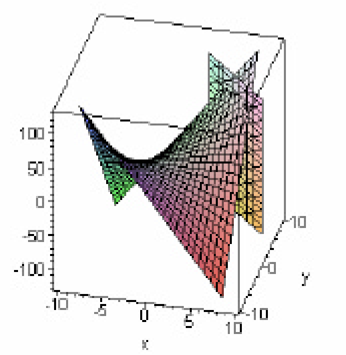
Lagrange Multiplier Problems

1. We want to find the relative maximum of $f(x,y)=2xy$ subject to the constraint $x+y=12$. As noted in class, this is the same as locating the intersection of a saddle and a plane. We found that the relative maximum occurs at $(x,y)=(6,6)$. The graph is shown below.



Recall out solution:

- Write Constraint: $g(x,y)=x+y-12=0.$
- Write F: $$F(x,y,l)=f(x,y)+l\,g(x,y)$$ $$=2xy+l\,(x+y-12).$$
- Write System of Equations:
  $$0=F_x=2y+l$$
  $$0=F_y=2x+l$$
  $$0=F_l=x+y-12.$$
- Solve System:
  $$2y+l=0\Rightarrow y=-\frac{l}{2}$$
  $$2x+l=0\Rightarrow x=-\frac{l}{2}$$
  $$x+y-12=0\Rightarrow -\frac{l}{2}-\frac{l}{2}-12=0\Rightarrow l=-12.$$
  Therefore, we find $x=6, y=6.$
- Write Solution: $(x,y)=(6,6).$

2. The other problem we set up was the one in which we designed a can to hold $250p$ cubic inches of candy with the maximum volume.

The volume of the can: $V=p\,r^2h=250p$.

The Surface Area = Top + Bottom + Side: $S(r,h)=2p\,r^2+2p\,rh.$

- Write Constraint: $g(r,h)=p\,r^2h-250p=0.$
- Write F: $$F(r,h,l)=S(r,g)+l\,g(r,h)$$ $$=2p\,r^2+2p\,rh+l\,(p\,r^2h-250p).$$
- Write System of Equations:

$$0 = F_r = 4p r + 2p h + 2p rh l$$
$$0 = F_h = 2p r + p r^2 l$$
$$0 = F_l = p r^2 h - 250p.$$

- Solve System:
  First clean up the equations, then attempt to solve them.
  $$2r + h + rh l = 0$$
  $$(2 + r l)r = 0$$
  $$r^2 h - 250 = 0.$$
  The second equation tells us that $r = 0$ or $2 + r l = 0$. Since r cannot vanish, we can use the other expression. Solving for $l = -\frac{2}{r}$, we can eliminate it from the first equation leaving two equations for the two unknowns:
  $$2r + h - rh\left(\frac{2}{r}\right) = 0 \Rightarrow 2r - h = 0 \Rightarrow h = 2r.$$
  $$r^2 h - 250 = 0.$$

  So, the third equation becomes $2r^3 = 250$, or $r^3 = 125 \Rightarrow r = 25$.

- Write Solution: Thus, our can has dimensions $r = 25$ in. and $h = 50$ in. How would this problem be different if we designed a can with no lid?

3. A Third Problem: Find positive x and y such that $x + y = 18$ and xy^2 is maximized.

   Here $f(x, y) = xy^2$.

   - Write Constraint: $g(x, y) = x + y - 18 = 0.$
   - Write F: $$F(x, y, l) = f(x, y) + l\, g(x, y)$$ $$= xy^2 + l(x + y - 18).$$
   - Write System of Equations:
     $$0 = F_x = y^2 + l$$
     $$0 = F_y = 2xy + l$$
     $$0 = F_l = x + y - 18.$$
   - Solve System: We need to eliminate one variable and find two equations and two unknowns. Let's eliminate l from the first two equations. From the first equation we quickly find $l = -y^2$. Substituting into the second equation, we are left with
     $$2xy - y^2 = 0$$
     $$x + y - 18 = 0$$
     Now, solve this nonlinear system. Solving for x in the last equation, and substituting into the first, one finds that
     $$2(18 - y)y - y^2 = 0 \Rightarrow 36y - 3y^2 = 0 \Rightarrow y = 0{,}12.$$

     Therefore, we find $x = 6, y = 12$. (Why do we not use y=0?)

   - Write Solution: $(x, y) = (6{,}12)$.