Mixing Problems

In such cases there is the possibility that the integral may not exist; i.e., the integral may diverge. The simplest example is

We consider mixing problems involving brine in a tank. Brine is a mixture of salt and water. Typically there is an inflow of water plus salt into a tank of volume V. Also, there is an outflow of well mixed brine with the same concentration as that in the tank. Letting the amount of salt in the tank at time t be $y(t)$, we are interested in the equation that models the rate of change of the salt content in the tank. We will model several scenarios and use Maple to plot corresponding solutions.

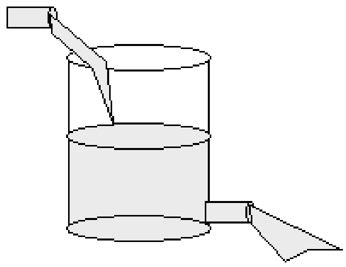

Figure 1: Mixing Problem

The main equation is given by

$$\frac{dy}{dt} = \text{Rate in} \; - \; \text{Rate Out}. \tag{1}$$

This equation says that the rate of change of the salt content is due to the rate at which salt enters the system less the rate at which it leaves. One needs to look at the units in order to model any particular situation. We will assume that y is given in pounds and time in minutes. Thus, the rate of change has units of pounds per minute. The rates will also need to have these units for consistency in the units.

System 1: A 50 gallon tank of brine initially has 5.0 lb of salt. Pure water enters the tank at 3.0 gal/min. We assume that the contents of the tank are continually mixed so as to maintain a uniform solution. The well mixed solution leaves through a spout at 3.0 gal/min. How much salt is in the tank after 10.0 minutes?

We need to determine the rates in and out of the tank. Since no salt is entering the tank, the Rate In is zero. The rate out is not zero as a mixture of brine is leaving the tank. However, we do not know the pounds per minute. We only know the number of gallons per minute. (Note: Gallons is a measure of volume.) So, how do you find the Rate Out? Look at the units:

$$\frac{\text{lb}}{\text{min}} = \frac{\text{lb}}{\text{gal}} \frac{\text{gal}}{\text{min}} \tag{2}$$

Thus, the rate is given as a product of the concentration ($\frac{\text{lb}}{\text{gal}}$) times the flow rate ($\frac{\text{gal}}{\text{min}}$). The outgoing concentration at any time t is the amount of salt per volume, y lb/50 gal.

Applying this to our general equation, we have

$$\frac{dy}{dt} = -\frac{y \text{ lb}}{50 \text{ gal}} \times 3 \frac{\text{gal}}{\text{min}}.$$

or,

$$\frac{dy}{dt} = -\frac{3y}{50}. \tag{3}$$

Since we initially have 5.0 lbs, we can provide the initial condition $y(0) = 5.0$.

▷ **Exercise 1** Open the worksheet **ODEs.mw** and note that it is set up for a first order differential equation of the form

$$\frac{dy}{dt} = f(t, y).$$

A direction field is shown for $a \leq t \leq b$ and $c \leq y \leq d$.

▷ **Exercise 2** Look at the direction field. What happens for different initial amounts of salt in the system?

Maple can solve the initial value problem. In the **dsolve** command there is a place to change the initial condition, **y(0)=2**.

▷ **Exercise 3** Look at the solution plot for an initial concentration of 5.0 lbs. How much salt is in the tank after 10 minutes? How much is left after waiting a very long time?

System 2: Now consider how the problem changes when salt water with a concentration of 2.0 lbs/gal enters the tank at 3.0 gal/min.

▷ **Exercise 4** Set up the initial value problem.

▷ **Exercise 5** Enter the appropriate information into the **ODEs.mw** worksheet. Look at the direction field. What happens for different initial amounts of salt in the system?

▷ **Exercise 6** Look at the solution plot for an initial concentration of 5.0 lbs. How much salt is in the tank after 10 minutes? How much is left after waiting a long time?

System 3: Now consider what happens when salt water enters with a concentration of 2.0 lbs/gal enters the tank at 4.0 gal/min.

▷ **Exercise 7** In this case more fluid flows into the system than flows out of it. How does this affect the volume? You need to determine the volume as a function of time. With this in mind, set up the initial value problem.

▷ **Exercise 8** How does this affect your answers in System 2?