Complex Analysis Notes

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Poisson Integral Formula

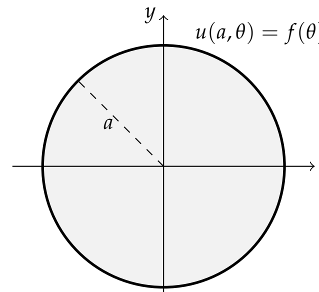

Figure 1: The disk of radius a with boundary condition along the edge at $r = a$.

The solution of Laplace's equation, $\nabla^2 u = 0$, in polar coordinates on the disk of radius a shown in Figure 1 with a fixed boundary condition, $u(a,\theta) = f(\theta)$, is given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n, \tag{1}$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta, \quad n = 0, 1, \ldots, \tag{2}$$

$$b_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta \quad n = 1, 2 \ldots. \tag{3}$$

We can put the solution in a more compact form by inserting the Fourier coefficients into the general solution. Doing this, we have

$$\begin{aligned} u(r,\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\theta + b_n \sin n\theta\right) r^n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, d\phi \\ &\quad + \frac{1}{\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \left[\cos n\phi \cos n\theta + \sin n\phi \sin n\theta\right] \left(\frac{r}{a}\right)^n f(\phi)\, d\phi \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n\right] f(\phi)\, d\phi. \end{aligned} \tag{4}$$

The term in the brackets can be summed. We note that

$$\begin{aligned} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n &= \operatorname{Re}\left(e^{in(\theta-\phi)} \left(\frac{r}{a}\right)^n\right) \\ &= \operatorname{Re}\left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n. \end{aligned} \tag{5}$$

Therefore,

$$\sum_{n=1}^{\infty} \cos n(\theta - \phi) \left(\frac{r}{a}\right)^n = \operatorname{Re}\left(\sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n\right).$$

The right hand side of this equation is a geometric series with common ratio of $\frac{r}{a} e^{i(\theta-\phi)}$, which is also the first term of the series. Since $\left|\frac{r}{a} e^{i(\theta-\phi)}\right| = \frac{r}{a} < 1$, the series converges. Summing the series, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{r}{a} e^{i(\theta-\phi)}\right)^n &= \frac{\frac{r}{a} e^{i(\theta-\phi)}}{1 - \frac{r}{a} e^{i(\theta-\phi)}} \\ &= \frac{r e^{i(\theta-\phi)}}{a - r e^{i(\theta-\phi)}} \end{aligned} \tag{6}$$

We need to rewrite this result so that we can easily take the real part. Thus, we multiply and divide by the complex conjugate of the denominator to obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n &= \frac{re^{i(\theta-\phi)}}{a-re^{i(\theta-\phi)}}\frac{a-re^{-i(\theta-\phi)}}{a-re^{-i(\theta-\phi)}} \\ &= \frac{are^{-i(\theta-\phi)}-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}.\end{aligned} \tag{7}$$

The real part of the sum is given as

$$\operatorname{Re}\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta-\phi)}\right)^n\right) = \frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}.$$

Therefore, the factor in the brackets under the integral in Equation (4) is

$$\begin{aligned}\frac{1}{2}+\sum_{n=1}^{\infty}\cos n(\theta-\phi)\left(\frac{r}{a}\right)^n &= \frac{1}{2}+\frac{ar\cos(\theta-\phi)-r^2}{a^2+r^2-2ar\cos(\theta-\phi)} \\ &= \frac{a^2-r^2}{2(a^2+r^2-2ar\cos(\theta-\phi))}.\end{aligned} \tag{8}$$

Thus, we have shown that the solution of Laplace's equation on a disk of radius a with boundary condition $u(a,\theta) = f(\theta)$ can be written in the closed form

Poisson Integral Formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}f(\phi)\,d\phi. \tag{9}$$

This result is called the Poisson Integral Formula and

$$K(\theta,\phi) = \frac{a^2-r^2}{a^2+r^2-2ar\cos(\theta-\phi)}$$

is called the Poisson kernel.

Example 1. Evaluate the solution (9) at the center of the disk.

We insert $r = 0$ into the solution (9) to obtain

$$u(0,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(\phi)\,d\phi.$$

Recalling that the average of a function $g(x)$ on $[a,b]$ is given by

$$g_{ave} = \frac{1}{b-a}\int_a^b g(x)\,dx,$$

we see that the value of the solution u at the center of the disk is the average of the boundary values. This is sometimes referred to as the mean value theorem.

Laplace's Equation in 2D - Complex Methods

HARMONIC FUNCTIONS ARE SOLUTIONS OF LAPLACE'S EQUATION. We have seen that the real and imaginary parts of a holomorphic function are harmonic. So, there must be a connection between complex functions and solutions of the two-dimensional Laplace equation. In this section we will describe how conformal mapping can be used to find solutions of Laplace's equation in two dimensional regions.

We can derive Laplace's equation for an incompressible, $\nabla \cdot \mathbf{v} = 0$, irrotational, , $\nabla \times \mathbf{v} = 0$, fluid flow. From well-known vector identities, we know that $\nabla \times \nabla\phi = 0$ for a scalar function, ϕ. Therefore, we can introduce a velocity potential, ϕ, such that $\mathbf{v} = \nabla\phi$. Thus, $\nabla \cdot \mathbf{v} = 0$ implies $\nabla^2\phi = 0$. So, the velocity potential satisfies Laplace's equation.

Fluid flow is probably the simplest and most interesting application of complex variable techniques for solving Laplace's equation. The study of fluid flow and conformal mappings dates back to Euler, Riemann, and others.[1] The method was further elaborated upon by physicists like Lord Rayleigh (1877) and applications to airfoil theory we presented in papers by Kutta (1902) and Joukowski (1906) on later to be improved upon by others. Conformal mappings have been used to study two-dimensional ideal fluid flow, leading to the study of airfoil design.

[1] "On the Use of Conformal Mapping in Shaping Wing Profiles," MAA lecture by R. S. Burington, 1939, published (1940) in ... 362-373

We begin by considering the fluid flow across a curve, C as shown in Figure 2. We assume that it is an ideal fluid with zero viscosity (i.e., does not flow like molasses) and is incompressible. It is a continuous, homogeneous flow with a constant thickness and represented by a velocity $\mathbf{U} = (u(x,y), v(x,y))$, where u and v are the horizontal components of the flow as shown in Figure 2.

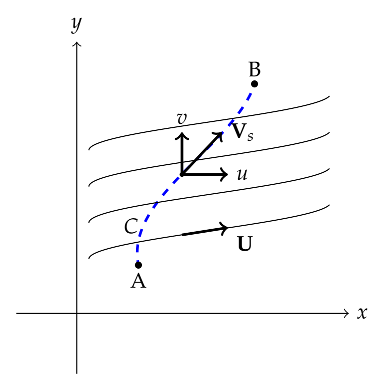

Figure 2: Fluid flow U across curve C between the points A and B.

We are interested in the flow of fluid across a given curve which crosses several streamlines. Therefore, for a unit thickness the mass flow rate is given by

$$\frac{dm}{dt} = \rho(u\,dy - v\,dx).$$

Since the total mass flowing across ds in time dt is given by $dm = \rho dV$, for constant density, this also gives the volume flow rate,

$$\frac{dV}{dt} = u\,dy - v\,dx,$$

over a section of the curve. The total volume flow over C is therefore

$$\frac{dV}{dt}\Big|_{\text{total}} = \int_C u\,dy - v\,dx.$$

If this flow is independent of the curve, i.e., the path, then we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}.$$

[This is just a consequence of Green's Theorem in the Plane.] Another way to say this is that there exists a function, $\psi(x,t)$, such that $d\psi = u\,dy - v\,dx$. Then,

$$\int_C u\,dy - v\,dx = \int_A^B d\psi = \psi_B - \psi_A.$$

However, from the calculus of several variables, we know that

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = u\,dy - v\,dx.$$

Therefore,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

It follows that if $\psi(x,y)$ has continuous second derivatives, then $u_x = -v_y$. This function is called the streamline function.

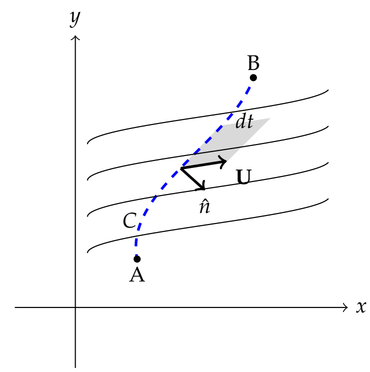

Figure 3: An amount of fluid crossing curve c in unit time.

Streamline functions.

Furthermore, for constant density, we have

$$\begin{aligned} \nabla \cdot (\rho \mathbf{U}) &= \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &= \rho \left(\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \right) = 0. \end{aligned} \tag{10}$$

This is the conservation of mass formula for constant density fluid flow.

Velocity potential curves.

We can also assume that the flow is irrotational. This means that the vorticity of the flow vanishes; i.e., $\nabla \times \mathbf{U} = \mathbf{0}$. Since the curl of the velocity field is zero, we can assume that the velocity is the gradient of a scalar function, $\mathbf{U} = \nabla \phi$. Then, a standard vector identity automatically gives

$$\nabla \times \mathbf{U} = \nabla \times \nabla \phi = 0.$$

For the two-dimensional flow with $\mathbf{U} = (u, v)$, we have

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}.$$

This is the velocity potential function for the flow.

Let's place the two-dimensional flow in the complex plane. Let an arbitrary point be $z = (x,y)$. Then, we have found two real-valued functions, $\psi(x,y)$ and $\psi(x,y)$, satisfying the relations

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \\ v &= \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \end{aligned} \tag{11}$$

These are the Cauchy-Riemann relations for the real and imaginary parts of a complex differentiable function,

$$F(z(x,y) = \phi(x,y) + i\psi(x,y).$$

From its form, $\frac{dF}{dz}$ is called the complex velocity and $\sqrt{\left|\frac{dF}{dz}\right|} = \sqrt{u^2+v^2}$ is the flow speed.

Furthermore, we have

$$\frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv.$$

Integrating, we have

$$\begin{aligned} F &= \int_C (u - iv)\, dz \\ \phi(x,y) + i\psi(x,y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x,y)\, dx + v(x,y)\, dy] \\ &\quad + i\int_{(x_0,y_0)}^{(x,y)} [-v(x,y)\, dx + u(x,y)\, dy]. \end{aligned} \tag{12}$$

Therefore, the streamline and potential functions are given by the integral forms

$$\begin{aligned} \phi(x,y) &= \int_{(x_0,y_0)}^{(x,y)} [u(x,y)\, dx + v(x,y)\, dy], \\ \psi(x,y) &= \int_{(x_0,y_0)}^{(x,y)} [-v(x,y)\, dx + u(x,y)\, dy]. \end{aligned} \tag{13}$$

These integrals give the circulation $\int_C V_s\, ds = \int_C u\, dx + v\, dy$ and the fluid flow per time, $\int_C -v\, dx + u\, dy$.

The streamlines are given by the level curves $\psi(x,y) = c_1$ and the potential lines are given by the level curves $\phi(x,y) = c_2$. These are two orthogonal families of curves; i.e., these families of curves intersect each other orthogonally at each point as we will see in the examples. Note that these families of curves also provide the field lines and equipotential curves for electrostatic problems.

Streamliners and potential curves are orthogonal families of curves.

Example 2. Show that $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ are an orthogonal family of curves when $F(z) = \phi(x,y) + i\psi(x,y)$ is holomorphic.

In order to show that these curves are orthogonal, we need to find the slopes of the curves at an arbitrary point, (x,y). For $\phi(x,y) = c_1$, we recall from multivaribale calculus that

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = 0.$$

So, the slope is found as

$$\frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}.$$

Similarly, we have

$$\frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}.$$

Since $F(z)$ is differentiable, we can use the Cauchy-Riemann equations to find the product of the slopes satisfy

$$\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}} \frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = -1.$$

Therefore, $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$ form an orthogonal family of curves.

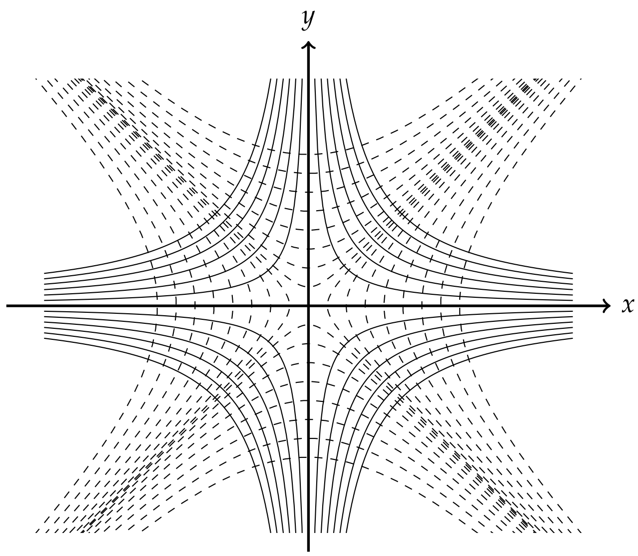

Figure 4: Plot of the orthogonal families $\phi = x^2 - y^2 = c_1$ (dashed) and $\phi(x,y) = 2xy = c_2$.

Example 3. For $F(z) = z^2 = x^2 - y^2 + 2ixy$, show that the level curves for $Re(F)$ and $Im(F)$ are orthogonal.

For this problem, $\phi(x,y) = x^2 - y^2$ and $\psi(x,y) = 2xy$. The slopes of the families of curves are given by

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \\
&= -\frac{2x}{-2y} = \frac{x}{y}. \\
\frac{dy}{dx} &= -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} \\
&= -\frac{2y}{2x} = -\frac{y}{x}.
\end{aligned} \tag{14}$$

The products of these slopes is -1, proving that the level curves for $Re(F)$ and $Im(F)$ are orthogonal. These orthogonal families are depicted in Figure 4.

We will now turn to some typical examples by writing down some differentiable functions, $F(z)$, and determining the types of flows that result from these examples. We will then turn in the next section to using these basic forms to solve problems in slightly different domains through the use of conformal mappings.

Example 4. Describe the fluid flow associated with $F(z) = U_0 e^{-i\alpha} z$, where U_0 and α are real.

For this example, we have

$$\frac{dF}{dz} = U_0 e^{-i\alpha} = u - iv.$$

Thus, the velocity is constant,

$$\mathbf{U} = (U_0 \cos\alpha, U_0 \sin\alpha).$$

Thus, the velocity is a uniform flow at an angle of α.

Since

$$F(z) = U_0 e^{-i\alpha} z = U_0(x\cos\alpha + y\sin\alpha) + iU_0(y\cos\alpha - x\sin\alpha).$$

Thus, we have

$$\begin{aligned} \phi(x,y) &= U_0(x\cos\alpha + y\sin\alpha), \\ \psi(x,y) &= U_0(y\cos\alpha - x\sin\alpha). \end{aligned} \qquad (15)$$

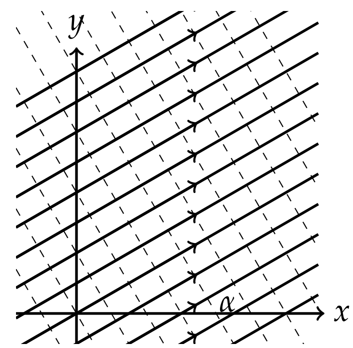

Figure 5: Stream lines (solid) and potential lines (dashed) for uniform flow at an angle of α, given by $F(z) = U_0 e^{-i\alpha} z$.

An example of this family of curves is shown in Figure **??**.

Example 5. Describe the flow given by the function $F(z) = \frac{U_0 e^{-i\alpha}}{z - z_0}$.

We write

$$\begin{aligned} F(z) &= \frac{U_0 e^{-i\alpha}}{z - z_0} \\ &= \frac{U_0(\cos\alpha + i\sin\alpha)}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0) - i(y-y_0)] \\ &= \frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0)\cos\alpha + (y-y_0)\sin\alpha] \\ &\quad + i\frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[-(y-y_0)\cos\alpha + (x-x_0)\sin\alpha]. \end{aligned} \qquad (16)$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[(x-x_0)\cos\alpha + (y-y_0)\sin\alpha] = c_1, \\ \psi(x,y) &= \frac{U_0}{(x-x_0)^2 + (y-y_0)^2}[-(y-y_0)\cos\alpha + (x-x_0)\sin\alpha] = c_2. \end{aligned} \qquad (17)$$

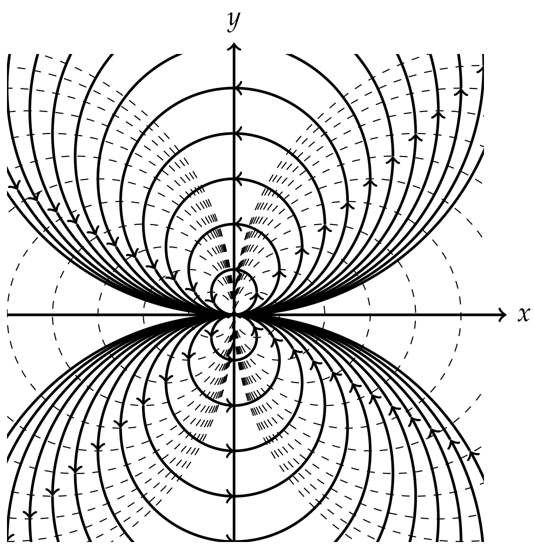

Figure 6: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z}$ for $\alpha = 0$.

The level curves for the stream and potential functions satisfy equations of the form

$$\beta_i(\Delta x^2 + \Delta y^2) - \cos(\alpha + \delta_i)\Delta x - \sin(\alpha + \delta_i)\Delta y = 0,$$

where $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\beta_i = \frac{c_i}{U_0}$, $\delta_1 = 0$, and $\delta_2 = \pi/2.$, These can be written in the more suggestive form

$$(\Delta x - \gamma_i \cos(\alpha - \delta_i))^2 + (\Delta y - \gamma_i \sin(\alpha - \delta_i))^2 = \gamma_i^2$$

for $\gamma_i = \frac{c_i}{2U_0}$, $i = 1, 2$. Thus, the stream and potential curves are circles with varying radii (γ_i) and centers $((x_0 + \gamma_i \cos(\alpha - \delta_i), y_0 + \gamma_i \sin(\alpha - \delta_i)))$. Examples of this family of curves is shown for $\alpha = 0$ in in Figure 6 and for $\alpha = \pi/6$ in in Figure 7.

The components of the velocity field for $\alpha = 0$ are found from

$$\begin{aligned}
\frac{dF}{dz} &= \frac{d}{dz}\left(\frac{U_0}{z - z_0}\right) \\
&= -\frac{U_0}{(z - z_0)^2} \\
&= -\frac{U_0[(x - x_0) - i(y - y_0)]^2}{[(x - x_0)^2 + (y - y_0)^2]^2} \\
&= -\frac{U_0[(x - x_0)^2 + (y - y_0)^2 - 2i(x - x_0)(y - y_0)]}{[(x - x_0)^2 + (y - y_0)^2]^2} \\
&= -\frac{U_0[(x - x_0)^2 + (y - y_0)^2]}{[(x - x_0)^2 + (y - y_0)^2]^2} + i\frac{U_0[2(x - x_0)(y - y_0)]}{[(x - x_0)^2 + (y - y_0)^2]^2} \\
&= -\frac{U_0}{[(x - x_0)^2 + (y - y_0)^2]} + i\frac{U_0[2(x - x_0)(y - y_0)]}{[(x - x_0)^2 + (y - y_0)^2]^2}. \quad (18)
\end{aligned}$$

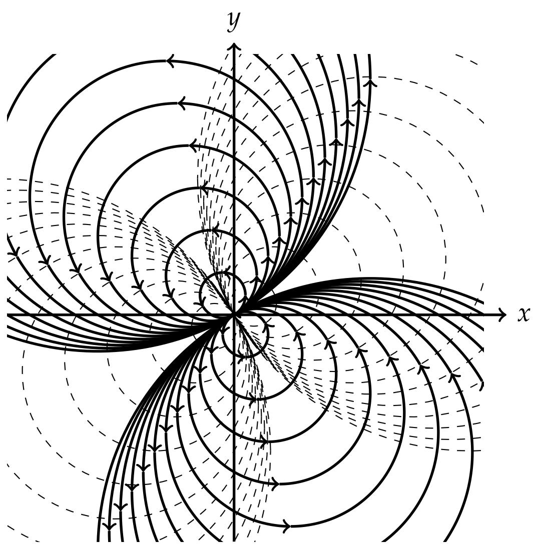

Figure 7: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{U_0 e^{-i\alpha}}{z}$ for $\alpha = \pi/6$.

Thus, we have

$$\begin{aligned} u &= -\frac{U_0}{[(x-x_0)^2 + (y-y_0)^2]}, \\ v &= \frac{U_0[2(x-x_0)(y-y_0)]}{[(x-x_0)^2 + (y-y_0)^2]^2}. \end{aligned} \tag{19}$$

Example 6. Describe the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$.

We write $F(z)$ in terms of its real and imaginary parts:

$$\begin{aligned} F(z) &= \frac{m}{2\pi}\ln(z - z_0) \\ &= \frac{m}{2\pi}\left[\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} + i\tan^{-1}\frac{y-y_0}{x-x_0}\right]. \end{aligned} \tag{20}$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{m}{2\pi}\ln\sqrt{(x-x_0)^2 + (y-y_0)^2} = c_1, \\ \psi(x,y) &= \frac{m}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_2. \end{aligned} \tag{21}$$

Rewriting these equations, we have

$$\begin{aligned} (x-x_0)^2 + (y-y_0)^2 &= e^{4\pi c_1/m}, \\ y - y_0 &= (x-x_0)\tan\frac{2\pi c_2}{m}. \end{aligned} \tag{22}$$

In Figure 8 we see that the stream lines are those for a source or sink depending if $m > 0$ or $m < 0$, respectively.

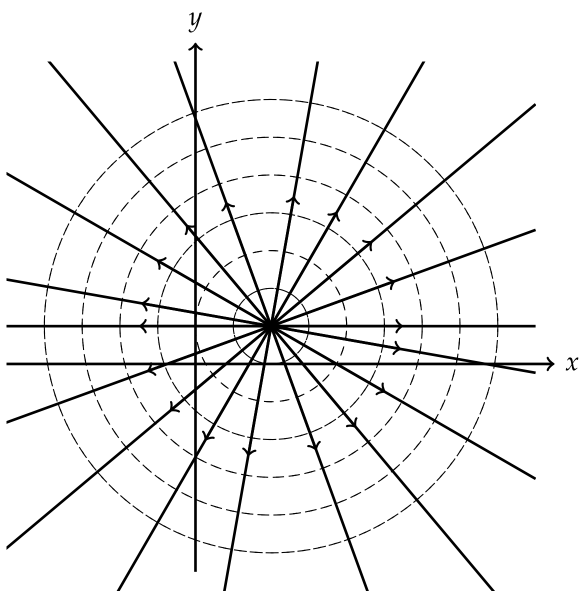

Figure 8: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = \frac{m}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

Example 7. Describe the flow given by $F(z) = -\frac{i\Gamma}{2\pi}\ln\frac{z-z_0}{a}$.

We write $F(z)$ in terms of its real and imaginary parts:

$$\begin{aligned} F(z) &= -\frac{i\Gamma}{2\pi}\ln\frac{z-z_0}{a} \\ &= -i\frac{\Gamma}{2\pi}\ln\sqrt{\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{a}\right)^2}+\frac{\Gamma}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} \end{aligned} \quad (23)$$

The level curves become

$$\begin{aligned} \phi(x,y) &= \frac{\Gamma}{2\pi}\tan^{-1}\frac{y-y_0}{x-x_0} = c_1, \\ \psi(x,y) &= -\frac{\Gamma}{2\pi}\ln\sqrt{\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{a}\right)^2} = c_2. \end{aligned} \quad (24)$$

Rewriting these equations, we have

$$\begin{aligned} y - y_0 &= (x-x_0)\tan\frac{2\pi c_1}{\Gamma}, \\ \left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{a}\right)^2 &= e^{-2\pi c_2/\Gamma}. \end{aligned} \quad (25)$$

In Figure 9 we see that the stream lines circles, indicating rotational motion. Therefore, we have a vortex of counterclockwise, or clockwise flow, depending if $\Gamma > 0$ or $\Gamma < 0$, respectively.

Example 8. Flow around a cylinder, $F(z) = U_0\left(z + \frac{a^2}{z}\right)$, $a, U_0 \in R$.

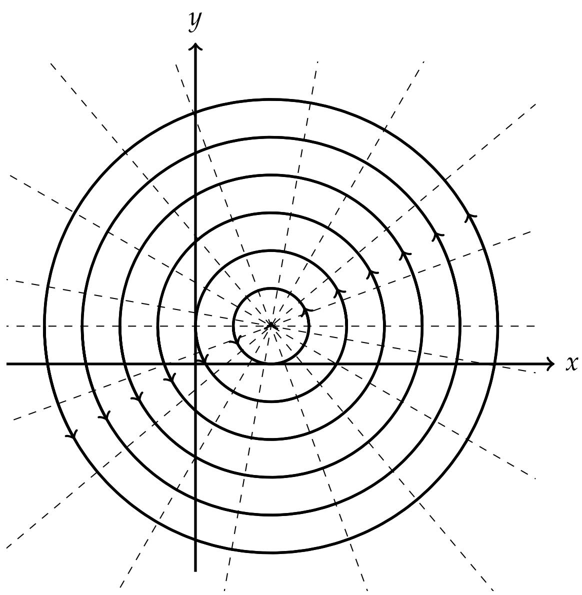

Figure 9: Stream lines (solid) and potential lines (dashed) for the flow given by $F(z) = -\frac{i\Gamma}{2\pi}\ln(z - z_0)$ for $(x_0, y_0) = (2, 1)$.

For this example, we have

$$\begin{aligned} F(z) &= U_0\left(z + \frac{a^2}{z}\right) \\ &= U_0\left(x + iy + \frac{a^2}{x + iy}\right) \\ &= U_0\left(x + iy + \frac{a^2}{x^2 + y^2}(x - iy)\right) \\ &= U_0 x\left(1 + \frac{a^2}{x^2 + y^2}\right) + iU_0 y\left(1 - \frac{a^2}{x^2 + y^2}\right). \end{aligned} \tag{26}$$

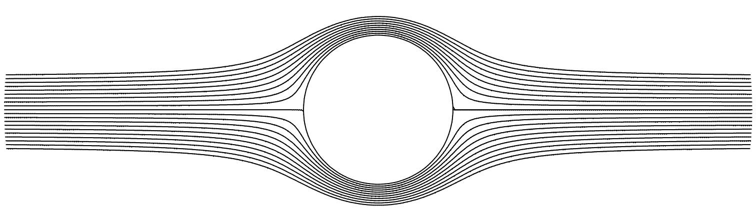

Figure 10: Stream lines for the flow given by $F(z) = U_0\left(z + \frac{a^2}{z}\right)$.

The level curves become

$$\begin{aligned} \phi(x, y) &= U_0 x\left(1 + \frac{a^2}{x^2 + y^2}\right) = c_1, \\ \psi(x, y) &= U_0 y\left(1 - \frac{a^2}{x^2 + y^2}\right) = c_2. \end{aligned} \tag{27}$$

Note that for the streamlines when $|z|$ is large, then $\psi \sim Vy$, or horizontal lines. For $x^2 + y^2 = a^2$, we have $\psi = 0$. This behavior is shown in Figure 10 where we have graphed the solution for $r \geq a$.

The level curves in Figure 10 can be obtained using the implicitplot feature of Maple. An example is shown below:

```
restart: with(plots):
k0:=20:
for k from 0 to k0 do
  P[k]:=implicitplot(sin(t)*(r-1/r)*1=(k0/2-k)/20, r=1..5,
  t=0..2*Pi, coords=polar,view=[-2..2, -1..1], axes=none,
  grid=[150,150],color=black):
  od:
display({seq(P[k],k=1..k0)},scaling=constrained);
```

A slight modification of the last example is if a circulation term is added:

$$F(z) = U_0\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln\frac{r}{a}.$$

The combination of the two terms gives the streamlines,

$$\psi(x, y) = U_0 y\left(1 - \frac{a^2}{x^2 + y^2}\right) - \frac{\Gamma}{2\pi}\ln\frac{r}{a},$$

which are seen in Figure 11. We can see interesting features in this flow including what is called a stagnation point. A stagnation point is a point where the flow speed, $\left|\frac{dF}{dz}\right| = 0$.

Stagnation points.

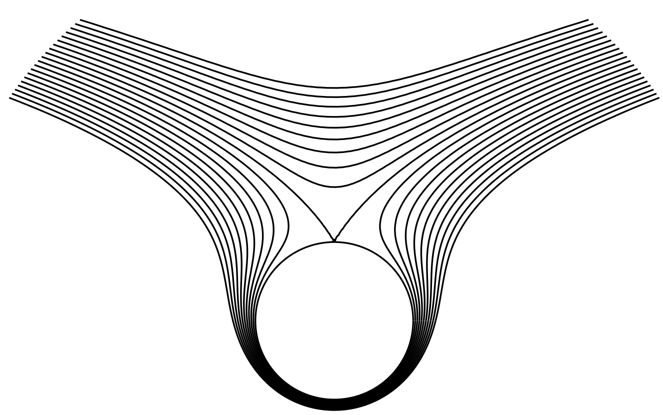

Figure 11: Stream lines for the flow given by $F(z) = U_0\left(z + \frac{a^2}{z}\right) - \frac{\Gamma}{2\pi}\ln\frac{z}{a}$.

Example 9. Find the stagnation point for the flow $F(z) = \left(z + \frac{1}{z}\right) - i\ln z$.

Since the flow speed vanishes at the stagnation points, we consider

$$\frac{dF}{dz} = 1 - \frac{1}{z^2} - \frac{i}{z} = 0.$$

This can be rewritten as

$$z^2 - iz - 1 = 0.$$

The solutions are $z = \frac{1}{2}(i \pm \sqrt{3})$. Thus, there are two stagnation points on the cylinder about which the flow is circulating. These are shown in Figure 12.

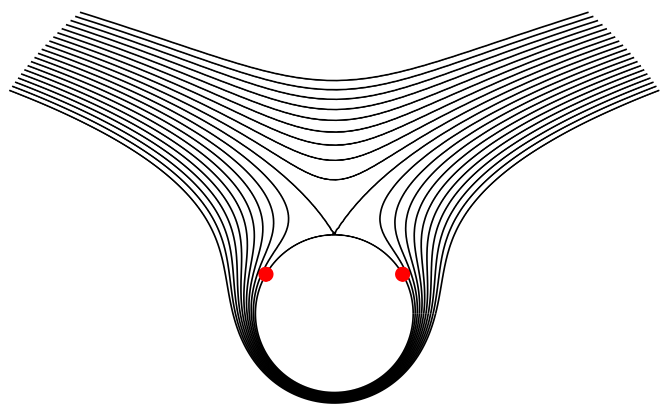

Figure 12: Stagnation points (red) on the cylinder are shown for the flow given by $F(z) = \left(z + \frac{1}{z}\right) - i \ln z$.