Lab 2

Improper Integrals

Purpose

To explore the convergence properties of improper integrals.

2.1 The Computation of Improper Integrals

In many applications one is faced with integrals which have one of the following properties:

- The integrand becomes infinite at one or more points of the interval of integration.
- One or both of the limits of integration is infinite.

In such cases there is the possibility that the integral may not exist; i.e., the integral may diverge. The simplest example is

$$\int_0^1 \frac{dx}{x}. \tag{2.1}$$

The integrand becomes infinite at the endpoint $x = 0$ To see if this integral exists, we cut out this endpoint from the integral and start the integration at $x = b$ for $0 < b < 1$, and then investigate what happens when b approaches 0. This is summarized as

$$\lim_{b \to 0^+} \int_b^1 \frac{dx}{x} = \lim_{b \to 0^+} \big(\ln \frac{1}{b}\big) = +\infty. \tag{2.2}$$

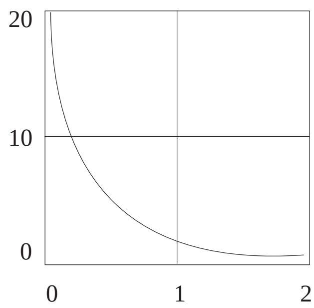

Figure 2.1: Divergent

Therefore, we say that the original integral diverges.

Another type of improper integral is one in which the integrand becomes infinite at a point inside the integration interval. An example of this is

$$\int_0^2 \frac{x}{x^2 - 1}\, dx. \tag{2.3}$$

In such cases the integral can be evaluated by noting that

$$\int_0^2 \frac{x}{x^2 - 1}\, dx = \int_0^1 \frac{x}{x^2 - 1}\, dx + \int_1^2 \frac{x}{x^2 - 1}\, dx. \tag{2.4}$$

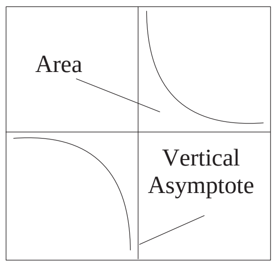

Figure 2.2: Convergent

The problem is now reduced to looking at two integrals in which the integrand becomes infinite at the endpoints of the integration interval. Each of these can be evaluated using the above analysis.

2.2 Method of Computation

In the following exercises you will be investigating the convergence properties of several integrals. In each case you will be evaluating integrals by computing limits such as

$$\lim_{b \to L} \int_a^b f(x)\,dx. \tag{2.5}$$

You can define such an integral as a function:

$$\mathbf{F} := \mathbf{b} -> \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)}\,\mathbf{dx} \tag{2.6}$$

Evaluating this function at various values of b as b approaches L, you can determine if the integral converges (exists) or diverges. If it converges, then you will obtain a finite value for the integral. Actually, you may only get an approximation to the integral.

The various test values for b can be set up manually by first defining **b:=10** before defining the integral. Then, you can change the values of b to larger values. You can also plot $F(b)$ vs b to see if the integral is converging as b approaches L. For example, you can type **plot(F(x), x = 1 .. b)**. For large values of b you can determine if the integral approaches some finite value in most cases.

Exercises

▷ **Exercise 2.1** Consider the integral

$$\int_1^\infty \frac{1}{x^p}\,dx. \tag{2.7}$$

Determine if this integral converges for the following values of p : 0.5, .95, 1.0, 1.5, 2.0, 3.0. What relationship do you see between convergence of the integral and the value of p? [**Note:** You will need to compute the integrals for each value of p separately.]

▷ **Exercise 2.2** Consider the integral

$$\int_0^2 \frac{dx}{(x-1)^{2/3}}. \tag{2.8}$$

Does it converge, or diverge? If it converges, to what does it converge?

[**Note:** Do the computation using Maple by splitting the integral into two integrals and taking the appropriate limits. Also, Maple actually computes the integrand as $[(x-1)^{1/3}]^2$. This results in complex values, when $x < 1$. (You should verify this.) You can avoid this by using $|x-1|^{2/3}$ as your integrand.]