Lab 4

Direction Fields and Euler's Method

Purpose

To investigate direction fields and to learn a simple numerical technique to solve first order differential equations.

Files

Euler.mws, DirectionFields.doc

4.1 Theory

Many physical phenomena are modelled using differential equations. A differential equation is an equation for an unknown function, $y(x)$, which involves the unknown function and its derivatives. The simplest type of differential equation involves only the first derivative. Formally, we have

$$F(\frac{dy}{dx}, y, x) = 0.$$

Such an equation is called a first order differential equation. Solving for the derivative, this can be written as

$$\frac{dy}{dx} = f(x, y). \tag{4.1}$$

A simple example is

$$\frac{dy}{dx} = 2x. \tag{4.2}$$

By a simple integration, we know that there are many possible solutions to Equation (4.2),

$$y(x) = x^2 + C.$$

This is called the general solution. It can be verified that this is a solution by inserting $y(x)$ into the Equation (4.2) and seeing that the solution makes the differential equation true.

If we want one member of this family of solutions, then we have to specify a condition on $y(x)$. For example, we could seek a solution satisfying Equation (4.2) and the condition $y(0) = 1$. Inserting $x = 0$ into the solution, we have $1 = C$. So, the resulting solution is

$$y(x) = x^2 + 1.$$

Conditions $y(x_0) = y_0$ are called initial conditions. Problems in which a differential equation and an initial condition are given are called initial value problems.

More general first order differential equations are not as easy to solve in terms of simple functions. We will learn how to solve several classes of first order differential equations in the class lectures. In the absence of these analytical methods, we can still obtain information about the solutions through the use of direction fields or numerical methods. We will explore these methods for the first order differential equation

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

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4.1.1 Direction Fields

We will investigate initial value problems of the form

$$y' = \frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \tag{4.3}$$

We first look at what are called direction fields. These give us sort of a snapshot of all of the solutions in a region of the xy-plane.

We sketch the graph of a typical solution. It is a curve that passes through the point (x_0, y_0). We would like to determine the shape of the solution curve. The differential equation actually provides us with the slope of the tangent to the solution curve at each point. In the figure we see an example of a solution curve with the tangents superimposed. Since we have the initial condition, we can see that the slope at that point is given by $f(x_0, y_0)$. However, we do not know any other points. In the next section we will see how we can get approximate values for the solution of the initial value problem. However, we do not have these values at this time.

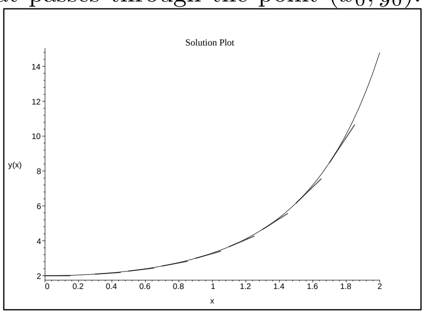

Figure 4.1: Plotting pieces of tangent lines to a solution curve.

For now, there is a way around our problem. We could just plot pieces of tangent lines at all points in the plane. This will give us what are called direction fields. An example of a direction field is given in the next figure. Here you can see a pattern developing. You can almost envision the above solution superimposed on the plot. In fact, given other initial conditions, you could sketch the solution passing through the curve. For example, given $y(1) = 1$, what do you think the solution curve would look like? How about the initial condition $y(-1) = -1$? You will be given a few examples of direction fields and will be asked to sketch in families of solutions.

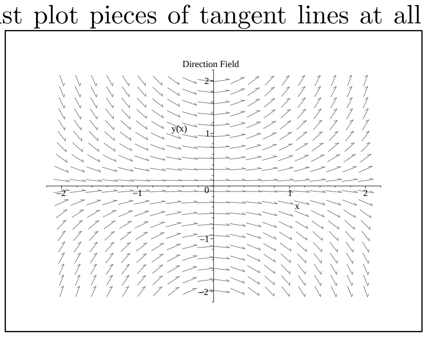

Figure 4.2: A direction field plot.

▷ **Exercise 4.1**

You are given six direction fields corresponding to six different first order differential equations. Discuss these plots with your lab partner. Note any patterns you see. What types of curves do you think the solutions take on? For each direction field, sketch three solution curves in different parts of the plot. Indicate for each a point on the graph, (x_0, y_0) , that would be an initial condition that would give that solution.

4.1.2 Euler's Method

Euler's method is a technique to numerically solve first order differential equations of the form in Equation (4.3).

The method is not very efficient, but it is simple and easy to implement on a computer. We may think of Euler's method as the quantitative version of the method of finding solution curves by direction fields. Basically, the method consists in flowing on straight lines by small increments along the direction field to approximate the solution. The solution of the differential equation, with the given initial condition, is a curve which passes through the point (x_0, y_0). Since y' represents the slope of the tangent to the curve, we see from (4.5) that the value $f(x_0, y_0)$ is precisely the slope of this tangent. Using the point-slope formula, we find the equation of this tangent line, and use it to approximate the solution curve near (x_0, y_0). We then increment the x-coordinate by a small amount and calculate the y-coordinate coordinate of this new point on the tangent line. At the new point, (x_1, y_1) we repeat the entire process, iterating n times, to get a sequence of successive approximations y_n to the solution. If more accuracy is desired, one can decrease the increment size and increase the number of iterations

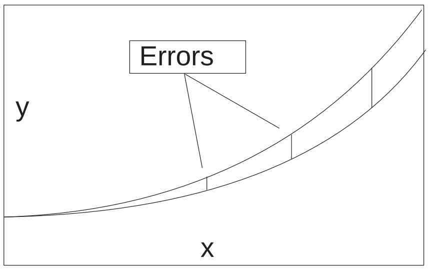

Figure 4.3: Euler's Method

Suppose that we are interested in the value of the solution to (4.5) at $x = b$. Let

$$\{x_0, x_1, \dots x_n = b,\}$$

with $x_0 < x_1 \dots < x_n = b$, be a partition of the interval (x_0, b), into n pieces of size h. Thus, $x_1 = x + h$ be a point nearby. The approximation y_1 to the solution at x_1 is given by

$$y_1 = y_0 + hf(x_0, y_0). \tag{4.4}$$

We now consider (x_1, y_1) as a new starting point and repeat the process. In general, the solution is obtained by iterating the formula

$$y_{n+1} = y_n + hf(x_n, y_n). \tag{4.5}$$

Clearly, the smaller we choose the value of h, the better the approximation. The trade off is that the smaller the value of h we pick, the larger the number of iterations needed. The approximation to the solution curve consists of a polygonal line joining the points (x_n, y_n). With the help of Maple it is possible to pick fairly small values of h without a noticeable waiting time. The polygonal approximation can then be made to look rather smooth. We suggest below some exercises to try in Maple, but it would be very worthwhile and instructive to try the first iterations by hand, with the help of a pocket calculator.

Exercises

Consider the differential equation

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

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▷ **Exercise 4.2** The Maple worksheet implements Euler's method to compute the solution for $0 \leq x \leq 1$. The worksheet generates the sequence of points (x_n, y_n) for $n = 1, 2, \ldots, N$ which approximates the solution of the differential equation. Open the worksheet and change the increments, h. Make observations as to how well the sequence of points (x_n, y_n) approximates the exact solution for the following:

1. $h = 1/2$, 2 iterations.
2. $h = 1/4$, 4 iterations.
3. $h = 1/10$, 10 iterations.
4. $h = 1/100$, 100 iterations.

▷ **Exercise 4.3** Show that $y = 2e^x - x - 1$ is the exact solution. Namely, insert this function (by hand) into the differential equation and show that the left side of the equation is the same as the right side.