Lab 10

Easy as Pi

Purpose

To investigate infinite series, which lead to the value of π.

10.1 The Search for π

It was known since Babylonian times that the ratio of the circumference of a circle to its diameter is the same for all circles. Also, it was known that the ratio of the area of a circle to the square of its radius was constant. However, it was not always known that these two constants were the same!

Much of what we know about early mathematics comes from writings such as those found in the Egyptian Rhind papyrus (2000-1800 B.C.) and cuneiform tablets from the Babylonian age of Hamurabi (1800-166 B.C.). In one problem in the Rhind papyrus, it is suggested that the area of a circle can be computed as

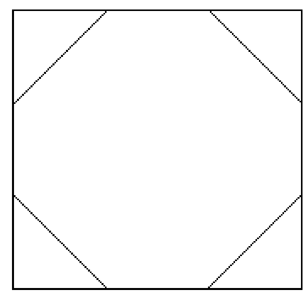

Figure 10.1: The Egyptian Rhind papyrus computation.

$$A \approx \left(\frac{8}{9}d\right)^2. \tag{10.1}$$

This could possibly be due to the following derivation [1]:

Take a square with sides of length d and trisect each side. Draw the line segments, as shown in the figure above. The resulting polygon approximates a circle. Its area can be computed as

$$A = \frac{7}{9}d^2 = \frac{63}{81}d^2 \tag{10.2}$$

$$A \approx \frac{64}{81}d^2 = \left(\frac{8}{9}d\right)^2 \tag{10.3}$$

This value corresponds to the approximation $\pi \approx 3.16$.

Even though the Babylonians were more advanced than the Egyptians, they had used $3r^2$ for the area of a circle. It is guessed that they had obtained this value as the average of the areas of the

[1] From *The Historical Development of Calculus,* C.H. Edwards, Jr., Springer-Verlag, p. 2 (1979)

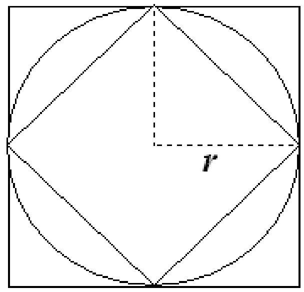

Figure 10.2: The Babylonians used squares.

inscribed and circumscribed squares, that are shown in Figure 10.2[2]. The area of the circumscribed square is clearly seen to be $4r^2$. The sides of the inscribed square have length $\sqrt{2}r$, giving its area as $2r^2$. So, the average is $3r^2$.

In his *Measurement of a Circle* Archimedes (240 B.C.) had shown that the above constants were actually the same.[3] He based his proof on the relation

$$A = \frac{1}{2} rC. \tag{10.4}$$

Furthermore, using polygonal figures with more sides than the above examples, he proved that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}. \tag{10.5}$$

Claudius Ptolemy of Alexandria (150 A.D.) had further improved this estimate, obtaining

$$\pi \approx \frac{377}{120} \approx 3.1416. \tag{10.6}$$

Since then, the accuracy in determining π has been improved considerably. Ludolph van Ceulen of Leyden (1540-1610) computed π to 35 decimal places and Zacharias Dase of Hamburg (1824-1861) had computed it to 200 places. Of course, this was all done without a computer! We now know π to millions of decimal places. However, we only need about 4-10 decimal places for practical applications. For your reference, here are a couple of digits:

$$\pi = 3.14159 \quad 26535 \quad 89793 \quad 23846 \quad 26433 \quad 83279 \quad 50288 \quad 41971 \quad 69399 \quad 37510\ldots. \tag{10.7}$$

Recently there has been an effort to compute the dth digit of π and other transcendental numbers in various bases without computing all of the preceding digits. Such computations are based on a number of identities introduced by D. Bailey, P. Borwein and S. Plouffe [4]. The identity used to obtain the binary and hexidecimal digits of π is given by

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right). \tag{10.8}$$

[2]Ibid., p. 4

[3]Ibid., p. 31

[4]D. H. Bailey, P. B. Borwein and S. Plouffe, On the Rapid Computation of Various Polylogarithmic Constants, Mathematics Computation, 1997.

Other such formulae have been proposed and F. Bellard has introduced an identity which has recently been used to produce the 40 trillionth bit of π [5]. The resulting algorithm was 43% faster than that using the above formula.

10.2 Series Approximations

There is an art to determining π. The current approximations of π are based on calculus. Archimedes' polygonal approach was correct, but he had no notion of limits, which are at the heart of calculus. We can now determine π using a variety of series.

One of these series was derived by Leibniz (1646-1716):

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots. \tag{10.9}$$

He derived this series from the computation of the area of a quarter circle.[6] He based this on his *transmutation theorem,*

$$\int_a^b y(x)\,dx = \frac{1}{2}\left(xy(x)|_a^b + \int_a^b z(x)\,dx \right), \tag{10.10}$$

where

$$z(x) = y(x) - x\frac{dy}{dx}. \tag{10.11}$$

It is interesting to note that by substituting the formula for $z(x)$ into the above integral, one arrives at the integration by parts formula.

Now, consider the semicircle Figure 10.3, which is the graph of $y(x) = \sqrt{2x - x^2}$. Integrating from 0 to 1, we would expect to get $\pi/4$. In order to apply Leibniz's transmutation theorem, one needs to compute $z(x)$. It is found as

$$z = \sqrt{\frac{x}{2-x}}. \tag{10.12}$$

Using (10.10), we have

$$\frac{\pi}{4} = \int_0^1 y(x)\,dx = \frac{1}{2}\left(1 + \int_0^1 z(x)\,dx\right). \tag{10.13}$$

The one trick in this derivation is to recognize that

$$\int_0^1 z(x)\,dx = 1 - \int_0^1 x dz, \tag{10.14}$$

as shown in Figure 10.4. Replacing the integral in (10.13), and solving equation (10.12) for x, we obtain

$$\frac{\pi}{4} = 1 - \int_0^1 \frac{z^2}{1+z^2}\,dz. \tag{10.15}$$

The computation is completed by inserting the geometric series

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \ldots, \qquad |z| < 1 \tag{10.16}$$

into the integral and integrating term by term.

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[5] F. Bellard, A New Formula to Compute the nth Binary Digit of Pi, http://www-stud.enst.fr/ bellard/pi/pi-bin.html, 1997

[6] Edwards, op. cit., p. 245-248

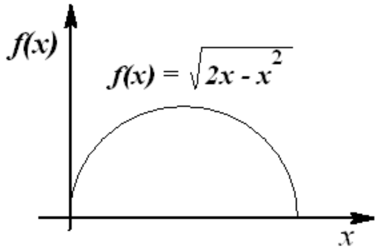

Figure 10.3: Graph of a semicircle.

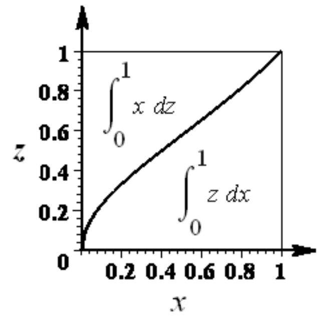

Figure 10.4: Note the areas are the same.

A quicker way to get this series approximation is to note that

$$\frac{\pi}{4} = \tan^{-1}(1) \tag{10.17}$$

and to use the series expansion

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots, \qquad |x| \le 1. \tag{10.18}$$

Instructions

There are many other representations for π. In this lab you will look at a few of these infinite series. For example, to evaluate the series $\sum_{n=0}^{\infty} a_n$, one looks at a sequence of partial sums, $S_N = \sum_{n=0}^{N} a_n$, and seeks a limit for large N. It is easy to type the summation in Maple.

In order to *see* the convergence of a series, you can evaluate an individual partial sum and you can plot the series of partial sums. A Maple worksheet can be set up using the following:

```
restart: with(plots):
S:= put in series:
N:=100:
evalf(S);
plot([n,S,n=1..N],style=point);
```

However, you will see that the series plots are not interesting. Instead, you can look at the error, $R_n = \Pi - S_N$, by plotting instead **plot([n,Pi-S,n=1..N],style=point);**. [Sometimes the plot might not appear, so you will need to go to the menu and select the menu items **Edit - Execute - Worksheet**].

The series representations, which you will use are:

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \tag{10.19}$$

$$\frac{1}{2} - \frac{\pi}{8} = \sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n+1)} \tag{10.20}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \tag{10.21}$$

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) \tag{10.22}$$

In order to record your observations, you should create a table

Series	S_{100}	S_{500}	S_{1000}	S_{5000}	N_5

1. Set up your worksheet for the series (10.19).
   (a) Compute the partial sums, S_{100}, S_{500}, S_{1000}, and S_{5000}.
   (b) Record the approximate values in your table keeping only the number of decimal places that the particular partial sum can be trusted.
   (c) Plot the corresponding partial sums and note the rate of convergence.
   (d) Now look at the plot of the errors.
   (e) Determine the value of π to five places if possible. How many terms are needed? Record this in the table as N_5.
   (f) Use the Alternating Series Estimation Theorem to determine the number of terms needed for five place accuracy. How does this compare to what you determined in the last step?
2. Repeat steps (a), (b), (d) and (e) for the remaining three series in (10.20)-(10.22). Record your results.
3. Arrange the series in order of fastest to slowest convergence.

Exercises

▷ **Exercise 10.1** Verify that (10.1) leads to $\pi \approx 3.16$.

▷ **Exercise 10.2** Prove (10.4), using what you know about circles.

▷ **Exercise 10.3** Using the methods of integration, which you have learned, compute the area of the quarter circle:

$$A = \int_0^1 \sqrt{2x - x^2}\, dx.$$

In your spare time you should go back and fill in the missing steps in Leibniz's derivation in Section 10.2.