Homework – pg 26

HW #2 – Length of Paths in $\mathbb{C}$

2a) $z = a\cos 2\pi t + ib\sin 2\pi t$

$x = a\cos 2\pi t,\ y = b\sin 2\pi t$

$\frac{x}{a} = \cos 2\pi t,\ \frac{y}{b} = \sin 2\pi t \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$L = \int_0^1 \sqrt{a^2\sin^2 2\pi t + b^2\cos^2 2\pi t}\, 2\pi\, dt, \quad \text{Let } \theta = 2\pi t$$

$= b\,E(2\pi, e)$, $b > a$ – Elliptic function

$$\longrightarrow \int_0^\phi \sqrt{1 - k^2\sin^2\theta}\, d\theta \equiv E(\phi, k)$$

Elliptic Function (Incomplete)

$$e = \frac{\sqrt{b^2 - a^2}}{b}$$

(2b) $|z| = 1 - \cos(\arg z)$ $\quad 0 \le \arg z \le 2\pi$

Polar form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$|z| = r,\ \arg z = \theta \Rightarrow r = 1 - \cos\theta$ cardiod

$x = r\cos\theta = (1 - \cos\theta)\cos\theta$
$y = r\sin\theta = (1 - \cos\theta)\sin\theta$

Note: you will need $\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$

(2c) $z = te^{it}$

$x = t\cos t,\ y = t\sin t$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{1 + t^2}\, dt, \quad t = \tan\theta$$

(2d) $x = t - \sin t$
$y = 1 - \cos t$ $\quad 0 \le t \le 2\pi$

Transcendental Functions

Thursday, February 08, 2007
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Algebraic Functions — $az+b$, polynomial, rational, z^2

Transcendental Functions — $\sin z$, $\cosh z$, e^z, $\ln z$, ...

① e^z:
$$\begin{cases} f(z) \text{ - analytic} \\ f'(z) = f(z) \\ f(x) = e^x \end{cases} \qquad f = u + iv$$

$u_x + iv_x = u + iv$, etc.

$\Rightarrow u_x = u$ or $u = e^x g(y)$, $v = e^x h(y)$

But $u_x = v_y$

$v_x = -u_y$

$$\Rightarrow \begin{cases} e^x g = e^x h' \\ e^x h = -e^x g' \end{cases} \quad \text{or} \quad \begin{cases} g = h' \\ h = -g' = -h'' \end{cases}$$

or $h'' + h = 0$

$h(y) = a\cos y + b\sin y$

$g(y) = -a\sin y + b\cos y$

$f(x) = e^x \Rightarrow g(0) = 1,\ h(0) = 0 \Rightarrow a = 0,\ b = 1$

$f(z) = e^x(\cos y + i\sin y) = e^{x+iy} = e^z$

Properties — $|e^z| = |e^x e^{iy}| = |e^x| = e^x$

$\overline{e^z} = e^{\bar z}$

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$\sin z = \sin(x+iy)$

$= \sin x\cos(iy) + \sin(iy)\cos x$

$= \sin x\cosh y + i\sinh y\cos x$

since $\sin(iy) = \dfrac{e^{-y}-e^{y}}{2i} = \dfrac{-1}{i}\sinh y$

$\cosh z = \dfrac{e^z+e^{-z}}{2}$

$\sinh z = \dfrac{e^z-e^{-z}}{2}$

<u>Logarithm</u> $\quad w = \ln z \iff z = e^w$

$re^{i\theta} = e^w$

$2\pi ki$ $\ln r + i\theta$ ...

$$re^{i\theta} = e^{w}$$

$$e^{2\pi K i} e^{\ln r + i\theta} = e^{w} \qquad K\text{-integer}$$

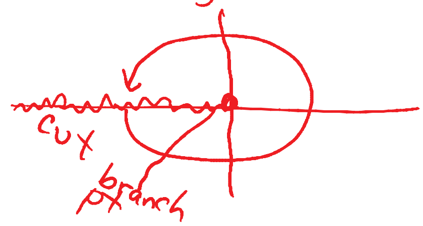
$$w = \ln r + i(\theta + 2\pi K)$$

$$= \ln\sqrt{x^2+y^2} + i\left(\tan^{-1}(y/x) + 2\pi K\right)$$

multivalued

Each $K \Rightarrow$ different branch

$-\pi \leq \theta < \pi$, $K=0$ Principal Branch



$$\operatorname{Log} z = \ln\sqrt{x^2+y^2} + i\tan^{-1} y/x$$

Inverse functions $\cos^{-1} z$, $\sinh^{-1} z$, etc.

$$w = \sin^{-1} z \iff z = \sin w$$

$$= \frac{e^{iw} - e^{-iw}}{2i}$$

$$2iz = e^{iw} - \frac{1}{e^{iw}}$$

$$2ize^{iw} = (e^{iw})^2 - 1$$

or $(e^{iw})^2 - 2ize^{iw} - 1 = 0$

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2+4}}{2} = iz \pm \sqrt{1-z^2}$$

$$iw = \ln\left(iz \pm \sqrt{1-z^2}\right)$$

Integration in the Complex Plane

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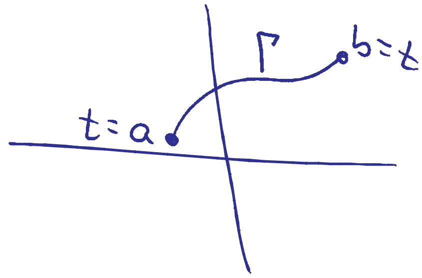
$$\int_\Gamma f(z)\,dz \qquad \text{path integral}$$

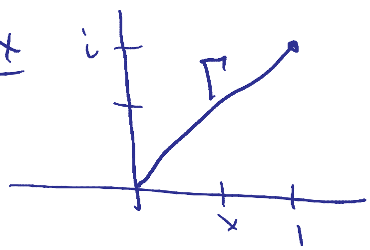


$$\Gamma = \{(x(t),y(t)) \mid a \le t \le b\}$$

$$z = x(t) + iy(t)$$

$$\int_\Gamma f(z)\,dz = \int_a^b f(x(t)+iy(t))[x'+iy']\,dt$$

Ex



$$I = \int_\Gamma z^2\,dz$$

$$z = x+ix, \quad x \in [0,1]$$

$$dz = (1+i)\,dx$$

$$I = \int_0^1 (1+i)^2 x^2 (1+i)\,dx$$

$$= (1+i)^3\left(\frac{1}{3}\right) = \frac{(i-1)}{3}2$$

$$(1+i)^3 = 1+3i+3i^2+i^3 = -2+2i$$

Dettman - Ch 3

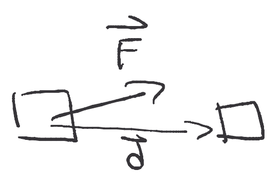
Starts with path integrals $\int_C P\,dx + Q\,dy$

where P, Q are real valued functions of two variables

C - path in $\mathbb{R}^2$

Application - Work $\quad W = F \cdot \text{dist}$

$$W = \vec{F}\cdot\vec{d}$$



$$\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$

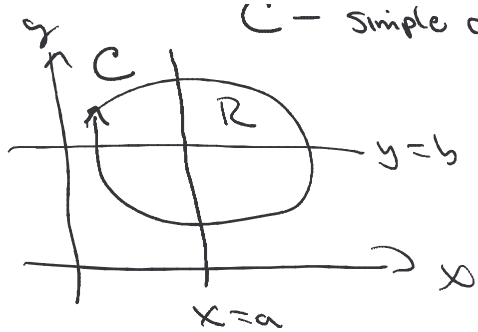
$$d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$$

$$W = \int_C \vec{F}\cdot d\vec{r}$$

Green's Lemma

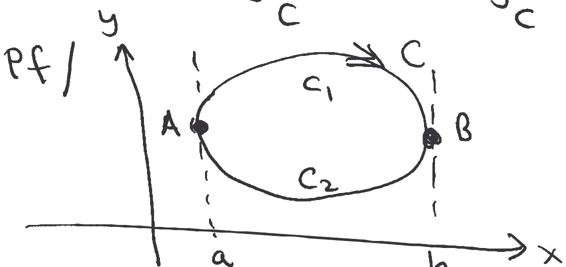
C - simple closed contour and $x=a$, $y=b$ intersect C in at most 2

C – simple closed contour and $x=a$, $y=b$ intersect C in at most 2 points. $P(x,y)$, $Q(x,y)$ are real valued, continuous and have continuous first partial derivatives in $\underbrace{C \cup \text{interior of } C}_{R}$



Then $\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \iint_R (Q_x - P_y)\,dx\,dy$

Pf/



Consider

$$-\iint_R \frac{\partial P}{\partial y}\,dy\,dx \equiv I$$

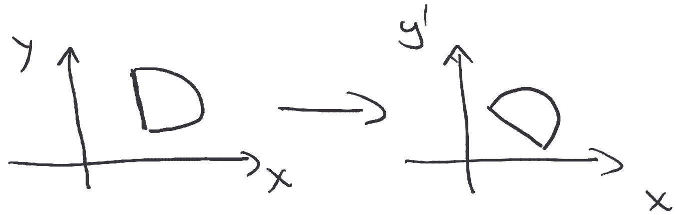
Then $I = -\int_a^b \left(\int_{g(x)}^{h(x)} \frac{\partial P}{\partial y} dy \right) dx = -\int_a^b [P(x,h) - P(x,g)]\,dx$

$= \int_C P(x,y)\,dx$ being careful of path direction.

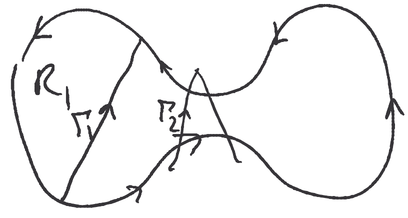
Similarly $\iint_R \frac{\partial Q}{\partial x}\,dy\,dx = \int_C Q\,dy$

Variations

① Rotate axes



② More Complex Regions



$\int_C P\,dx + Q\,dy \overset{?}{=} \iint_R (Q_x - P_y)\,dx\,dy$

But $\int_{C_1} P\,dx + Q\,dy = \iint_{R_1} (Q_x - P_y)\,dx\,dy$

$$\Rightarrow \sum_{i=1}^{n}\int_{C_i} P\,dx + Q\,dy = \sum_{i=1}^{n}\iint_{R_i}(Q_x - P_y)\,dx\,dy = \iint_R (\quad)\,dx\,dy$$

Schematically,

$$\int_{C_1} = \int_{C_1'} + \int_{\Gamma_1}$$

$$\int_{C_2} = \int_{C_2'} + \int_{\Gamma_2} - \int_{\Gamma_1}$$

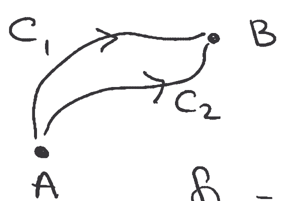
$$\vdots$$

$$\int_{C_n} = \int_{C^{n'}} - \int_{\Gamma_{n-1}}$$

Note: the path integral from a to b is negated as the direction is reversed.

$$\text{Sum} \quad \sum_{i=1}^{n}\int_{C_i} = \int_C$$

<u>Path Independence</u> $\int_C P\,dx + Q\,dy$ is the same for any path between two given pts.



Path Indep $\Rightarrow \oint_C P\,dx + Q\,dy = 0$ $\forall$ simple closed C

$$\oint_C = \oint_{C_1} - \oint_{C_2} = 0$$

<u>Contours in z-plane</u>

Let $f(z) = u + iv$ in a simply connected domain D
u, v have continuous 1st derivatives and
$u_x = v_y$, $u_y = -v_x$.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= -u_y\,dx + u_x\,dy \Rightarrow v = \int dv = \int -u_y\,dx + u_x\,dy$$

Note if $\nabla^2 u = 0$ then one can get v from u for v the harmonic conjugate

<u>3.2</u> $\int_a^b f(z)\,dz$ If $f(z)$ continuous on a smooth arc C,

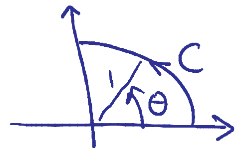
3.2 $\int_a f(z)\,dz$ If $f(z)$ continuous on a smooth arc C, The $\int_C f\,dz$ exists.

Ex $\int_a^b f(z)\,dz = \int_{t_a}^{t_b} (u+iv)(\dot{x}+i\dot{y})\,dt$ $\quad z(t) = x(t)+y(t)i$

$dz = (\dot{x}+i\dot{y})\,dt$

$$= \int_{t_a}^{t_b} \left[(u\dot{x}-v\dot{y})+i(v\dot{x}+u\dot{y})\right]dt$$

Ex $\int_C \bar{z}\,dz$



arc of unit circle

Method I $\quad x = \cos\theta$

$y = \sin\theta$

$\bar{z} = \cos\theta - i\sin\theta$, $z = \cos\theta + i\sin\theta$

$dz = (-\sin\theta + i\cos\theta)\,d\theta$ — finish ...

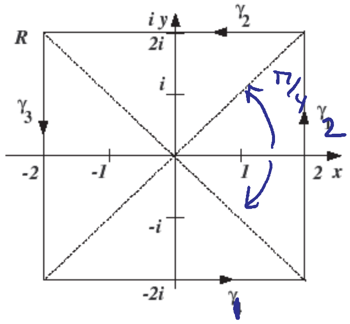
Method II $\quad$ On unit circle, $z = e^{i\theta}$

$\Rightarrow \bar{z} = e^{-i\theta}$, $dz = ie^{i\theta}d\theta$

$$\int_C \bar{z}\,dz = \int_0^{\pi/2} (e^{-i\theta})\, i e^{i\theta}\,d\theta = i\int_0^{\pi/2} d\theta$$

So far – $\int_C f(z)\,dz$ for $z=z(t)$ along C

dz/z



γ_1: $z = 2+iy$ $\quad y \in [-2, 2]$

$dz = i\,dy$

$$\int_{\gamma_1} \frac{dz}{z} = \int_{-2}^{2} \frac{i\,dy}{2+iy} = \ln(2+iy)\Big|_{-2}^{2}$$

$$= \ln(2+2i) - \ln(2-2i)$$

$$= \ln(2\sqrt{2}\,e^{i\pi/4}) - \ln(2\sqrt{2}\,e^{-i\pi/4})$$

$$= i\pi/2 \quad \checkmark$$

$$\begin{aligned}
\oint_{\gamma 1} \frac{dz}{z} &= \int_{-2}^{2} \frac{dx}{x-2i} \\
&= \ln|x-2i|_{-2}^{2} \\
&= (\ln(2\sqrt{2}) + \frac{7\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{5\pi i}{4}) \\
&= \frac{\pi i}{2}.
\end{aligned}$$

Similarly, the integral along the top segment is computed as

$$\begin{aligned}
\oint_{\gamma 3} \frac{dz}{z} &= \int_{2}^{-2} \frac{dx}{x+2i} \\
&= \ln|x+2i|_{2}^{-2} \\
&= (\ln(2\sqrt{2}) + \frac{3\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{\pi i}{4})
\end{aligned}$$

$$= \frac{\pi i}{2}.$$

The integral over the right side is

$$\begin{aligned}\oint_{\gamma_2} \frac{dz}{z} &= \int_{-2}^{2} \frac{idy}{2+iy} \\ &= \ln|2+iy|_{-2}^{2} \\ &= (\ln(2\sqrt{2}) + \frac{\pi i}{4}) - (\ln(2\sqrt{2}) - \frac{\pi i}{4}) \\ &= \frac{\pi i}{2}.\end{aligned}$$

Finally, the integral over the left side is

$$\begin{aligned}\oint_{\gamma_4} \frac{dz}{z} &= \int_{2}^{-2} \frac{idy}{-2+iy} \\ &= \ln|-2+iy|_{-2}^{2} \\ &= (\ln(2\sqrt{2}) + \frac{5\pi i}{4}) - (\ln(2\sqrt{2}) + \frac{3\pi i}{4}) \\ &= \frac{\pi i}{2}.\end{aligned}$$

Total $\int_C \frac{dz}{z} = 2\pi i$

There is an easier way! – Cauchy's Thm

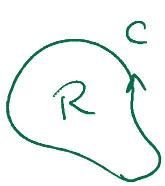
Section 3.3

Thm – Let $f(z)$ be analytic in a simply connected domain D. Then $\oint_C f(z)dz = 0$ for C a simple closed contour.

Version I Let $f(z) = u + iv$ and $z = x+iy$. Assume u, v satisfy the CR equations. Then $\oint_C f(z)dz = 0$

Pf/ $\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$

$$= \oint_C u\,dx - v\,dy + i\oint_C u\,dy + v\,dx$$



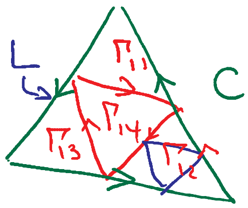
Consider $\oint_C u\,dx - v\,dy = \iint_R \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{0 \text{ by CR}} dxdy = 0$

Similarly $\oint_C v\,dx + u\,dy = 0$. ✓

Version II Our text – 3 Steps

1. Triangles
2. Closed Polygons
3. General Contours.

Step ① Lemma –



Assume $\int_C f(z)\,dz = I \neq 0$

Connect midpoints on sides

$$\int_C = \int_{\Gamma_{11}} + \int_{\Gamma_{12}} + \int_{\Gamma_{13}} + \int_{\Gamma_{14}}$$

$\int_C \neq 0 \Rightarrow \int_{\Gamma_{1j}} \neq 0$ for some j

Let $J_1 = \max_j \left| \int_{\Gamma_{1j}} f\,dz \right| \Rightarrow \left| \int_C f\,dz \right| \leq 4 J_1$

Repeat process for triangle $\Gamma_{1j} \equiv \Gamma_1$

$$\Rightarrow \left| \int_C f\,dz \right| \leq 4^n \left| \int_{\Gamma_n} f\,dz \right|$$

Interiors $C^*, \Gamma_1^*, \Gamma_2^*, \ldots$

& $R_0 = C^* \cup C$, $R_1 = \Gamma_1^* \cup \Gamma_1$

So, $R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots$ Closed, nested sequence of sets.

Intersection = point $z_0 \in D$ (Thm 1.4.2)

$d(R) = \text{diam}(R)$

$d(R_0) \leq L/2$

$d(R_1) \leq L/4$

etc.

$$d(R_n) = \frac{L}{2^{n+1}}$$

Consider $f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta(z,z_0)(z-z_0)$

given $\epsilon > 0 \;\exists\; \delta(\epsilon) \ni.$ $|\eta| < \epsilon$ whenever $|z-z_0| < \delta$

$$\left| \int_{\Gamma_n} f(z)\,dz \right| = \left| \int_{\Gamma_n} [f(z_0) + f'(z_0)(z-z_0) + \eta(z,z_0)(z-z_0)]\,dz \right|$$

$$= \left| \int_{\Gamma_n} \eta(z,z_0)(z-z_0)\,dz \right|$$

$$\leq \int_{\Gamma_n} |\eta(z,z_0)(z-z_0)|\,|dz|$$

$d(R_n) < \delta$

$$= \int_{\Gamma_n} |\eta|\,\underline{|z-z_0|}\,|dz| \leq \epsilon \frac{L}{2^{n+1}} \frac{L}{2^n} = \epsilon \frac{L^2}{2 \cdot 4^n}$$

Need 3.2.2 $\int_a^b dz = b-a$

3.2.3 $\int_\alpha^\beta z\,dz = \frac{1}{2}(\beta^2 - \alpha^2)$

$\Big\}$ $\int_{\Gamma_n} dz = 0$, $\int_{\Gamma_n} z\,dz = 0$

$$|I| = \left| \oint_C f(z)\,dz \right| \leq 4^n \left| \int_{\Gamma_n} f(z)\,dz \right| \leq \cancel{4^n} \left(\frac{\epsilon L^2}{2 \cdot \cancel{4^n}} \right) = \frac{\epsilon L^2}{2}$$

$|I| \le \frac{L^2}{2}\epsilon$ for any $\epsilon > 0$. $\Rightarrow \underline{\underline{|I| = 0}}$

Implications

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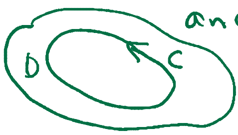
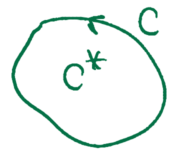
3.3.1 $f(z)$ is analytic in a simply connected domain D and $C \subset D$, simple closed contour, Then



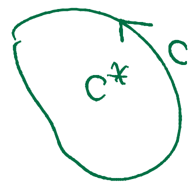
$$\int_C f(z)\,dz = 0 \qquad \text{(pg 86)}$$

3.4.2 $f(z)$ is analytic within C and continuous in $C \cup C^*$, Then $\int_C f(z)\,dz = 0$

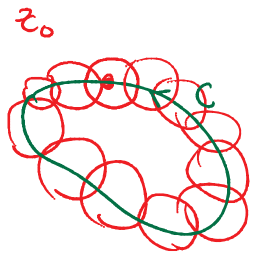


3.4.1 $f(z)$ analytic inside and on $C \Rightarrow$

$$\int_C f(z)\,dz = 0$$



Pf/



① f anal at z_0 $\Rightarrow$ f anal in Disk about z_0

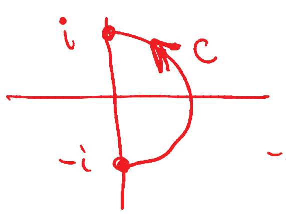
② ∞ family of disks covering C

③ Heine-Borel – we can find a finite sub covering

$$D = C^* \cup \left[\bigcup_\alpha D_\alpha\right]$$

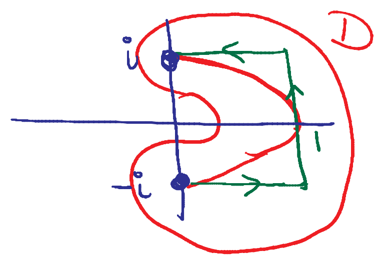
3.4.3 $f(z)$ is analytic in a simply connected domain Then $\int_a^b f(z)\,dz$ is path (contour) independent for contours from a to b that lie in D.

Ex 3.4.1 pg 93 $\qquad \int_{-i}^{i} \frac{dz}{z}$



$z = e^{i\theta}$

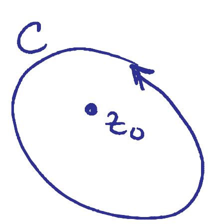
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



3.4.4 C – simple closed contour

$$\int_C \frac{dz}{z - z_0} = \pm 2\pi i$$

depending on the direction



Uses – HW Ex 3.2.6 (pg 86)

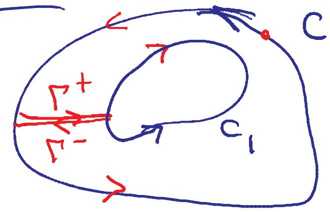
Uses – HW Ex 3.2.6 (pg 86)

Pf $I=\int_C \frac{dz}{z-z_0}=2\pi i$ for $C: |z-z_0|=\rho$

Let $z-z_0=\rho e^{i\theta}$ & $dz = i\rho e^{i\theta}\,d\theta$

$$I=\int_0^{2\pi}\frac{i\rho e^{i\theta}\,d\theta}{\rho e^{i\theta}} = i\theta\Big|_0^{2\pi} = 2\pi i$$

3.4.5 Let $f(z)$ be analytic on C_1, C_2 and between C_1 & C_2.

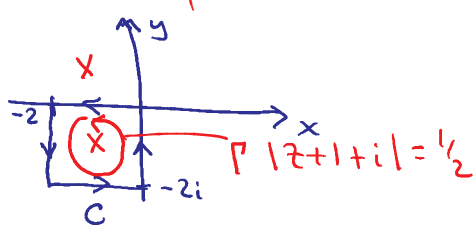


Then $\int_{C_{2+}} f(z)\,dz = \int_{C_{1+}} f(z)\,dz$

$$\int_{C_2}+\int_{C_1^-}+\overbrace{\cancel{\int_{\Gamma^+}+\int_{\Gamma^-}}}^{0}=0$$

$$\int_{C_2}-\int_{C_1}=0 \Rightarrow \int_{C_2}=\int_{C_1}$$

Ex 3.4.3 $\int_C \frac{dz}{z^2+2z+2}$



$0=z^2+2z+2=(z-z_+)(z-z_-)$

$=z^2+2z+1+1$

$0=(z+1)^2+1$

$z+1=\pm i$

$z=-1\pm i$

So, $z^2+2z+2=(z+1-i)(z+1+i)$

Note $\frac{1}{(z+1-i)(z+1+i)} = \left[\frac{1}{z+1-i}+\frac{-1}{z+1+i}\right]\frac{1}{2i}$

$= \frac{z+1+i-(z+1-i)}{(z+1-i)(z+1+i)}\frac{1}{2i}$

$$\int_C \frac{dz}{z^2+2z+2} = \frac{1}{2i}\underbrace{\int_C \frac{dz}{z+1-i}}_{z-(-1+i)} - \frac{1}{2i}\int_C \frac{dz}{\underbrace{z+1+i}_{z-(-1-i)}}$$

$z-(-1+i)$ → vanishes by Cauchy's Thm

$$\int \frac{dz}{z+1+i} = \int \frac{dz}{z+1+i} = \int \frac{dz}{z-(-1-i)} = 2\pi i$$

$$\int_C \frac{dz}{z+1+i} = \int_\Gamma \frac{dz}{z+1+i} = {}^{0}\int_\Gamma \frac{dz}{z-(-1-i)} = 2\pi i$$

So $$\int_C \frac{dz}{z^2+2z+2} = -\frac{1}{2i}(2\pi i) = \boxed{-\pi}$$

<u>Ex</u> <u>3.4.4</u> $$\int_{\substack{C_+ \\ |z|=2}} \frac{dz}{z^2-1}$$