As we wind down the course, there are still systems to explore, especially numerically. Some MATLAB code is provided on the second page.

Do 3 of the following:

1. Consider the Hamiltonian $H(x,y) = \frac{1}{2}(x^2+y^2) - xy^3$.
   (a) Write the corresponding first order system. [Recall that $\dot{x} = \frac{\partial H}{\partial y}$ and $\dot{y} = -\frac{\partial H}{\partial x}$.]
   (b) Find the equilibrium solutions and classify them.
   (c) Confirm your results using MATLAB to plot a direction field and a selection of orbits.
   (d) Show that $\frac{dH}{dt} = 0$. Plot the the curves $H(x_0, y_0) =$ const. and describe their importance for this system.

2. Consider the van der Pol oscillator equation

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0.$$

   (a) Write the corresponding first order system.
   (b) Plot a direction field and a selection of orbits for different values of μ.
   (c) Sketch a bifurcation diagram for this system.

3. Describe the features of the polar system $\dot{r} = r(1-r)(2-r)(3-r)$, $\dot{\theta} = -1$.

4. The Lorenz system is given by

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= x(\rho-z)-y\\ \dot{z} &= xy-\beta z.\end{aligned}$$

   Typically, one selects $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$. Numerically explore the behavior of this system for different values of $\rho : \rho = 10$, $\rho = 22$, $\rho = 24.5$, $\rho = 28$, $\rho = 100$. Describe the types of behavior which result.

Here are some MATLAB code snippets. First, a quiverplot can be used to plot planar direction fields. A grid is setup to locate the arrows. The system is introduced as a vector field.

```
% Using quiver to plot planar system

[x,y] = meshgrid(-0.5:0.1:0.5,-0.5:0.1:0.5);
dx = y;
dy = x-3*a*x.^2;
figure(1)
% Normalize arrow size with variable r
r = ( dx.^2 + dy.^2 ).^0.5;
h=quiver(x,y,dx./r,dy./r);
set(h,'AutoScale','on', 'AutoScaleFactor',0.5)
axis([-.5 .5 -.5 .5])
```

Now, orbits can be added using an ODE solver. This even shows how separatrices are added to the plot in this example.

```
% Enter the system as an anonymous function
a=1.0;
fhomo = @(t,y) [y(2); y(1)-3*a*(y(1)).^2];

hold on

% Plot a single orbit where [0:0.1:20] gives time interval from
% t=0 to t=20 in steps of 0.1 and [.1,0] is the initial condition
[t,x]=ode45(fhomo,[0:0.1:20],[.1,0]);
plot(x(:,1),x(:,2),'k')

% Plot multiple solutions - Add a loop to plot several solutions compactly
for k=1:5
    [t,x]=ode45(fhomo,[0:0.1:20],[.1*k,0]);
    plot(x(:,1),x(:,2),'k')
    [t,x]=ode45(fhomo,[0:0.1:2.85],[-.1*k,.5]);
plot(x(:,1),x(:,2),'k')
end

% Plot the separatrices y = +/- x* sqrt(1-2ax) from H(x,y)=0.
x=-.5:.01:.5;
plot(x,x.*sqrt(1-2*a*x),'g')
plot(x,-x.*sqrt(1-2*a*x),'g')

hold off
axis square
```