Complex Numbers

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MAT 415-515

Introduction to Complex Variables

$x^2+1=0 \Rightarrow x^2=-1$

or $x=\pm\sqrt{-1} \equiv \pm i$

Complex Numbers: $a+bi$, $a,b\in\mathbb{R}$

... of ordered pairs (x,y) $x,y\in\mathbb{R}$

Let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$

Equality $z_1=z_2$ iff $x_1=x_2$ and $y_1=y_2$

Equivalence Relation:

1. $z_1=z_1$, $\forall z_1$
2. $z_1=z_2 \Rightarrow z_2=z_1$ $\quad \forall z_1,z_2$
3. $z_1=z_2$ ... $z_3 \Rightarrow z_1=$ ... $_3$ $\forall z$

...ion $\quad$ ... $_1+z_2$ $\quad$ ...ff $\quad x_1+x_2=x_3$ and y_1+y_2...

1. z_1 ... $_2=z_2+z_1$ (commutative)
2. z_1 ... $z_2+z_3)=(z_1+z_2)+z_3$
3. $\exists 0$ ... $\forall$ ...
 $\forall z$ $\exists$ ... z ... $)=0$

#3 $\quad z+(\alpha,\beta)=z$

$(x,y)+(\alpha,\beta)=(x,y)$

$x+\alpha=x$ and $y+\beta=y$

$0 = (0,0)$

Uniqueness of 0

Assume $0, 0'$ are additive identities

Then

$$\left.\begin{aligned} 0 - 0 &= 0' && \text{by } 3 \\ 0' + 0 &= 0 + 0' && \text{by } 1 \\ &= 0 && \text{by } 3 \end{aligned}\right\} \Rightarrow 0'$$

Subtraction $z_1 - z_2 = z_1 + (-z_2)$

Multiplication $z_1 z_2 = z_3$ iff

$$x_3 = x_1 x_2 - y_1 y_2$$
$$y_3 = x_1 y_2 + x_2 \quad ,$$

Properties

1. $z_1 z_2 = z_2 z$, $\forall z_1, z_2$
2. $z_1(z_2 z_3) = (z_1 z_2) z$, $\forall z_1 z_2 z_3$
3. $\exists$ identity $\underline{1} \ni \forall z,\ \underline{1} z = z$
4. $\forall z \neq 0, \exists z^{-1} \ni\ z^{-1} z = 1$

3 $\ ! \ - = z$

$(\alpha, \beta) z = z$

$$\Rightarrow \begin{aligned} \alpha x - \beta y &= x \\ \alpha y + \beta x &= y \end{aligned} \quad \Rightarrow \alpha = 1, \beta = 0$$

$$\underline{1} = (1, 0)$$

4. Find $z^{-1} = (\xi, \eta)$

$$(\xi, \eta) z = \underline{1}$$
$$\xi x - \eta y = 1$$

$$\xi = \frac{x}{x^2 + y^2}$$

$\xi y + \eta x = 0$ ⎫ $\eta = -\frac{y}{x^2+y^2}$

Note: $x^2+y^2 \neq 0 \Rightarrow x, y \neq 0$ or $z = 0$

Division: $\frac{z_1}{z_2} = z_1 z_2^{-1} = z_3$

$$x_3 = \frac{x_1 x_2 \qquad\qquad}{x_2^2+y_2^2}$$

$$y_3 = \frac{-x_1 y_2 + x_2 y_1}{x_2^2+y_2^2}$$

Collection of ordered pairs of reals with equality and the above addition, multiplicati

- algebraic field over $\mathbb{R}$

$$z_1(z_2+z_3) = z_1 z_2 + z_1 z_3 \quad \forall z_1, z_2, z_3 \in \mathbb{C}$$

Consider $z = (0,1)$

$$z^2 = (0^2 - 1^2, 0\cdot 1 + 1\cdot 0) = (-1, 0) = -(1,0) =$$

So $i = (0,1)$

$\mathbb{R} \subset \mathbb{C}$ since $(x,0) \in \mathbb{C}$, $x \in \mathbb{R}$

$\mathbb{R}$ is isomorphic to the set of reals.

- $\mathbb{R}$ is an ordered field.

Real Part of $z = x$ or $\text{Re}(z) = x$

Imaginary part of z $\text{Im}(z) = y$

Define $az = (ax, ay)$ $a \in \mathbb{R}, z \in \mathbb{C}$

1. $(a+b)z = az + bz$

2. $a(\quad {}_2) = az_1 + az_2$
3. $a(bz) = abz$
4. $1z = z$

+ Addition properties $\Rightarrow$ $\mathbb{C}$ is a linear vector space

Note $(c, b) = a(1,0) + b(0,1)$
$= a + bi$

Normally $z = x + iy$

Complex Modulus

$$|z| = \sqrt{x^2 + y^2} \geq 0$$

$$|z| = 0 \iff - = 0$$

(or $x = 0, y = 0$)

Complex Conjugate, $\bar{z} = x - iy$

Note $z\bar{z} = (x+iy)(x-iy)$
$= x^2 + y^2 = |z|^2$

Complex Plane

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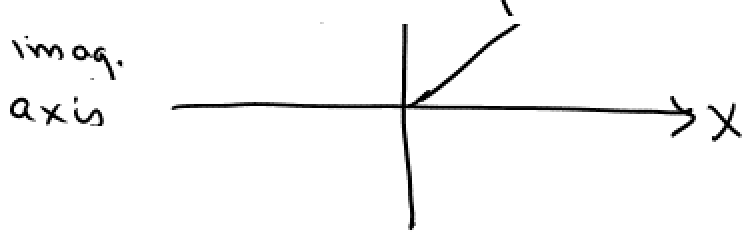
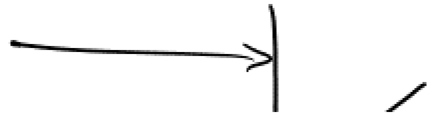
$\mathbb{C}$ - linear vector space over $\mathbb{R}$
- set ordered pairs $z = (x, y)$





— real axis

Polar Representation

$$z = x + iy$$
$$= r\cos\theta + i r\sin\theta$$

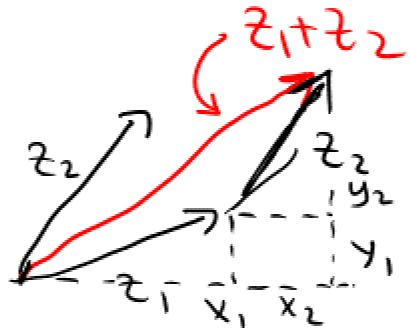
$|z| = r$

$\theta = \arg z$

$\quad - \tan^{-1}$

Geometry

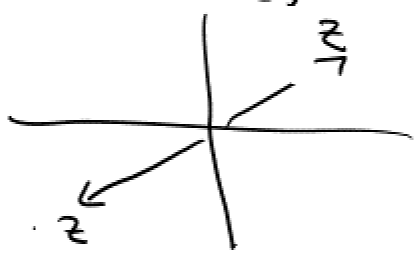
$z_1 + z_2$

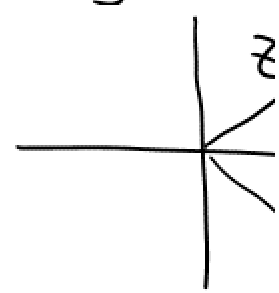


$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$

$\bar{z} = x - iy \quad = ($

$-z$





$\arg z = -\arg\bar{z}$

$\arg(-z) = \pi + \arg z$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's Formula)

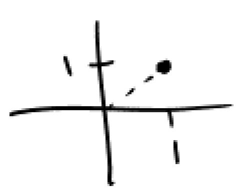
So $z = r(\cos\theta + i\sin\theta)$ — before

$\Rightarrow \boxed{}$

Ex $z = 1 + i$

$r = \sqrt{2}$

$\theta = \tan^{-1}(1)$



$\theta = \tan^{-1}(1)$
$= \pi/4$

$z = \sqrt{2}\, e^{i}$

<u>Ex</u> $t = \sqrt{1+i} = \left(\sqrt{2}\, e^{i\pi/4}\right)^{1/2}$

$t^2 = 1+i$

$- (1+i) = 0 \Rightarrow 2$ solutions

<u>Need</u> $e^{2\pi i} = \cos\pi + i\sin 2\pi = 1$

$e^{4\pi i} = 1$

$\Longrightarrow e^{2K\pi i} = 1$, K-integer

Now, $\sqrt{1+i} = \left(\sqrt{2}\, e^{i\pi/4} \cdot 1\right)^{1/2}$

$= \left(\sqrt{2}\, e^{i\pi/4} e^{2K\pi i}\right)^{1/2}$

$= \sqrt[4]{2}\, e^{i} \quad e^{\pi K i}$

$= \sqrt[4]{2}\, e^{i\pi/8}, -\sqrt[4]{2}\, e^{i\pi/8}$

Other Properties:

$^{i\theta} - \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$|e^{i\theta}| = \sqrt{e^{i\theta} e^{-i\theta}} = 1$

$e^{i} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$

$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i0}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}, \quad z_2 \neq 0$$

Inequalities

$Re(z) \leq |z|$, $Im(z) \leq |z|$ (   )

Pf/ ez) $x \leq \sqrt{x^2} \leq \sqrt{x^2+y^2} = |z|$

Triangle Inequality

$$|z_1+z_2| \leq |z_1| + |z_2|$$

Pf/ $|z_1+z_2|^2 - (\quad \overline{z_1+z_2})$

$= (z_1 + \quad {}_2)(\bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + \underbrace{z_1 z_2 \quad z_1}$

Note: $\overline{z\,\bar{z}_2} = \bar{z}\,\bar{\bar{z}}_2 = z_2\bar{z}_1$

and $z + \bar{z} = (x+iy) + (x-iy) = 2$

So, $|z_1+z_2|^2 = |z_1|^2 + 2\,Re(z_1\bar{z}_2) + |z_2|^2$

$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$

$= |z_1|^2 + 2 \quad z_2| + |z_2|^2$

$$= (|z_1|+|z_2|)^2$$

Others $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$

$$\Rightarrow |z_1| - |z_2| \le |z_1 - z_2|$$

Similarly, $|z_2| - |z_1| \le |z_1 - z_2|$

$$\Rightarrow |z_1 - z_2| \ge \big|\,|z_1| - |z_2|\,\big|$$

Point Sets

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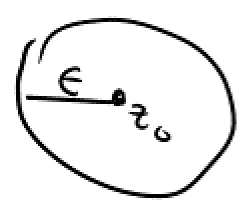
Point Sets	S
Complement	
Subset	**if**
Union	
Intersection	
ε-neighborhood of z_0	
Limit Point of S	**If every ε-neighborhood of z_0 contains at least one point different from z_0**
Interior Point of S	**If some ε-neighborhood of z_0 contains only points in S**
Open	**If S contains only interior points**
Closed	**If S contains all of its limit points or if it has no limit points**
Boundary Point of S	**If every ε-neighborhood of z_0 contains both points in S and in $C(S)$.**
Boundary of S	**Collection of boundary points.**
Closure	**for the set of limit points**
Empty Set	ϕ
Bounded	**If there exists M such that**



Comments

Proper Subset ,

Equality and implies

Disjoint

z_0 is a limit point of S every ε-neighborhood of z_0 contains an infinite number c ✳ points.

Finite sets are closed.

The complement of an open set is closed.

There are sets that are both open and closed. There are sets that are neither.

Compact Sets If every infinite subset has a limit point.
Compact sets are closed and bounded

Bolzano-Weierstrass Thm Every bounded infinite set has at least on limit point.

Covering Collection of sets such that

Heine-Borel Thm If S is closed and bound and has an open covering, then there is a finite subcovering of S.

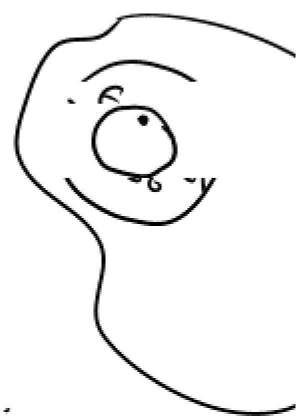
✳ z_0 is a limit point

Let $|z_0 - z_1| < \epsilon_1$

Pick $\epsilon_2 = \frac{1}{2}|z_0 - z_1|$

$\exists\, z_2 \ni |z_0 - z_2| < \epsilon_2$

Repeat $\Rightarrow z_1, z_2, z_3, \ldots$

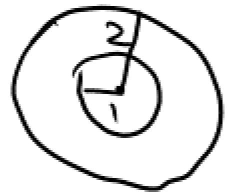


Examples: Open/Closed Sets

Open $\phi, \mathbb{C}, \{z \mid |z| < 1\}, Im(z) > 0$ H - upper ...

Closed $\mathbb{C}$, ϕ, $\{z \mid |z| \leq 1\}$. $Re(z) \geq 0$

Neither $\{z \mid 1 < |z| \leq 2\}$



Compact Sets

- if every infinite subset has a limit pt.

$\equiv$ Closed and bounded

Pf/ 1) S is finite or ϕ

There are no infinite subsets $\Rightarrow$ no li

trivial - closed, bounded $\Rightarrow$ com

2) S is infinite & compact.

Let z_0 be a limit point of

$\exists\, z_1 \in N_{\epsilon_1}(z_0)$ or $|z_1 - z_0| < \epsilon_1$

$\neq$

Let $\frac{1}{2}|z_1 - z_0| = \epsilon_2$ Then $\exists\, z_2 \in N_{\epsilon}$

with $z_2 \neq z_0, z_1$

repeating this process $\Rightarrow$

$z_1, z_2, z_3, \ldots$ an infinite sequenc

of distinct z's $\subseteq S$ with

limit point

S compact $\Rightarrow$ $z_0 \in S$

So all limit points $\in S' \Rightarrow S$ closed.

Then $\exists\, z_1 \in S \ni.\ |z_1| > 1$

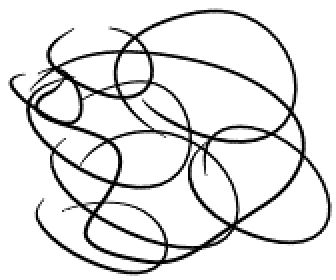
and $\exists\, z_2 \in S \ni.\ |z_2| > 2$

.... $\exists\, z_n \in S \ni.\ |z_n| > n$.

for any integer n

$z_1, \; , \; , 3 \ldots$ w/o limit pt

Coverings



Stereographic Projection

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Stereographic Projection

$\mathbb{C}$ - not compact

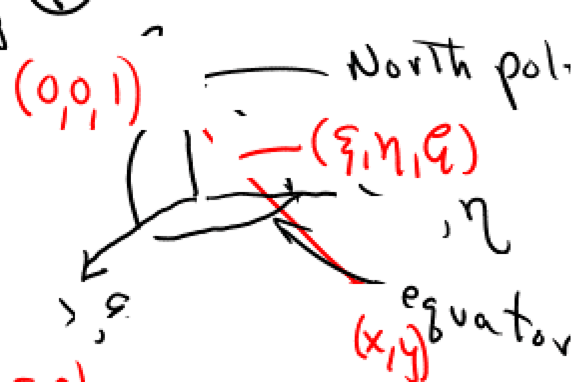
Consider $z = 1, 2, 3, \ldots$ has no limit pt.

Want to compactify $\mathbb{C}$

Riemann Sphere

$z = x + iy$

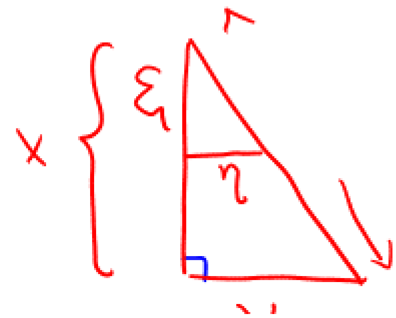
$\xi^2 + \eta^2 + \zeta^2 = 1$

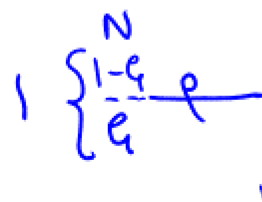


z is mapped to the sphere

(ξ, η, ζ) – image of z
is almost 1-1 Not North Pole.
To include The North Pole - add The point at ∞. $\Rightarrow$ extended complex

Transformations





$$\frac{\xi}{x} = \frac{\eta}{y} = \frac{\rho}{r} = \frac{1-\zeta}{1}$$

where $r = \sqrt{x^2 + y^2}$ and $\rho = \sqrt{\xi^2 + \eta^2}$. Fr

$$x = \frac{\xi}{1-\zeta},\; y = \frac{\eta}{1-\zeta},\; \xi = \frac{2x}{r^2+1},\; \eta =$$

$\xi^2 + \eta^2 + \zeta^2 = 1$

$\rho^2 + \zeta^2 = 1$

$x^2 + y^2 = \dfrac{\xi^2 + \eta^2}{(1-\zeta)^2}$

$r^2 = \dfrac{\rho^2}{(\ \ )}$

$1 + r^2 = 1 + \dfrac{1-\zeta}{\ \ \zeta)}$

$\alpha z \bar{z} + \hat{\beta} z + \hat{\gamma} \bar{z} + \delta = 0$

$$\frac{\ }{(1-\zeta)^2} + \frac{\ }{1-\zeta} + \frac{\ }{1-\zeta} + \delta = 0,$$

$$+ \zeta) + \beta\xi + \gamma\eta + \delta(1-\zeta) = 0,$$

$$\beta\xi + \gamma\eta + (\alpha - \delta)\zeta + \alpha + \delta = 0.$$

stere c projection. For example, *straight lines and circles are mapped into circles.* The general equation of a circle in the complex plane is

So

$$\alpha x^2 + \alpha y^2 + \beta x + \gamma y + \delta = 0.$$

$1-\zeta$

Under phic projection, we have

$$\alpha \frac{\xi^2 + \eta^2}{(1-\zeta)^2} + \frac{\beta\xi}{1-\zeta} + \frac{\gamma\eta}{1-\zeta} + \delta = 0,$$

$1 - \zeta^2 \qquad \beta\xi \qquad \gamma\eta$

plane in

Need the intersection of the plane and the uni

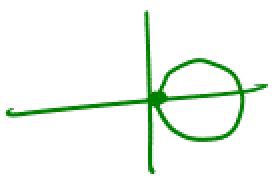
= circle or a pt.

Ex $\alpha = 0$ $\quad \beta\xi + \gamma\eta - \delta\zeta + \delta = 0$ & $\xi^2+\eta^2+\zeta^2 = 1$

and $\quad \beta x + \gamma y + \delta = 0$ — line

Note the circles pass through $N: (\xi, \eta, \zeta)$

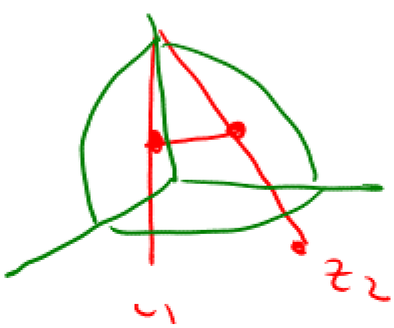
Ex $\delta = 0, \alpha \neq 0$



$\alpha x^2 + \alpha y^2 + \beta x + \gamma y = 0$ passes

But $(0,0) \longrightarrow$ South Pole (

Chordal Metric



$\rho(z_1, z_2) = \sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 -}$

$= \sqrt{2(1 - \xi_1\xi_2 - \eta}$

$\sqrt{}$

$\sqrt{1+r_1^2} \quad \sqrt{1+r_2^2}$

$\rho(z_1, z_2) \qquad \dfrac{2|z_1 - z_2|}{}$

Ex $\rho(z_1,\infty) = \sqrt{\xi^2+\eta_1^2+(\zeta_1-1)^2} = \sqrt{2}$

$$= \frac{2}{\sqrt{1+r_1^2}} = \frac{2}{\sqrt{1+|z_1|^2}}$$

Metric

$$\rho(z_1,z_2) = \rho(z_2,z_1)$$
$$\rho(z_1,z_2) \le \rho(z_1,z_3)+\rho(z_3,z_2)$$
$$\rho(z_1,z_2) \ge 0$$
$$\rho(z_1,z_2) = 0 \text{ iff } z_1 = z_2$$

$$\rho(z_1,z_2) \le 2|z_2-z_1| \le (1+M^2)\,\rho($$
$$|z|<M$$

Curves & Regions
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Curves

Simple Jordan arc – set of points given by

$z(t) = x(t)+iy(t),\ 0<t<$

$x(t), y(t)$ continuous, rea

$\ni.\ t_1 \ne t_2 \Rightarrow z(t_1) \ne$

Simple Smooth arc – Jordan arc $\ni.$ $\dot{x}(t),$

& $\dot{x}^2+\dot{y}^2 \ne 0$

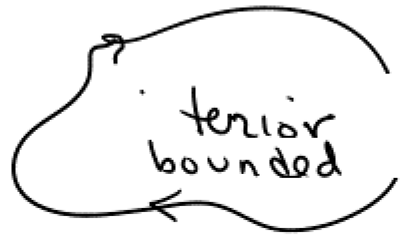
$\Rightarrow$ continuously turni

Simple closed Jordan curve – Jordan simple arc

$z(t_1) = z(t_2)$ iff $t_1 = 0$

$t_1 = 1$,

Jordan Curve Thm — Every simple closed curve in $\mathbb{C}$ divides $\mathbb{C}$ into 2 regi
open sets)



exterior (unbounde

Simple piecewise smooth curve

$t_0 = \quad t_1 < t_2 < \cdots < t_n \cdot 1 \cdot t_{n+1}$

$\dot{x}, \dot{y}$ are piecewise smooth, $\dot{x}^2 +$

i.e. $\dot{x}, \dot{y}$ are smooth for $t_j <$

j

Needed for integration

for example — the length of a curve

$$L = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

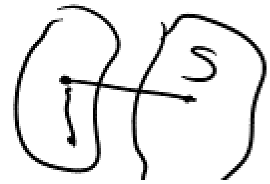
$$L = \int ds = \int \sqrt{dx^2 + dy^2}$$

$$= \int \sqrt{\frac{dx}{dt} \qquad dt)}$$

Regions - S

Connected - if every pair of points can be joined by a simple Jordan arc lying in S

Connected



not connected

Domain - nonempty, open connected set

Region - may include part or all of the bou

D Simply connected - if closed Jordan c
their interior in D

Functions:

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Let $z \in S$

$z \longmapsto w = f(z)$ $\Rightarrow$ pairs

$z \in S \subset z$-plane

$w = f(z) \subset w$-plane

S is mapped to the w-plane

z is mapped to w

f is a mapping

Set of $f(z)$ = Range = R.

Let $f: S \to R$ - onto

↳ i.e., for every
$z \in S \ \exists$

One to One (1-1) [injection]

If $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$

1–1 and Onto is a bijection ⟹ ∃ an inverse func

$$z = g(w) \ni \quad g[f(z)] = z$$

$$g : R \text{ onto } S$$

<u>Ex</u> $w = z^2$ or $f(z) = z^2$, $\{z \mid |z| \le 1,\ 0 \le \arg z < \pi\}$

Inverse $z = \sqrt{w}$ $\quad w = re^{i\theta}$

$g(w) =$ $\quad \sqrt{w} = \pm\sqrt{r}\, e^{i\theta/2}$

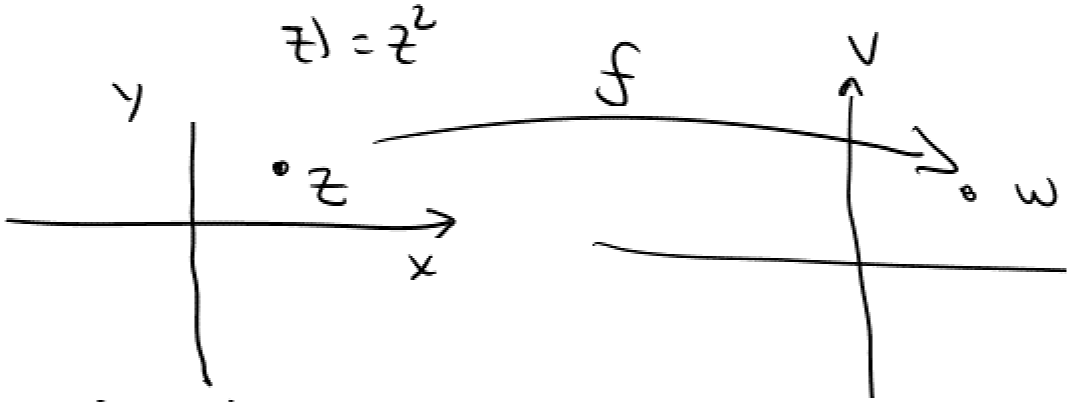
$f(z) = z^2$

$$g[f(z)] = \sqrt{z^2} \stackrel{?}{=} z$$

<u>Ex</u> $f(z) = \bar{z}$

<u>Ex</u> $f(z) = \dfrac{az+b}{cz+d}$ linear fractional transformat

<u>Ex</u> w $\quad z) = z^2$



z-plane $\quad$ w-plane

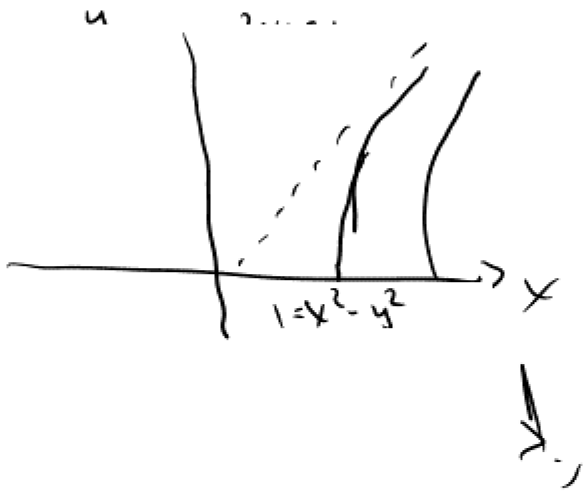
$z = x + iy$ $\quad w = u + iv$

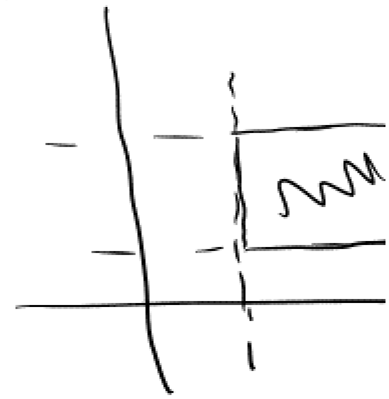
$f(z) = u + iv$

$w = f(z) = u(x,y) + iv(x,y)$

$w = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$

$u = x^2 - y^2, \quad v = 2xy$





Limits and Continuity

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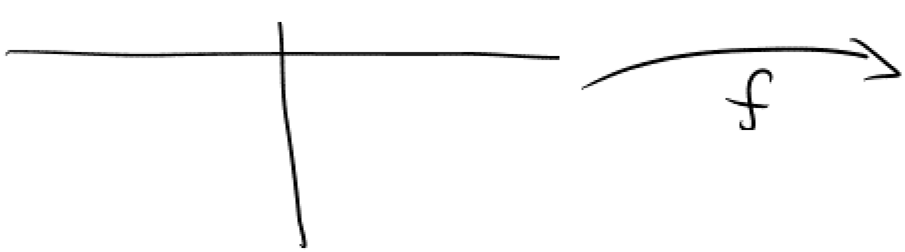
Goal - differentiability - we need limit

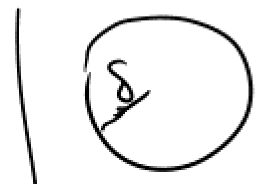
Let $w = f(z)$, $z \in S$ and let z_0 be a limi

If $\exists\, L \ni. \ \forall \epsilon > 0 \ \exists\, \delta \ni. \ |f(z) - L| <$

satisfying $0 < |z - z_0| < \delta$, Then

$\lim_{z \to z_0} f(z) = L$.





z - plane w - pl

① f not necessarily defined at z_0
② not necessarily defined in $0<|z-z_0|<$
③ May extend our definition to include
i.e., $\lim_{z\to\infty}$ (or $L=\infty$ i.e.
But need to use chordal metric: ρ(

Ex $f(z)=z^2$ in $S=\{z \mid |z|\le 1\}$

$\lim_{z\to 1} f(z)=1$

Pf/ Pick $\epsilon>0$
Let $|z^2-1|<\epsilon$ when $0<|z-1|<\delta$ and

$|z-1||\_\_|<\epsilon$ Note $|z+1|$

Since $|z+1|<$

$|z^2-1| = |z-1||z+1| < 2|z-1|$

$< 2\delta$

Choose $\delta=\epsilon/2 \Rightarrow |z^2-1|<\epsilon$

for $x\in\mathbb{R}$, $\lim_{x\to x_0} f(x)=f(x_0)$

Let $w=f(z)$ with domain S. f is c

if $f(z_0) \neq \infty$ and $\forall \epsilon > 0 \ \exists \delta \ldots \ |f| \epsilon| -$
$\forall z \in S$, with $|z - z_0| < \delta$

<u>f is continuous in S</u> if it is continuous $\forall z \in S$

<u>Ex</u> $f(z) = z^2$ continuous everywhere in fini

<u>Ex</u> $f(z) = \frac{1}{z}$ continuous in extended $\mathbb{C}$
text - proves for $z_0 = \infty$

<u>Thm</u> $w = f(z)$ is continuous in closed S
then $f(z)$ is bounded.
[Bounded - $\exists M > 0 \ni |f(z)$

Note - in general $\delta = \delta(\epsilon, z_0)$
If δ doesn't depend on z_0, then f is u

<u>Thm</u> - f is continuous in closed S, then f is

$\ast$

Derivatives
Monday, January 29, 2007
5:56 PM

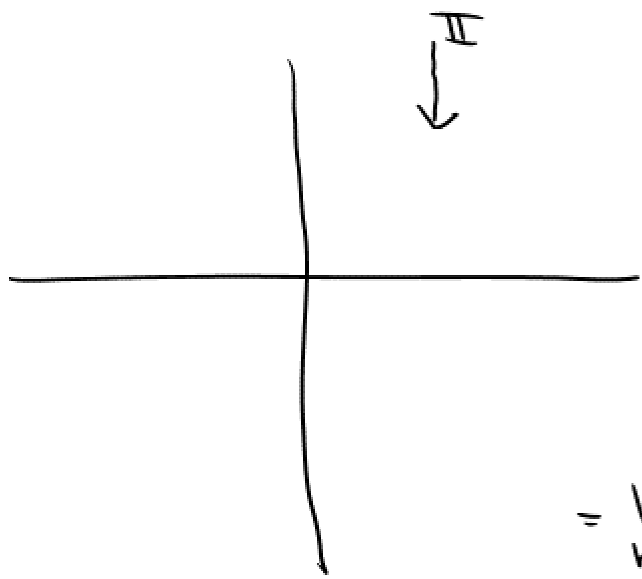
Let $w = f(z)$ in $N_\epsilon(z_0)$ with $f(z_0) \neq \infty$.

Then
$$f(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
is the derivative of $f(z)$ at z_0 if the limit

exists and is not ∞. Or, we say f is differen[tiable]
at z

Book - does examples - z^2 | $\frac{(z_0+h)^2 - z_0^2}{h} =$
and $\bar{z}$ $|z|^2$ ...
$\longrightarrow 2z_0$
Usual formulae - ✓

$f(z) = u(x,y) + iv(x,$
should get same $f'($
for what path i



$y = y_0 = \text{const}$
$x = x_0 + h$ or $h = x$

$\frac{f(z_0+h) - f(z_0)}{h}$

$= \lim\limits_{h \to 0} \frac{\quad}{h}$

$= \lim\limits_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i$ ——

$$f'(z_0) = \frac{\partial u}{\partial x}\Big|_{(x_0,y_0)} + i \frac{\partial v}{\partial x}\Big|_{(x_0,y_0)}$$

II. Let $x = x_0 = \text{const}$
$y = y_0 + h$
$(z = x_0 + i(y_0 + h)$
$= z_0 + ih)$

$$f'(z_0) = \frac{1}{i}\Big(\qquad \partial y \Big)\Big|_{(x_0,0)}$$

So we have:

and $f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$ $\Rightarrow$ Need $\left[\frac{\partial u}{\partial x} = \frac{\partial}{\partial} \quad \frac{\partial v}{\partial x} = - \right.$

Differentiable
olomorphic
Analytic

$\Rightarrow$ -ire - diff for a

Ex $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u(x,y) = x^2 - y^2$
$v(x,y) = 2xy$

$u_x = 2x \quad u_y = -$
$v_x = 2y \quad v_y = 2$

So $u_x = v_y$ & $v_x = -$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2yi = 2(x+iy)$

Thm Sufficient Conditions –
Let $f(z) = u + iv$ in $N_r(z_0)$ and $(x$
u_x, u_y, v_x, v_y continuous in $N_\epsilon(z_0)$
and u, v satisfy the CR equations
Then f is differentiable at z_0.

Ex $f(z) = \bar{z} = x - iy$

$u(x,y) = x$
$v(x,y) = -y$

$u_x = 1$
$v_y = -1$ so $u_x \neq$

Ex $f(z) = \underbrace{\ln\sqrt{x^2+y^2}}_{u(x,y)} + i\,\underbrace{\tan^{-1}(y|x)}_{v(x,y)}$, $-\frac{\pi}{2} < \tan$ or $\mathrm{Re}(z$

Test CR

$$\frac{\partial u}{\partial x} = \frac{2x}{2(\sqrt{x^2+y^2})^2} \overset{?}{=} \frac{\partial v}{\partial y} = \frac{1}{1+(\ \ x)^2}$$

$$= \frac{x}{x^2+y^2} \qquad\qquad = \frac{x}{x^2+y^2}$$

$$y \quad = \quad x^2+y^2 \qquad\qquad \frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2}$$

$$f'(z) = u_x + i v_x$$

$$= \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} =$$

$$= \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$$

Note: $f(z) = \ln z = \ln\sqrt{x^2+y^2} + i\tan$

if $z = re^{i\theta}$, then $\ln z = \ln r + i\theta$

(at some point we will see $\ln z = \ln r$

$u(x,y)$, realvalued

u is harmonic iff $\nabla^2 u = 0$

or $u_{xx} + u_{yy} = 0$

∇ - Laplacian, $\nabla = \partial$

$2 \quad \partial^2 \quad \partial^2$

Given $\nabla^2 u = 0$, let v be the harmonic

∂ the

Let $f = u + iv$ be diff., with u_x, u_y, v_x, v_y cont.

CR conditions hold and

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) =$$

$\Rightarrow \nabla^2 u = 0$ ✓ [Similarly, $\nabla^2 v =$

Ex Consider $u(x,y) = e^x \cos y$. a) Show $\nabla^2 u$

b) Find harm

function

a) $u_x = e^x \cos y$

$$\left.\begin{array}{l} u_{xx} = e^x \cos y \\ u_y = -e^x \sin y \\ u_{yy} = -e^x \cos y \end{array}\right\} u_{xx} + u_{yy} = 0 \checkmark$$

b) Find $v(x,y) \ni f(z) = u + iv$ is diff.

i) $u_x = v_y \rightarrow$

$$v_y = e^x \cos y$$

$$\Rightarrow v = \int e^x \cos y \, dy = \boxed{}$$

ii) $u_y = -v_x$

$\leftarrow$

Need $C'(x) = 0$ or $C = $ con

$V(x,y) = e^x \sin y + C$

c) $f(z) = u + iv$

$= e^x \cos y + i e^x \sin y$

$= e^x(\cos y + i \sin y) = e^x e^{iy} = e^z$

d) $f'(z) = u_x + i v_x$

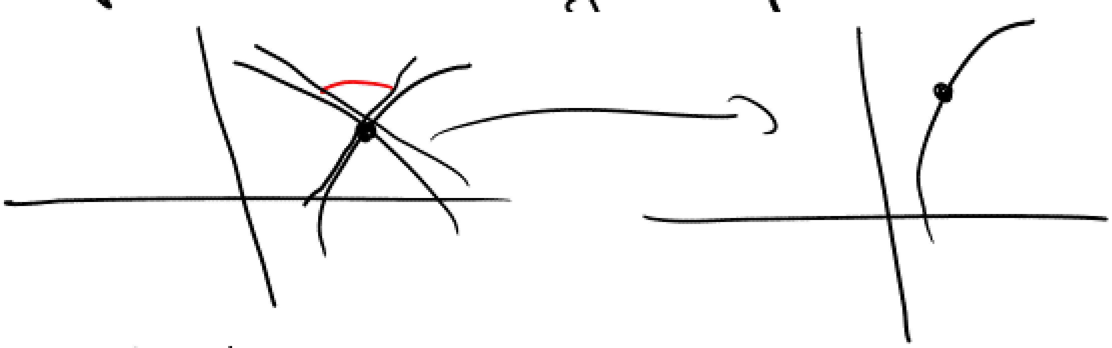
$= e^x(\cos y + i \sin y) = e^z$

Linear Fractional Transformations

Thursday, February 01, 2007
6:02 PM

LFT's are special maps: $f(z) = \frac{az+b}{cz+d}$

general properties of analytic maps:



z-plane w-plane

Conformal maps - preserve angles.

Let $w = f(z)$, $z = z_0 + \Delta z$

Let $w = f(z)$ $z = z_0 + \Delta z$

$$\Delta w = f'(z_0)\Delta z + \eta(z_0, \Delta z)\Delta z$$

where $\lim_{\Delta z \to 0} \eta(z_0, \Delta z) = 0$

Note $f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(}{}$

$$= \lim_{\Delta z \to 0} \frac{f'(z_0)\Delta z + \eta \Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(f'(z_0) + \eta(z_0, \Delta \right.$$

$$\underbrace{\arg(\Delta w)}_{\phi} = \arg\left(\Delta z\left(f'(z_0) + \eta(z_0, \Delta z)\right)\right)$$

$$= \arg(\underbrace{\Delta z}_{\theta}) + \arg\left(f'(z_0) + \eta(\right.$$

Let $\Delta z \to 0$.

$$\phi = \theta + \arg(f'(z_0))$$