This exam has 12 problems on 5 pages with two problems designated graduate students only. Do as many problems as you can. You can use the back of a page but note this in the problem space. There are no calculators, phones, or other electronic devices allowed during this exam. Be sure to show all your work.

Name: KEY

Score:

Problem 1. (12 pts) Let X and Y be sets and $A \subset X$. Define the following:

(a) Topology on X. Collection of subsets of X satisfying X open, ϕ open, arbitrary unions of open sets are open, and finite intersections of open sets are open.

(b) Subspace topology on A. Subset of A is open if it is $A \cap U$ for some open U in X.

(c) Discrete topology on X. All subsets of X are open

(d) X is connected. if there are no nonempty open sets of X such That They are disjoint $[U \cap V = \phi]$ and cover X $[U \cup V = T]$.

(e) X is compact. if every open cover of X admits a finite refinement.

(f) X is Hausdorff. if for any two distinct points $x, y \in X$, $\exists$ open $U, V \subset X$ such That $x \in U$, $y \in V$, and $U \cap V = \phi$.

Problem 2. (2 pts) State the Heine-Borel Theorem.

Subspace $T \subset \mathbb{R}^n$ is compact iff T is closed and bounded.

Problem 3. (4 pts) Let S and T be two topologies on a set X.

a. Is their union, $S \cup T$, a topology on X? Why?

No. If $S \neq T$, $\exists\, U \in S, U \notin T$. Let $V \in T \cap S$, $V \neq \phi$
Then $V \in S$, $U \cap V \in S$ but if $V \in T$, $V \notin T \cap S$
Then $U \cap V \notin S \cup T$.

b. Is their intersection, $S \cap T$, a topology on X? Why?

Yes. S, T both contain ϕ, X. If $U \in S, U \in T$, $U \neq \phi, X$
and $V \in S$, $V \in T$. Then, $U \cap V \in S, T$. $\Rightarrow U \cap V \in S \cap T$

Problem 4. (6 pts) Let $X = \{1, 2, 3, 4\}$ be equipped with the topology

$$T = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$

Let $Y = \{a, b, c\}$.

a. Let $f : X \to Y$ be the function sending $1 \to a$, $2 \to a$, $3 \to b$, and $4 \to c$. Find the quotient topology Q on Y defined by f.

1, 2 > a
3 — b
4 — c

Q: $\{\phi, \{a\}, Y\}$

Need $f^{-1}(U)$ open in X; i.e. in T

b. Let $g : X \to Y$ be the function sending $1 \to a$, $2 \to b$, $3 \to b$, and $4 \to c$. Find the quotient topology P on Y defined by g.

1 \ a
2, 3 > b
4 / c

P: $\{\phi, \{a\}, Y\}$

c. Are the spaces (Y, Q) and (Y, P) homeomorphic? If yes, write down the homeomorphism. If not, explain why not.

Yes, use identity

Problem 5. (6 pts) Consider the real numbers $\mathbb{R}$ with the standard topology and the closed interval $[-1, 2] \subset \mathbb{R}$ with the subset topology. Consider the following subsets of $[-1, 2]$. Which are open in $\mathbb{R}$? Which are open in $[-1, 2]$?

a. $A = (0, 1)$, Both

b. $B = (1, 2]$, Not in $\mathbb{R}$ but $B = [-1,2] \cap (1,3)$

c. $C = \{-1\} \cup (1, 2]$? Neither

Problem 6. (5 pts) Let X and Y be topological spaces and let $y_0 \in Y$ be a point.

a. Give a basis for the product topology on $X \times Y$.

Opensets are of the form $U \times V$ for $U \subset X$ open $V \subset Y$ open

b. Prove that the function $f : X \to X \times Y$, defined by $f(x) = (x, y_0)$, is continuous when $X \times Y$ is given the product topology.

Need to show $f^{-1}(U \times V)$ is open for basis element $U \times V$

$f^{-1}(U \times V) = \phi$ if $y_0 \notin V$

$f^{-1}(U \times V) = U$ if $y_0 \in V$

Both ϕ, U are open in X, so f is continuous

Problem 7. (3 pts) Which of the following are connected? $(0, 1)$, $\mathbb{R} - \{0\}$, $\mathbb{R}^2 - \{0\}$.

$(0,1)$, $\mathbb{R}^2 - \{0\}$

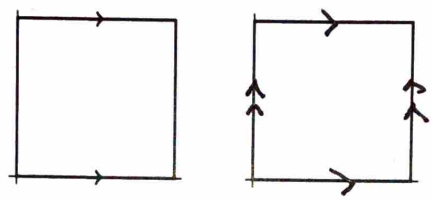
Problem 8. (3 pts) Describe how one shows that $D^2/\partial D^2 \cong S^2$. $D^2/\partial D^2 = D^2 - \partial D^2 + \{*\}$

1) $D^2 - \partial D^2 \cong \mathbb{R}^2$
2) $\mathbb{R}^2 \cong S^2 - \{(0,0,1)\}$ by stereographic projection
gives $D^2 - \partial D^2 \longrightarrow S^2 - \{(0,0,1)\}$
Now extend to $D^2/\partial D^2 \longrightarrow S^2$ by identifying $* \mapsto (0,0,1)$

Problem 9. (5 pts) In Figure (a) one has the identification on $[0,1]^2$ given by the equivalence relation

$$(x,y) \sim (x',y') \Leftrightarrow x = x' \text{ and } y - y' \in \mathbb{Z}.$$

Describe the quotient space $[0,1]^2/\sim$.



(a) (b)

(a) cylinder $[0,1] \times S^1$

(b) torus $S^1 \times S^1$

On Figure (b) draw the identification given by

$$(x,y) \sim (x',y') \Leftrightarrow x - x' \in \mathbb{Z} \text{ and } y - y' \in \mathbb{Z}.$$

Describe the quotient space $[0,1]^2/\sim$.

Problem 10. (4 pts) Describe the space $\mathbb{R}P^2$. How can it be defined as a quotient space?

Real projective plane - set of all straight lines through origin

$\mathbb{R}P^2 \cong S^2/\sim$ where $\vec{x}, \vec{y} \in S^2$, $\vec{x} \sim \vec{y} \Leftrightarrow \vec{x} = \pm\vec{y}$ (antipodal pts)

The following are for graduate students only.

Problem 11. (5 pts) Consider the diagram below.

$$\begin{tikzcd}
\mathbb{R}^3 - \{0\} \arrow[r, "g"] \arrow[d, "\pi_1"] & S^2 \arrow[d, "\pi_2"] \\
\mathbb{R}P^2 \arrow[r, "f"] & S^2/\sim
\end{tikzcd}$$

a. Describe the spaces and the functions g, π_1 and π_2 in the diagram.

π_1 $\vec{x} = \lambda \vec{y}$ stereographic proj.

π_2 antipodal map

$\mathbb{R}^3 - \{0\}$ punctured space

$\mathbb{R}P^2$ projective plane

S^2 surf. of sphere

$S^2/\sim$ quotient space

b. Let U be open in $\mathbb{R}P^2$. Prove that $f(U)$ is open.

$f(U) = \pi_2(g(\pi_1^{-1}(U)))$ show indiv. maps continuous

Problem 12. (5 pts) Let X be a topological space and define the diagonal map $\Delta : X \to X \times X$ be defined as $\Delta(x) = (x, x)$ for $x \in X$.

a. Show that the diagonal map is continuous.

$\forall$ open $U \subset X$, $\Delta^{-1}(U \times U) = \{x \mid (x,x) \in U \times U\} = U$ is open.

b. Prove that X is Hausdorff if and only if $\Delta(X) = \{(x, x) : x \in X\}$ is closed in $X \times X$.

$\Delta(X)$ closed. Let $x \neq y \in X$. Then $(x,y) \notin \Delta(X)$

So $(x,y) \in \Delta(X) \setminus \Delta(X)$ [open] From product topology,

$x \in U, y \in V$, U, V open in X. But $(U \cap V) \cap \Delta(X) = \phi \Rightarrow U \cap V = \phi$

Thus X is Hausdorff.

Work backwards to show X Hausdorff $\Rightarrow \Delta(X)$ closed.