Lab 11

Function Approximation

Purpose

To investigate the approximation of functions by Taylor polynomials and their errors.

11.1 Taylor's Theorem

Many functions can be represented as the sum of their Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{11.1}$$

where $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$. As with any series, the partial sums of this series are approximations to the sum of the series, $f(x)$. In this case, the partial sums are given by

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{11.2}$$

Notice that $P_k(x)$ is a kth degree polynomial in x. It is called the kth-degree Taylor polynomial of $f(x)$ at a.

If $f(x)$ is the sum of its Taylor series, then $P_k(x) \to f(x)$ as $k \to \infty$. So, $P_k(x)$ can be used as an approximation of $f(x)$. When approximating a function by a Taylor polynomial, it is useful to know how good the approximation is. The answer to this question is provided by Taylor's formula:

Taylor's Theorem: *If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,*

$$f(x) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \tag{11.3}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{11.4}$$

for some c between a and x. Note that $R_n(x)$ has a form similar to the $(n+1)$st term of the Taylor series. The difference is that the derivative is evaluated at c.

A simpler way to view equation (11.3) is that for each x in I,

$$f(x) = P_n(x) + R_n(x). \tag{11.5}$$

Thus, we can approximate $f(x)$ by its nth order Taylor polynomial at a, while making an error of magnitude $|R_n(x)|$. We can estimate this error for many functions by using the Remainder Estimation Theorem:

The Remainder Estimation Theorem: *If there are positive constants M and r such that*

$$|f^{(n+1)}(t)| \leq Mr^{n+1} \tag{11.6}$$

for all t between a and x, inclusive, then the remainder term $R_n(x)$ satisfies the inequality

$$|R_n(x)| \leq M\frac{r^{n+1}|x-a|^{n+1}}{(n+1)!}. \tag{11.7}$$

If these conditions hold for every n and all other conditions of Taylor's theorem are satisfied by f, then the series converges to $f(x)$.

Instructions

In this lab you will explore Taylor series approximations of several functions. This is easy to do in Maple. You will first create a worksheet to study the series expansion for $f(x) = e^x$ about $x = a$. We will see in class that this expansion can be obtained by hand as

$$e^x = \sum_{n=0}^{\infty} \frac{e^a}{n!}(x-a)^n. \tag{11.8}$$

We obtain the Nth order Taylor polynomial at a, as

$$P_N(x) = e^a + e^a(x-a) + \frac{1}{2}e^a(x-a)^2 + \frac{1}{3!}e^a(x-a)^3 + \ldots + \frac{1}{n!}e^a(x-a)^N. \tag{11.9}$$

This can be used to approximate $f(x) = e^x$ in the neighborhood of $x = a$. How good is the approximation? How many terms (what N) will be good enough?

We can investigate this using the following Maple code:

```
> restart:
> f := exp(x);
> a := 0: N := 1:
> T := taylor(f, x = a, N+1);
> P := convert(T, polynom);
> plot({f, P}, x = -2 .. 2, color = ([blue, black]));
> R := f-P:
> plot(R, x = -2 .. 2);
```

The function is entered in **f** with the expansion point and degree of the Taylor polynomial. The function **taylor** is a built-in function that gives the Taylor series approximation. However, Maple also outputs the order, i.e., $O(x^2)$. This tells us that the first nonzero terms involve powers of x^2. But, we cannot graph this function. So, we need to strip the result of the order symbol. **convert(T, polynom)** will do this as the output shows.

Now you can plot the original function and the Taylor series approximation. How good is the approximation? One could eyeball it, but a more accurate determination would be to compute the remainder $R_N(x) = f(x) - P_N(x)$. We do this next and then plot the result to see what the error is over the given interval, [-2,2].

Maple comes with other nice features. The Student package for Calculus1 will produce the Taylor approximation without the order symbol and can also provide the same comparison plot. However, for values of $a \neq 0$, it produces an expanded version of the polynomial and does not display the result in terms of powers of $x - a$. You can add the following lines to the worksheet if you want to see how it works.

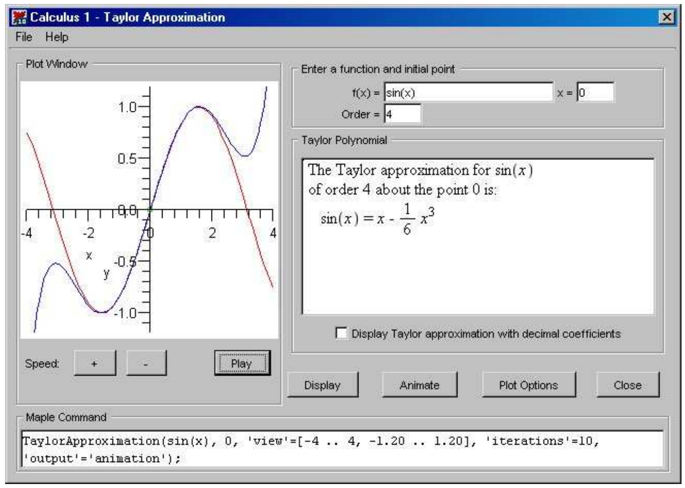

Figure 11.1: The Maplet for Taylor Approximations.

```
> with(Student[Calculus1]):
> TaylorApproximation(f, x = a, order = N);
> expand(P);
> TaylorApproximation(f, x = a, order = N, output = plot);
```

- Create a worksheet with the first Maple code above.
- For $f(x) = e^x$ and $a = 0$, look at the output for $N = 1, 2, 3, 4$. Describe each of these approximations. Record the order of the polynomial and the maximum magnitude of the error over the interval [-2,2] in each case.
- Set $a = 1$ for $f(x) = e^x$ and repeat the last steps for $x \in [-1, 3]$. By changing N determine the order of the Taylor polynomial that is needed for the error to be bounded by 0.001?
- For $N = 2$, what is the largest interval over which the error has a magnitude no larger than 0.001?

Maple has a tutorial on Taylor Approximations. Go to the menu and select the menu items **Tools - Tutors - Calculus single Variable - Taylor Approximations**. This will bring up a Maplet as shown in Figure.

- Explore this Maplet. The window shows $f(x) = \sin(x)$, $a = 0$ and $N = 4$. Change N and observe what happens. Record your observations. Why are there no even powers of x?
- Change a to **Pi/2**. How does this change the Maplet results?
- Consider $f(x) = \ln(1+x)$ and $x = 0$ in the Maplet. Increase N until you think you know what is happening. How good is the approximation? For what values of x does the series converge to $f(x)$?

Series Application

All bodies emit and absorb radiation. At the end of the nineteenth century physicists were able to predict the energy emitted by a body at a given temperature as a function of the wavelength of the radiation. Bodies which absorb all incident radiation are called blackbodies. The energy per volume radiated by a body at temperature T (in Kelvin) of wavlength λ was given by the Rayleigh-Jeans formula

$$u(\lambda) = \frac{8\pi kT}{\lambda^4}, \tag{11.10}$$

where $k = 1.381 \times 10^{-23}$ J/K is Boltzmann's constant. However, this law was only good for long wavelengths but predicted very large values for short wavelengths. In 1900 Max Planck, having introduced assumptions which lead to the quantum revolution, derived a new radiation law. Planck's radiation law takes the form

$$u(\lambda) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kt} - 1}, \tag{11.11}$$

where $h = 6.626 \times 10^{-34}$ Js is Planck's constant and $c = 2.998 \times 10^8$ m/s is the speed of light in a vacuum.

Since the Rayleigh-Jeans Law works for large wavelengths, then the two radiation theories ought to agree for large λ, or small $\frac{1}{\lambda}$. We will rewrite these two laws in a form more palatable to you. We write the wavelengths in units of micrometers as $\lambda = 10^{-6}/x$. (x is actually a rescaled light frequency, but we will not go into that here.) Inserting all constants and letting $u(\lambda) = f(x)$, we have the new forms to three significant digits:

$$f(x) = 347Tx^4, \tag{11.12}$$

$$f(x) = \frac{5.00 \times 10^6 x^5}{e^{14400x/T} - 1}. \tag{11.13}$$

These laws should agree for small x.

▷ **Exercise 11.1** Assuming that $14400x/T$ is small, write the first Taylor polynomial for $e^{14400x/T}$. [You need $e^x \approx 1 + x$.] Insert this expression into equation (11.13). Does the resulting expression agree to three significant digits with the Rayleigh-Jeans formula in (11.12)?

▷ **Exercise 11.2** Letting $T = 5700$ K (the average temperature of the sun), plot these energy density functions on the same axes for $x = 0$ to $x = 1$. How well do the laws agree? [Define **f1** and f2 and use **plot({f1,f2},x=0..1)**.]

▷ **Exercise 11.3** Now plot only the Planck radiation Law, **f2**. Locate the maximum value by increasing the x-interval. Recalling that $\lambda = 1/x$ micrometers, what is the wavelength corresponding to the peak energy radiated?