\mathbf{x}''$, then what matrix, in simplified form, rotates directly from $\mathbf{x}$ to $\mathbf{x}''$. That is, evaluate $\tilde{R}_\phi \tilde{R}_\theta$ and simplify.

***Problem* 6:** Are the following curves ellipses or hyperbolae? Find the quadratic form $\mathbf{x}^T Q\mathbf{x}$ and investigate the eigenvalue problem for Q. From these results find the conic equation in standard form in an unrotated system, determine the angle of rotation in each case and sketch the graph of the curve on the same axes.

a. $3x^2 + 4xy + y^2 = 1$.

b. $3x^2 + 4xy + 2y^2 = 1$.

Set II.

***Problem* 7:** Show that the two factorizations are algebraically correct. Which one is an SVD? For that case, write down σ_1, u_1 and v_1. Note: the third matrix in the first row was transposed.

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

***Problem* 8:** Solve a 2×2 eigenvalue problem to find the singular values of

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$$

***Problem* 9:** Let $A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}$.

a. Write out the Rayleigh quotient, $R_A(\mathbf{x})$ in terms of x_1 and x_2, where $\mathbf{x} = [x_1, x_2]^T$.

b. Find $R_A(\mathbf{x})$ for $x_1 = 1$ and $x_2 = 2$.

c. Find the gradient vector $\nabla R_A(\mathbf{x})$.

- Show that the gradient vector is zero when $x_1 = 1$ and $x_2 = 2$.

***Problem* 10:** Consider the matrix

$$A = \begin{bmatrix} -149 & -50 & -154 \\ 537 & 180 & 546 \\ -27 & -9 & -25 \end{bmatrix}.$$

Apply the QR algorithm for more than a half dozen iterations until you think you know what the eigenvalues are. What are the eigenvalues? How many iterations did you perform? What is the EVD of A?

***Problem* 11:** Use the middle square method for 20 steps in each of the following seeds [Pay attention to the number of digits in the seed.]: 1009, 3043, 123456.

Are these good pseudonumber generators? For the last one produce plots (histograms, grayscale images) to display the behavior for 10,000 values.

***Problem* 12:** Consider a linear congruential method with $a = 13$, $c = 0$, and $m = 64$. Show that there are cycles which depend on the seed by finding the lengths of cycles starting with $1, 2, 3$, or 4.

***Problem* 13:** Consider the PRNGs generated by small Mersenne primes. How large does p have to be so that $2^p - 1$ generates good sets of random numbers to simulate 100 tosses of a die?

***Problem* 14:** Starting from a uniformly generated set of 1000 random numbers, produce a normally distributed set of 1000 random numbers with mean $\mu = 1.2$ and standard deviation $\sigma = 0.1$. Determine the meant and standard deviations of these two random sets. How good are they? What happens if you increase the number of random numbers?

***Problem* 15:** A pair of dice are to be continually rolled until all the possible outcomes 2, 3,..., 12 have occurred at least once. Develop a simulation study to estimate the expected number of dice rolls that are needed using a built-in PRNG.