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a. Let $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, and $\mathbf{w} = \mathbf{i} - \mathbf{k}$.

i. Find the area of the parallelogram formed by the vectors **v** and **w**.

$\vec{v}\times\vec{w} = (\hat{i}+4\hat{j}+2\hat{k})\times(\hat{i}-\hat{k})$
$= -\hat{i}\times\hat{k} + 4\hat{j}\times\hat{i} - 4\hat{j}\times\hat{k} + 2\hat{k}\times\hat{i}$
$= \hat{j} - 4\hat{k} - 4\hat{i} + 2\hat{j} = -4\hat{i}+3\hat{j}-4\hat{k}$

$A = |\vec{v}\times\vec{w}| = \sqrt{4^2+3^2+4^2} = \boxed{2\sqrt{10}}$

ii. Compute $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & -1 \\ 1 & 4 & 2 \end{vmatrix}$

$= \begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = \boxed{-5}$

iii. Find the angle between **u** and **w**.

$\cos\theta = \frac{\vec{u}\cdot\vec{w}}{uw} = \frac{2}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{3}}$

$\theta = \boxed{54.7^\circ}$

iv. Determine the parametric equations for the line through (1,2,3) in the direction of **v**.

$\vec{r} = \vec{r}_0 + t\vec{v} = \langle 1,2,3\rangle + t\langle 1,4,2\rangle$

$\begin{cases} x = 1+t \\ y = 2+4t \\ z = 3+2t \end{cases}$

b. Let a particle have an acceleration of $\mathbf{a} = t\mathbf{i} - 5\mathbf{j}$ and at the origin have an initial velocity of $\mathbf{v}_0 = \mathbf{i} + \mathbf{j}$. Determine the particle's position and velocity at time $t = 2$?

$\vec{v} = \int \vec{a}\,dt = \frac{1}{2}t^2\hat{i} - 5t\hat{j} + \hat{i} + \hat{j}$

$\vec{r} = (\frac{t^3}{6} + t)\hat{i} + (-\frac{5}{2}t^2 + t)\hat{j}$

$\vec{r}(2) = (\frac{8}{6}+2)\hat{i} + (-10+2)\hat{j}$

$= \boxed{\frac{10}{3}\hat{i} - 8\hat{j}}$

Page	Points	Score
1	29	
2	24	
3	23	
4	24	
Total	100	

c. State the Fundamental Theorem of Line Integrals.

$\int_C \vec{\nabla}f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

d. Suppose $f(x,y) = x^2y + \ln y$ and $x = s^2 + t^2$, $y = st$. Find $\frac{\partial f}{\partial s}$ at $(s,t) = (1,2)$. $x = 1^2+2^2 = 5,\ y = 2$

$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$

$= 2x(2s) + (x^2 + \frac{1}{y})t$

$= 2(5)(2) + (5 + \frac{1}{2})2 = \boxed{21}$

e. Let $f(x,y) = y^2\sin(2x)$. Compute df in terms of dx and dy.

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$= +2y^2\cos(2x)\,dx + 2y\sin(2x)\,dy$

f. Define $\mathbf{F}(x,y,z) = (x^2+y)\mathbf{i} + (y^2+z)\mathbf{j} + (z^2+x)\mathbf{k}$.

i. Compute $\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x^2+y) + \frac{\partial}{\partial y}(y^2+z) + \frac{\partial}{\partial z}(z^2+x)$

$= \boxed{2x+2y+2z}$

ii. Compute $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2+y & y^2+z & z^2+x \end{vmatrix}$

$= \boxed{-\hat{i} - \hat{j} - \hat{k}}$

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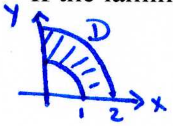
a. Consider a lamina oriented in the xy-plane occupying the region that lies between two quarter circles with radii one and two in the first quadrant. If the lamina density is $\rho = x^2 + y^2$, what is its mass?



$M = \iint_D \rho \, dA$

$= \iint (x^2+y^2)\, dA$

$= \int_0^{\pi/2}\int_1^2 r^3 \, dr\, d\theta$

$= \frac{\pi}{2}\left(\frac{r^4}{4}\right)\Big|_1^2 = 2\pi - \frac{\pi}{8} = \boxed{\frac{15\pi}{16}}$

b. Evaluate $\iint_D 3xe^{x-3y}\, dxdy$ over the region bounded by the lines $2x+3y=1$, $2x+3y=0$, $x-3y=1$, $x-3y=0$ using the transformation $u = 2x+3y$, $v = x-3y$.

$u+v = 3x$

$u-2v = 9y$ $\Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{3} & -\frac{2}{9} \end{vmatrix} = -\frac{1}{9}$

$\iint_D 3xe^{x-3y}\, dxdy = \frac{1}{9}\int_0^1\int_0^1 (u+v)e^v \, du\, dv$

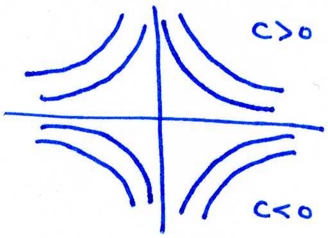
$= \frac{1}{9}\int_0^1 \left(\frac{1}{2}+v\right)e^v dv$

$= \frac{1}{9}\left[\frac{1}{2}e^v + ve^v - e^v\right]_0^1$

$= \boxed{\frac{1}{18}(e+1)}$

c. Sketch and label several level curves for $f(x,y) = 6xy$.

$6xy = c \Rightarrow y = \frac{c}{6x}$



d. Consider the cone $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u\rangle$ for $0 \le u \le 1$, $0 \le v \le 2\pi$. Find its surface area.

$A = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA = \int_0^{2\pi}\int_0^1 \sqrt{2}u \, du\, dv = \boxed{\sqrt{2}\pi}$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix}$

$= -u\cos v\,\hat{i} - u\sin v\,\hat{j} + u\hat{k}$

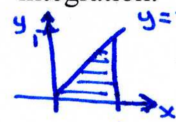
$|\vec{r}_u \times \vec{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}u$

e. Fido is on a mountain which has a shape described by $z = 100 - 25x^2 - 4y^2$. If Fido is at the point (1, 2, 59) on the mountain, in what the direction must Fido go to get down the mountain as fast as possible?

$\vec{\nabla} z = \langle -50x, -8y\rangle$

$-\vec{\nabla} z = \langle 50(1), 8(2)\rangle = \langle 50, 16\rangle$

f. Evaluate $\int_0^1\int_y^1 \sin x^2 \, dxdy$ by changing the order of integration.



$\int_0^1\int_0^x \sin x^2 \, dy\, dx = \int_0^1 x\sin x^2 \, dx$

$= -\frac{1}{2}\cos x^2\Big|_0^1 = \boxed{\frac{1}{2}(1-\cos 1)}$

g. Let $f(x,y) = x^2 + 2xy^2$. What is the rate of change of $f(x,y)$ in the direction $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ at $(x,y) = (-1,1)$?.

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}\,\Big|_{(-1,1)}$

$= \langle 2x+2y^2, 4xy\rangle \cdot \frac{\langle 2,-1\rangle}{\sqrt{5}}\Big|_{(-1,1)}$

$= \boxed{\frac{4}{\sqrt{5}}}$

h. Find the length of the curve $\mathbf{r}(t) = \langle t^2, 2t, \ln t\rangle$ for $1 \le t \le e$.

$L = \int_C |\vec{r}\,'|\, dt$

$= \int_1^e \sqrt{4t^2 + 4 + \frac{1}{t^2}}\, dt$

$= \int_1^e \frac{1}{t}\sqrt{4t^4 + 4t^2 + 1}\, dt$

$= \int_1^e \frac{2t^2+1}{t}\, dt = t^2 + \ln t\Big|_1^e = e^2+1-1 = \boxed{e^2}$

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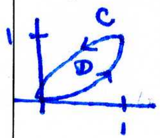
a. Let D be the region between $y = x^2$ and $x = y^2$ for $0 \le x \le 1$ and C the curve enclosing D. Compute $\int_C x^2 y\,dy + xy^2\,dx$. $= \iint_D \left[\frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA$



$= \iint_D (2xy - 2xy)\,dA = 0$

b. Find $\frac{\partial z}{\partial x}$ when $xz = \ln(yz)$. $= \ln y + \ln z$

$z + x\frac{\partial z}{\partial x} = \frac{1}{z}\frac{\partial z}{\partial x}$

$z^2 = (1 - xz)\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial x} = \frac{z^2}{1 - xz}$

c. Find and classify the relative extremum for $f(x, y) = 2x - x^2 + y^2 - 1$.

$\left.\begin{array}{l}\frac{\partial f}{\partial x} = 2 - 2x = 0 \\ \frac{\partial f}{\partial y} = 2y = 0\end{array}\right\} (x, y) = (1, 0)$

$D = f_{xx} f_{yy} - f_{xy}^2 = -2(2) - 0 < 0$

saddle

d. Find the extreme values of $f(x, y) = x + y$ on the curve $2x^2 + y^2 = 1$.

$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \Rightarrow 1 = 4\lambda x \Rightarrow x = \frac{1}{4\lambda}$

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \Rightarrow 1 = 2\lambda y \Rightarrow y = \frac{1}{2\lambda}$

$1 = 2x^2 + y^2$

$= \frac{1}{8\lambda^2} + \frac{1}{4\lambda^2} = \frac{3}{8}\frac{1}{\lambda^2} \Rightarrow \lambda^2 = \pm\sqrt{\frac{8}{3}} = \pm 2\sqrt{\frac{2}{3}}$

So, $x = \frac{1}{8}\sqrt{\frac{3}{2}}$, $y = \frac{1}{4}\sqrt{\frac{3}{2}}$ $\Rightarrow f(x,y) = \boxed{\frac{3}{8}\sqrt{\frac{3}{2}}}$

or $x = -\frac{1}{8}\sqrt{\frac{3}{2}}$, $y = -\frac{1}{4}\sqrt{\frac{3}{2}}$ $\Rightarrow f(x,y) = \boxed{-\frac{3}{8}\sqrt{\frac{3}{2}}}$

e. Let $\mathbf{F}(x, y) = (e^x \cos 3y)\mathbf{i} - (2y + 3e^x \sin 3y)\mathbf{j}$.

i. Find a function $f(x, y)$ such that $\nabla f = \mathbf{F}$.

$\frac{\partial f}{\partial x} = e^x \cos 3y$

$\Rightarrow f(x,y) = e^x \cos 3y + C(y)$

$\frac{\partial f}{\partial y} = -3e^x \sin 3y + C'(y) \Rightarrow C'(y) = -2y$

$C(y) = -y^2 + K$

$f(x,y) = \boxed{e^x \cos 3y - y^2 + K}$

ii. What is $\nabla \times \mathbf{F}$? $\vec{\nabla} \times \vec{\nabla} f = 0$ (identity)

f. Find the work done, $W = \int_C \mathbf{F} \cdot d\mathbf{r}$, by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the quarter circle $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, $-\frac{\pi}{2} \le t \le 0$.

$W = \int_{-\pi/2}^{0} \vec{F} \cdot \vec{r}\,'\,dt$

$= \int_{-\pi/2}^{0} (-\cos^2 t \sin t - \cos^2 t \sin t)\,dt$

$= -2\int_{-\pi/2}^{0} \cos^2 t \sin t\,dt$

$= 2\left.\frac{\cos^3 t}{3}\right|_{-\pi/2}^{0} = \boxed{\frac{2}{3}}$

g. For a general parametrized curve $\mathbf{r}(t)$, define the following using $\mathbf{r}$ and its derivatives:

i. Tangent vector. $\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$

ii. Normal vector. $\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|}$

iii. Curvature.

$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$

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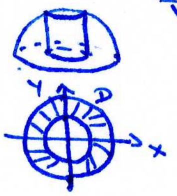
a. A hole of radius 1/2 is drilled vertically through the center of the hemisphere $x^2+y^2+z^2=1,\ z\ge 0$. Determine the volume integral of the remaining piece using cylindrical coordinates. $z=f(x,y)=\sqrt{1-r^2}$



$V=\iint_D f(x,y)\,dA$

$=\iint_D \sqrt{1-x^2-y^2}\,dA$

$=\int_0^{2\pi}\int_{1/2}^{1}\sqrt{1-r^2}\,r\,dr\,d\theta$

$=2\pi\left.\frac{(1-r^2)^{3/2}}{-3}\right|_{1/2}^{1}=\frac{2\pi}{3}\left(\frac{3}{4}\right)^{3/2}$

b. Consider the surface $z=2x^2-3y^2$.

i. Find a normal vector at the point (-2,1).

$\phi=2x^2-3y^2-z$

$\vec{N}=\vec{\nabla}\phi=\langle 4x,-6y,-1\rangle|_{(-2,1)}$

$=\langle -8,-6,-1\rangle$

ii. What is the equation of the tangent plane at (-2,1)?

$(x_0,y_0,z_0)=(-2,1,5)$

$8(x+2)+6(y-1)+z-5=0$

$\boxed{8x+6y+z=-5}$

c. Use Gauss' Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot d\mathbf{S}$, where S is the positively oriented surface of the unit sphere with its center at the origin and $\mathbf{F}(x,y,z)=(z-y)\mathbf{i}+(y^2+z^2)\mathbf{j}+(y-x^2)\mathbf{k}$.

$\vec{\nabla}\cdot\vec{F}=\frac{\partial}{\partial x}(z-y)+\frac{\partial}{\partial y}(y^2+z^2)+\frac{\partial}{\partial z}(y-x^2)=2y$

$\iint_S \vec{F}\cdot d\vec{S}=\iiint_{\text{Sphere}} 2y\,dV$

$=\int_0^{\pi}\int_0^{2\pi}\int_0^{1} 2\sin\theta\sin\phi\,\rho^3\,d\rho\,d\theta\,d\phi=0$

d. Use Stoke's Theorem to evaluate: $\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C y\,dx+(2x-z)\,dy+(z-x)\,dz$, for C the intersection of $y=\sqrt{x^2+z^2}$ and $y=1$.

$\int_C F\cdot dr=\int_C dx+(z-x)dz$

$=\iint dx\,dz=$ area of unit circle $=\boxed{\pi}$

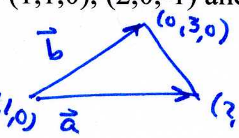
e. The volume of a right circular cone of height H and radius R is $V=\frac{1}{3}\pi R^2H$. If the radius and height are measured as $R=3.2\pm0.2$ in and $H=5.6\pm0.1$ in, what is the error is made in computing the volume?

$\Delta V=\frac{2}{3}\pi RH\Delta R+\frac{1}{3}\pi R^2\Delta H$

$=\frac{2}{3}\pi(3.2)(5.6)(.2)+\frac{1}{3}\pi(3.2)^2(.1)$

$=\boxed{8.58\text{ in}^3}$

f. Find the equation of the plane containing the points (1,1,0), (2,0,-1) and (0, 3, 0).



$\vec{b}=\langle 0,3,0\rangle-\langle 1,1,0\rangle=\langle -1,2,0\rangle$

$\vec{a}=\langle 2,0,-1\rangle-\langle 1,1,0\rangle=\langle 1,-1,-1\rangle$

~~$\vec{F}=\vec{r}_0+u\vec{a}+v\vec{b}$~~

$\vec{N}=\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&-1\\-1&2&0\end{vmatrix}=2\hat{i}+\hat{j}+\hat{k}$

$2(x-1)+(y-1)+z=0$

$\boxed{2x+y+z=3}$

g. Name the surface:

i. $3x^2+y=5$. parabolic cylinder

ii. $z=3x^2-y^2$. hyperbolic paraboloid / Saddle

iii. $x^2+y^2-z^2=-5$. hyperboloid of two sheets.