8.2 Induced Homomorphisms

Homotopy groups of homeomorphic spaces are isomorphic

group isomorphism - bijective group homomorphism

group homomorphism $h: G \to H$

such that $h(u \cdot v) = h(u) \cdot h(v)$, $\forall u, v \in G$.

So, identities $\mapsto$ identities, inverses $\mapsto$ inverses

Given pointed continuous map $f: X \to Y$

$\exists$ induced function $f_*: \pi_n(X) \to \pi_n(Y)$

Such that $f_*[j] = [f \circ j]$ for pointed $j: S^n \to X$.

Moreover, $[j] = [k] \Rightarrow f_*[j] = f_*[k]$.

Thm 8.12 For pointed, cont. $f: X \to Y$, f_* is a group homomorphism satisfying

1. $g: Y \to Z$, $(g \circ f)_* = g_* \circ f_*$
2. $i: X \to X$, identity $\Rightarrow$ i_* is an identity
3. $h: X \to Y$ homotopic to $f \Rightarrow h_* = f_*$
4. $c: X \to Y$ takes all pts to y_0, $c_* = 0$ (zero homomorphism)

Consequences

There is no continuous $f: D^2 \to S^1$ $\ni$. $f(x,y) = (x,y)$, $\forall (x,y) \in S^1$.

Inclusion map $i: S^1 \to D^2$ induces zero homomorphism on π_1.

$f: S^1 \to S^1$, $\ni$. $f \circ f = \text{const (base pt)} \Rightarrow f_*: \pi_1(S^1) \to \pi_1(S^1)$ is zero homomorphism.

S, T homotopically equivalent $\Rightarrow \pi_n(S), \pi_n(T)$ isomorphic

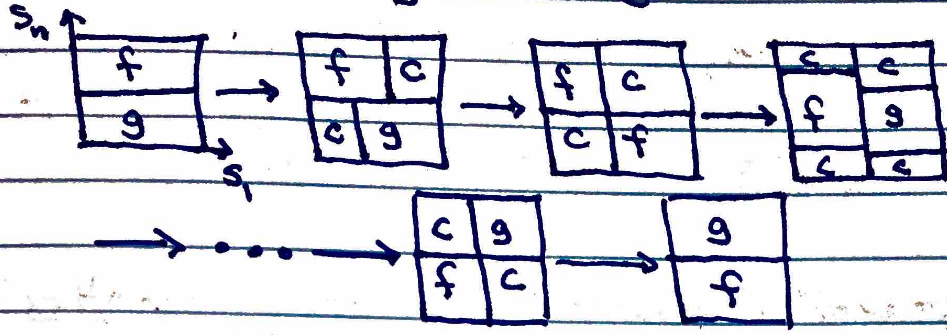
Ex $\mathbb{R}^2 - \{0\} \simeq S^1$

So $\pi_i(\mathbb{R}^2 - \{0\}) = 0$, $i \neq 1$, $\pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z}$

9.3 Fundamental group $\pi_1(x)$

Thm $n>1$, $\pi_n(X)$ is Abelian.

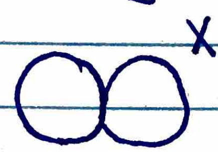
Pf/ By picture mixes s_1, s_n in homotopy function to show $g\#f \simeq f\#g$



However, $\pi_1(X)$ is not necessarily Abelian.

Ex Figure 8

$(|x|-1)^2+y^2=1$



Let $f: S^1 \to X$, $f(x,y)=(1-x,y)$ loop RH circle

$g: S^1 \to X$, $g(x,y)=(x-1,y)$ loop LH circle

Can show $f\#g \not\simeq g\#f$.

π_0? $\pi_0(X)=[S^0,X]$ may not be a group

$f: S^0 \to X$ where $f(1)=x_0 \in X$

so knowing $f(-1)$, know f.

$f,g: S^0 \to X$ are pointed homotopic if $\exists$ path from $f(-1)$ to $g(-1)$

Path Connected X is path connected if given $x,y \in X$ there exists cont. map $p:[0,1] \to X \ni$. $p(0)=x$, $p(1)=y$.

X is path connected iff $\pi_0(X)$ has only one element.

Path Connected $\Rightarrow$ Connected

If not path connected, form equivalence classes

$x \sim y$ if $\exists$ path from x to y

Set of equivalence classes $= [S_0, X]$

For $n > 0$, X a pointed top. space $\pi_n(X) = \pi_n(X_0)$

where X_0 is path component containing base pt.

<u>Homotopy Groups of $\mathbb{Q}$</u> $\pi_0(\mathbb{Q}) = \mathbb{Q}$. $\pi_i(\mathbb{Q}) = 0$, $i > 0$

$\mathbb{Z}$ isomorphic to $\mathbb{Q}$ $\Rightarrow \pi_0(\mathbb{Z}) = \mathbb{Z} = \mathbb{Q}$

<u>Van Kampen Thm</u> Hard to compute homotopy groups

can relate $\pi_1(X)$ to π_1(subspaces of X).

<u>Thm</u> Let $X = U \cup V$, U, V open in X

$x_0 \in U \cap V$ and $U \cap V$ is path connected.

Then $\forall \alpha \in \pi_1(x)$, $\alpha = \beta_1 + \cdots + \beta_n$

where $\beta_i \in j_*(\pi_1(U))$ or $\beta_i \in k_*(\pi_1(V))$

j_*, k_* induced homomorphisms of

inclusion maps $j: U \subset X$, $k: V \subset X$.

Pf/

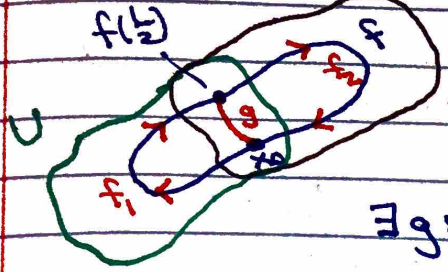
Let $\alpha \in \pi_1(X)$ be represented by $f: S^1 \to X$

with $f(0) = f(1) = x_0 \in X$ (gives loop)

$f(t) \in U$, $t \leq 1/2$

$f(t) \in V$, $t \geq 1/2$

$f(1/2) \in U \cap V$



$X = U \cup V$

$U \cap V$ pathconnected $\Rightarrow$

$\exists\, g : [0,1] \to U \cap V \ni.$

$g(0) = f(\frac{1}{2}),\ g(1) = x_0 \in U \cap V \subset X$

Split path:
$$f_1(s) = \begin{cases} f(s), & s \le \frac{1}{2} \\ g(2s-1), & s \ge \frac{1}{2} \end{cases}$$

$$f_2(s) = \begin{cases} g(1-2s), & s \le \frac{1}{2} \\ f(s), & s \ge \frac{1}{2} \end{cases}$$

$f_1(0) = f(0) = x_0 \qquad f_1(1) = g(1) = x_0$

$f_1(\frac{1}{2}) = f(\frac{1}{2}) = g(0)$

$f_2(0) = g(1) = x_0 \qquad f_2(1) = f(1) = x_0$

$f_2(\frac{1}{2}) = g(0) = f(\frac{1}{2})$

Homotopy $f_1 \# f_2 \longrightarrow f$

$$F(s,t) = \begin{cases} (f_1 \# f_2)(\frac{s}{1+t}), & s \le \frac{1}{2} \\ (f_1 \# f_2)(\frac{s+t}{1+t}), & s \ge \frac{1}{2} \end{cases}$$ Continuous

and

$$F(s,0) = (f_1 \# f_2)(s)$$

$$F(s,1) = \begin{cases} (f_1 \# f_2)(\frac{s}{2}) = f(s), & s \le \frac{1}{2} \\ (f_1 \# f_2)(\frac{s+1}{2}) = f(s), & s \ge \frac{1}{2} \end{cases}$$

[Can generalize via domain splitting]

$\exists\, g_1, g_2 \ni.\quad f_1 = j \circ g_1,\ f_2 = K \circ g_2,\ \text{Im} f_1 \subset U,\ \text{Im} f_2 \subset V$

$f_i : [0,1] \to U \cap V$

$g_1 : [0,1] \to U \qquad g_2 : [0,1] \to V$

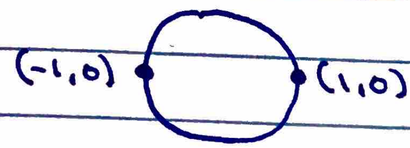
Then $[f_1] \in j_*(\pi_1(U))$ Induced homomorphisms

$[f_2] \in K_*(\pi_1(V))$

Ex $X = S^1, \quad U = V = X$

$\Rightarrow \pi_1(U) = \pi_1(V) = \pi_1(X) = \mathbb{Z}$

Ex $X = D^2, \quad U = D^2 - \{(-1,0)\}, \quad V = D^2 - \{(1,0)\}$



$U, V \simeq S^1$

$\pi_1(U) = \pi_1(V) = \mathbb{Z}$

But $X = D^2$ is contractible $\Rightarrow \pi_1(X) = 0$

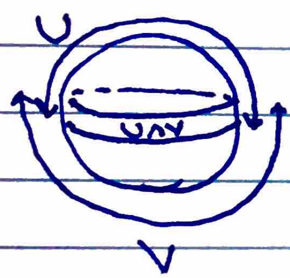
Can't find $\pi_1(X)$ knowing just $\pi_1(U), \pi_1(V)$

need to account for $U \cap V$.

Need stronger version of Thm.

Involves free groups, almalgamation, normal subgroups.

Ex

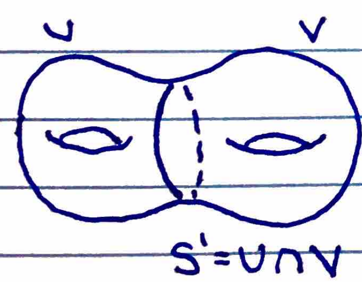


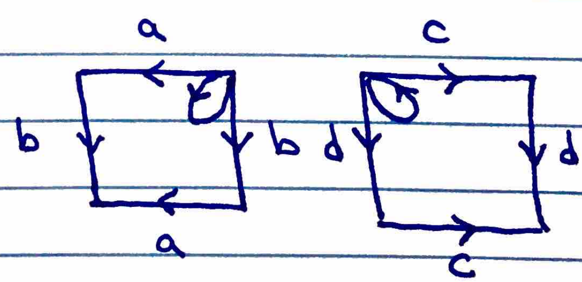
$\pi_1(S^2) = \pi_1(U) \underset{\pi_1(U\cap V)}{*} \pi_1(V)$

$\pi_1(U) = \{1\} = \pi_1(V)$

$\pi_1(U \cap V) \simeq \pi_1(S^1) = \mathbb{Z}$

Ex





$\pi_1(U \cap V) = \mathbb{Z}$

$\pi_1(U)$ $\qquad$ $\pi_1(V)$

$aba^{-1}b^{-1}$ $\qquad$ $dcd^{-1}c^{-1} = (cdc^{-1}d^{-1})^{-1}$

Combine

$aba^{-1}b^{-1} = (cdc^{-1}d^{-1})^{-1}$