Nonlinear Dynamics – Challenge 2 Hints

Note that the first 7 problems are fairly general. After that the problems are more algebraic.

1. Just explain what happens to points mapped by S. In particular, you should also show that $S\begin{pmatrix}1\\y\end{pmatrix}=S\begin{pmatrix}0\\y\end{pmatrix}$ and $S\begin{pmatrix}x\\1\end{pmatrix}=S\begin{pmatrix}x\\0\end{pmatrix}$.
2. This is difficult if you try to show everything algebraically.
   a. Set up $A=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ for integer entries. Compute $A\begin{pmatrix}x+n_1\\y+n_2\end{pmatrix}$ and apply the mod function by subtracting off all additive integers.
   b. Provide a geometric description of how points map under S^2 and A^2.
   c. Generalize your argument in part b.
3. Since $Sv=v$, then $Av=v+\begin{pmatrix}n_1\\n_2\end{pmatrix}$ for n_1 and n_2 integers. Writing this out, you have $\begin{pmatrix}av_1+bv_2\\cv_1+dv_2\end{pmatrix}=\begin{pmatrix}v_1+n_1\\v_2+n_2\end{pmatrix}$. Solve for v_1 and v_2. This is possible since you are to assume that $\det(A-I)\neq 0$.
4. A hint is given in the book. Since A^n is a matrix of integers, you can apply Problem 3 and Problem 2c.
5. A hint is given and the problem is done geometrically. The key is that a linear map will map a domain of the unit square (torus) into a new region whose area is $|\det A|$ times the area of the original domain.
6. You are told to use the result of problem 5 to $A-I$. Namely, We know a fixed point satisfies $Av=v.$ Rewriting this as $(A-I)v=0,$ we have the problem set up with the matrix $A-I$ with integer entries and $v_0=0.$ Thus, the number of solutions of $(A-I)v=0$ is $|\det(A-I)|$. The result follows simply from this.
7. Use Mathematical Induction on *n*. Prove that $A^1=A.$ Now assume it is true for *k-1*. $A^{k-1}=\begin{pmatrix}F_{2k-2} & F_{2k-3}\\F_{2k-3} & F_{2k-4}\end{pmatrix}$. Show that $A^{k-1}A=A^k$. Then by induction one can conclude that the form is true for all integers $n\geq 1.$ For example, the 1-1 entry of the product is $2F_{2k-2}+F_{2k-3}$. This can be simplified using the Fibonacci rule with $n=2k-1$: $F_{2k-1}=F_{2k-2}+F_{2k-3}$. Then $F_{2k-3}=F_{2k-1}-F_{2k-2}$. Inserting into the 1-1 expression and using the Fibonacci rule for $n=2k$, one gets the first part of the result.
8. This is a straight forward computation for fixed points and period two orbits as applied to this 2D cat map.
9. This is just a computation of a determinant.
10. Prove either by induction or manipulating the Fibonacci rule.