Lab 1

Numerical Integration of Functions

Purpose

To explore three common methods for numerical integration.

Files

INTFUN.MW

1.1 Approximating Integrals

In practice there are many integrals that cannot be computed through analytical techniques. In these cases one has to resort to numerical methods into order to arrive at an approximation to the answer that is sought. Often in science one is faced with the problem of having to integrate a function, which is given in tabular form. In this lab you will study three common methods of numerical integration: Euler's method, the trapezoidal rule and Simpson's rule. These methods will be applied to several functions, and you will study the accuracy of each method.

For the simplest methods, the interval $[a, b]$, over which the function is being integrated, is divided into several subintervals of equal size. We are interested in approximating the integral

$$\int_a^b f(x)\,dx. \tag{1.1}$$

Divide the interval $[a, b]$ into N subintervals of width

$$h = \frac{b-a}{N}, \tag{1.2}$$

and label these subintervals as

$$(x_0, x_1), \quad (x_1, x_2), \quad \ldots \quad (x_{N-1}, x_N), \qquad x_0 = a, \quad x_N = b. \tag{1.3}$$

The function $f(x)$ is then approximated over each subinterval by a polynomial function, $p(x)$, which can easily be integrated. These results are then added up to give an approximation to the original integral. The choice of the polynomial $p(x)$ will lead to the various rules for integration.

1.2 Euler's Method

For Euler's method we will approximate the value of the function on the subinterval (x_i, x_{i+1}) by its value at the left endpoint, $p(x) = f(x_i)$. Therefore, the integral over this subinterval can be approximated by

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} p(x)\,dx = hf(x_i). \tag{1.4}$$

Here we have used the fact that $h = x_{i+1} - x_i$.

Since this gives an approximation of the integral over each subinterval, we can add up the results to get

$$\int_a^b f(x)\,dx \approx h[f(x_0) + f(x_1) + \ldots + f(x_{N-1})]. \tag{1.5}$$

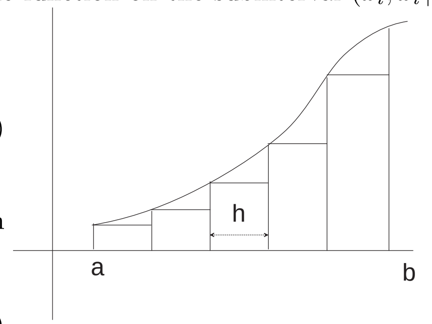

Figure 1.1: Euler's Method

Using the summation symbol, we can rewrite this result as

$$\int_a^b f(x)\,dx \approx h \sum_{k=0}^{N-1} f(x_k). \tag{1.6}$$

This method should remind you of the process of computing the Riemann sum. Recall that as the sizes of the subintervals are decreased, the approximation becomes much better. However, there are more terms to be added in the summation in equation (1.6). Therefore, it will take longer to compute the sum and will require more storage to save the data. One should then weigh the cost of using large values of N against the size of h, which is tied into the accuracy of the method.

1.3 The Trapezoidal Rule

One way to increase the accuracy of the method is to change the approximating polynomial to fit the function better over each subinterval. Instead of approximating the area in each subinterval by a rectangle, as we saw with Euler's method, we could approximate it with a trapezoid. This can be accomplished by letting $p(x)$ be a linear function, which agrees with $f(x)$ at the endpoints of the subinterval, $p(x_i) = f(x_i)$.

Let $p(x) = mx + b$. For the subinterval (x_i, x_{i+1}), we need to require that

$$f(x_i) = mx_i + b, \qquad f(x_{i+1}) = mx_{i+1} + b. \tag{1.7}$$

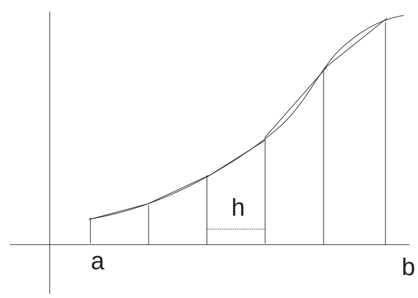

Figure 1.2: Trapezoidal Rule

Solving for m and b gives

$$m = \frac{f(x_{i+1}) - f(x_i)}{h}, \qquad b = f(x_i) - mx_i. \tag{1.8}$$

Integrating $p(x)$ over this interval, we find

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} p(x)\,dx = \frac{h}{2}[f(x_{i+1}) + f(x_i)]. \tag{1.9}$$

Adding all of the contributions from the subintervals yields

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(x_0) + \ldots + 2f(x_i) + \ldots + f(x_N)], \tag{1.10}$$

or

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\sum_{k=0}^{N-1}[f(x_k)+f(x_{k+1})]. \tag{1.11}$$

This is the so-called Trapezoidal Rule.

1.4 Simpson's Rule

Finally, we will consider approximating the function over a subinterval, using a quadratic function. In this case we let $p(x) = ax^2 + bx + c$ and require this polynomial to agree with $f(x)$ at the points x_{i-1}, x_i, and x_{i+1}. This leads to the results:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \int_{x_i}^{x_{i+1}} p(x)\,dx = \frac{h}{3}[f(x_{i+1}) + 4f(x_i) + f(x_{i-1})]. \tag{1.12}$$

$$\begin{aligned}\int_a^b f(x)\,dx \approx\ & \tfrac{h}{3}\{f(x_0) + f(x_N) + 2[f(x_2) + f(x_4) + \ldots + f(x_{N-2})] \\ & +4[f(x_1) + f(x_3) + \ldots + f(x_{N-1})]\},\end{aligned} \tag{1.13}$$

where N is an even integer. This result can be written using summation notation as

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[f(x_0) + f(x_N) + 2\sum_{k=1}^{(N-2)/2} f(x_{2k}) + 4\sum_{k=0}^{(N-2)/2} f(x_{2k+1})]. \tag{1.14}$$

Instructions

- Open the file **INTFUN.MW**. In this file you will find an implementation of the numerical methods of integration discussed above. **N** is the number of subintervals, which should be even for the correct computation using Simpson's Rule. Compare the worksheet with the above formulae for each method. Make sure you understand every step in the worksheet. Each partner should individually prepare a list of questions about the worksheet and the handout. When you are done, ask your partner these questions to make sure that you both understand the lab.
- For the first integral, let $f(x) = x^2$ and integrate this function from $a = 0$ to $b = 1$. Start with four subintervals ($N = 4$). Make sure that you have entered your name and the date at the top of the worksheet. Now, go to the end of the worksheet. Calculate the worksheet and note the results.
- Try several different values of **N**. Make sure that they are even! Record the smallest **N** needed for EACH method to get three place accuracy. [**Note:** $N > 2$ and you should have a minimum of four decimal places visible.]

Maple has several built-in tools for exploring topics in introductory mathematics courses. These are designed to bypass the need for complicated looking worksheets like the one you have been using. To do this, go to the Tools menu item in an open Maple session. In order to access these Maplets, move the cursor to **Tutors**, then **Calculus - Single Variable**, and finally select the desired Maplet. In this lab you need to select **Approximate Integrals ...**. After a short time you will see the new window as shown in Figure on the right.

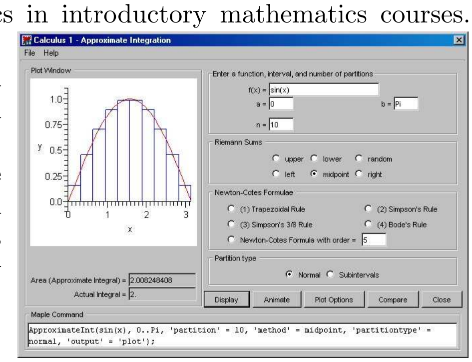

Figure 1.3: MapletInt

- Explore the Maplet. Click on **(1) Trapezoidal Rule**. Change the default entries for a, b and n and click on **Display** for each change. Record your observations. Do the same for **(2) Simpson's Rule**.
- Use the Maplet to repeat your results for integrating $f(x) = x^2$ from $a = 0$ to $b = 1$. Does the Maplet get the same results that were obtained using the worksheet?
- Complete the exercises below.

Exercises

▷ **Exercise 1.1** How do your results for $f(x) = x^2$ compare to the exact result? Do the exact integral by hand.

▷ **Exercise 1.2** Compute the following integrals using the integration Maplet for $N = 4, 10, 20$ using the best method:

$$\int_{-1}^{1} e^{-x^2}\, dx, \qquad \int_{0.01}^{3\pi} \frac{\sin x}{x}\, dx. \tag{1.15}$$

▷ **Exercise 1.3** A simple pendulum is a point mass hanging on a massless rod, or string. attached to a support. Releasing the mass from a given height, the mass is seen to swing back and forth, completing each cycle of its swing in a fixed time, called the period. In physics one learns that the period is given by $T = 2\pi\sqrt{\frac{L}{g}}$ where L is the length of the pendulum and $g = 9.8$ m/s^2 is the acceleration due to gravity. However, this is only true for small release angles. The exact period is given by

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}, \tag{1.16}$$

where $k = \sin(\frac{1}{2}\theta_0)$ and θ_0 is the angle that the pendulum makes to the vertical when released from rest.

Use your worksheet to determine the period to four decimal places using each method for $\theta_0 = 45^o$ and $L = 1.0$ m. (Do not forget to convert to radians!) How many terms are used in each case? How does this value compare to the small angle approximation that is given in introductory physics classes?

In order to carry this out you simply need to do the following integral:

$$T = \int_0^{\pi/2} 4\sqrt{\frac{1}{9.8}} \frac{1}{\sqrt{1 - \sin^2(\pi/8)\sin^2 x}}\, dx. \tag{1.17}$$

Compare this to $T = 2\pi\sqrt{\frac{1}{9.8}}$. **Note:** The familiar π can be entered by typing **Pi** or going to the Common Symbols palette and clicking on π.