(1)

MAT 261 Exam V

Name ____________________

a. Let $\mathbf{F}(x,y,z) = (x^2+y)\mathbf{i} + (y^2+z)\mathbf{j} + (z^2+x)\mathbf{k}$. Compute

i. $\nabla \cdot \mathbf{F}$. $= 2x + 2y + 2z$

ii. $\nabla \times \mathbf{F}$. $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2+y & y^2+z & z^2+x \end{vmatrix}$

$= -\hat{i} - \hat{j} - \hat{k}$

b. State the

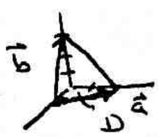
i. Fundamental Theorem of Calculus.

$\int_a^b F'(x)\,dx = F(b) - F(a)$

ii. Fundamental Theorem for line integrals.

$\int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

c. Find the area of the triangle with vertices at (1,0,0), (0,2,0),(0,0,3) using integration. [Sketch the triangle.]



$A = \iint_S dS = \iint_D \frac{dA}{\hat{n}\cdot\hat{k}}$

$= \iint_D \frac{dA}{\cos\theta}$

$\vec{a} = (0,2,0) - (1,0,0)$

$\vec{b} = (0,0,3) - (1,0,0)$

$\vec{r} = \langle x, y, \cancel{6-3y-2x} \rangle$

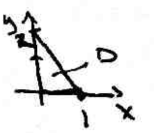
$ax + by + cz = 1$

$x + \frac{1}{2}y + \frac{1}{3}z = 1$

$\cos\theta = \frac{2}{7}$

$A = \frac{7}{2}\iint dA$

$= \boxed{\frac{7}{2}}$



$y = (x-1)(-2)$

$= 2 - 2x$

Instructions:

a. Do all of your work in this booklet. Do not tear off any sheets.
b. **Show all of your steps** in the problems for full credit.
c. **Be clear and neat** in your work. Any illegible work, or scribbling in the margins, will not be graded.
d. Place a **box around your answers.**
e. Place your **name on all of the pages.**
f. If you need more space, you may use the back of a page and write **On back of page #** in the problem space.

DO AS MANY PROBLEMS AS YOU CAN!

Page	Pts
1 (35 pts)	
2 (35 pts)	
3 (30 pts)	
Total (100 pts)	

d. Let $\nabla^2 f = 0$ in a region $D \subset R^2$. Prove using Green's Theorem that $\int_{\partial D} \frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy = 0$.

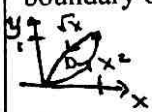
$\int_{\partial D} \underbrace{\frac{\partial f}{\partial y}}_{P} dx - \underbrace{\frac{\partial f}{\partial x}}_{Q} dy = \iint_D \left(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right) dA$

$= -\iint_D \nabla^2 f \, dA$

$= 0$

$\nabla^2 f = \vec{\nabla} \cdot \vec{\nabla} f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

e. Let D be the region between $y = x^2$ and $x = y^2$ for $0 \le x \le 1$. Compute $\int_C x^2 y\,dy + xy^2\,dx$ for C the boundary of the region followed counterclockwise.



$\int_C x^2y\,dy + xy^2\,dx = \iint_D \left[\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(x^2y)\right] dA$

$= \int_0^1 \int_{x^2}^{\sqrt{x}} (y^2 - x^2)\,dy\,dx$

$= \int_0^1 \left(\frac{y^3}{3} - x^2 y\right)\Big|_{x^2}^{\sqrt{x}} dx$

$= \int_0^1 \left(\frac{x^{3/2} - x^6}{3} - \frac{x^{5/2} - x^4}{2}\right) dx = \boxed{0}$

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MAT 261 Exam V

Name ______________________

a. Consider the surface:
$\mathbf{r}(u,v) = < v\cos u, v\sin u, 2v >$, $0 \le v \le 2$, $0 \le u \le 2\pi$.

i. Describe and/or draw the surface.

Helicoid

ii. Find a normal to the surface at $(u,v) = (0,1)$.

$\vec{r}_u = \langle -v\sin u, v\cos u, 0\rangle$

$\vec{r}_v = \langle \cos u, \sin u, 2\rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 2 \end{vmatrix}$$

$= 2v\cos u\,\hat{i} + 2v\sin u\,\hat{j} - v\hat{k}$

$\boxed{\vec{N} = 2\hat{i} - \hat{k}}$

iii. What is the equation of the tangent plane at this point? $\vec{r} = \langle 1, 0, 2\rangle$

$2(x-1) + 0(y-0) - (z-2) = 0$

$\boxed{2x - z = 0}$

iv. What is the area of this surface?

$$A = \int_0^2\int_0^{2\pi} |\vec{r}_u \times \vec{r}_v|\,du\,dv$$

$\sqrt{5v^2}$

$$= \int_0^2\int_0^{2\pi} \sqrt{5}\,v\,du\,dv$$

$$= 2\pi\sqrt{5}\int_0^2 v\,dv$$

$$= \boxed{4\pi\sqrt{5}}$$

Let $\mathbf{F}(x,y) = < e^x\cos 3y, -3e^x\sin 3y >$.

b. Find a function $f(x,y)$ such that $\nabla f = \mathbf{F}$ for all (x,y).

$\frac{\partial f}{\partial x} = e^x\cos 3y \Rightarrow f(x,y) = e^x\cos 3y + C(y)$

$\frac{\partial f}{\partial y} = -3e^x\sin 3y$ $\qquad \frac{\partial f}{\partial y} = -3e^x\sin 3y + c'(y)$

$c'(y) = 0$ or $C = K$

$f(x,y) = e^x\cos 3y + K$

c. Evaluate $\int \mathbf{F}\cdot d\mathbf{s}$ for the path $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. $= 0$

$\vec{F} = \vec{\nabla} f \Rightarrow \vec{F}$ conservative

from Fundamental Thm for line integrals

d. Compute $\nabla \times \mathbf{F}$. $= 0$

since $\vec{\nabla} \times (\vec{\nabla} f) = 0$

(2)

MAT 261 Exam V Name ____________________

a. Consider the surface:
$\mathbf{r}(u, v) = < v\cos u, v\sin u, 2v >$, $0 \le v \le 2$, $0 \le u \le 2\pi$.

i. Describe and/or draw the surface.

Helicoid

ii. Find a normal to the surface at $(u, v) = (0, 1)$.

$\vec{r}_u = \langle -v\sin u, v\cos u, 0 \rangle$

$\vec{r}_v = \langle \cos u, \sin u, 2 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 2 \end{vmatrix}$$

$= 2v\cos u\,\hat{i} + 2v\sin u\,\hat{j} - v\hat{k}$

$\boxed{\vec{N} = 2\hat{i} - \hat{k}}$

iii. What is the equation of the tangent plane at this point? $\vec{r} = \langle 1, 0, 2 \rangle$

$2(x-1) + 0(y-0) - (z-2) = 0$

$\boxed{2x - z = 0}$

iv. What is the area of this surface? $\sqrt{5v^2}$

$$A = \int_0^2 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$= \int_0^2 \int_0^{2\pi} \sqrt{5}\, v \, du\, dv$$

$$= 2\pi\sqrt{5} \int_0^2 v\, dv$$

$$= \boxed{4\pi\sqrt{5}}$$

Let $\mathbf{F}(x, y) = < e^x\cos 3y, -3e^x\sin 3y >$.

b. Find a function $f(x, y)$ such that $\nabla f = \mathbf{F}$ for all (x, y).

$\frac{\partial f}{\partial x} = e^x\cos 3y \Rightarrow f(x,y) = e^x\cos 3y + c(y)$

$\frac{\partial f}{\partial y} = -3e^x\sin 3y$ $\quad$ $\frac{\partial f}{\partial y} = -3e^x\sin 3y + c'(y)$

$c'(y) = 0$ or $C = K$

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c. Evaluate $\int \mathbf{F} \cdot d\mathbf{s}$ for the path $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. $= 0$

$\vec{F} = \vec{\nabla} f \Rightarrow \vec{F}$ conservative

from Fundamental Thm for line integrals

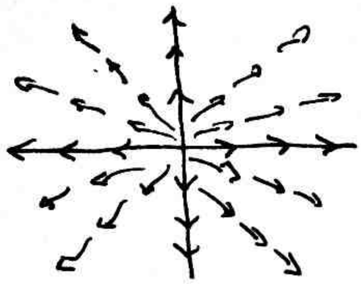
d. Compute $\nabla \times \mathbf{F}$. $= 0$

since $\vec{\nabla} \times (\vec{\nabla} f) = 0$

(3)

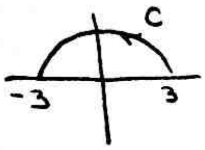
MAT 261 Exam V Name ______________________

a. Sketch the vector field $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$ ← unit vector



b. Compute $\int_C y\, ds$ for C the upper half of the circle $x^2+y^2=9$. .

$\int_C y\,ds = \int_0^{\pi} 9 \sin t\, dt = \boxed{18}$



$\vec{r} = \langle 3\cos t, 3\sin t\rangle$

$\vec{r}\,' = \langle -3\sin t, 3\cos t\rangle$

$|\vec{r}\,'| = 3$

c. Is $\int_C (2y^2 - 12x^2y^3)\, dx + (4xy - 9x^4y^2)\, dy$ path independent? Why?

$\frac{\partial}{\partial y}(2y^2 - 12x^2y^3) = 4y - 36x^2y^2$

$\frac{\partial}{\partial x}(4xy - 9x^4y^2) = 4y - 36x^3y^2$

No

d. Use Gauss' Theorem to prove that $\iint_S (f\nabla g - g\nabla f)\cdot \mathbf{n}\, dS = \iiint_E (f\nabla^2 g - g\nabla^2 f)\, dV$

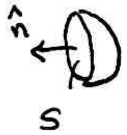
$= \iiint_E \nabla\cdot(f\nabla g - g\nabla f)\, dV$

$= \iint_S (f\nabla g - g\nabla f)\cdot\hat{n}\, dS$

e. Use Stoke's Theorem to evaluate: $\int_C y\, dx + (2x - z)\, dy + (z - x)\, dz$, where C is the intersection of $x^2+y^2+z^2=4$ and $y = 1$.

$\vec{F} = \langle y, 2x-z, z-x\rangle$

$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & 2x-z & z-x \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$



$I = \iint_S \vec{\nabla}\times\vec{F}\cdot\hat{n}\, d\vec{S}$ $\quad \hat{n} = -\hat{j}$

$= -\iint dA = \boxed{-3\pi}$

$x^2+y^2=3$

$\Rightarrow r^2=3$

f. Prove $\nabla\times(\nabla f) = 0$.

$\vec{\nabla}\times\vec{\nabla}f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \vec{0}$