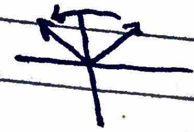
Hopf map - Need Quaternions

Rotation in $\mathbb{R}^2$



Use complex numbers

$z = a+bi = re^{i\theta}$, $a,b \in \mathbb{R}$, $i^2=-1$

What algebraic structure for $\mathbb{R}^3$?

1843 William Rowan Hamilton [tried $a+bi+cj$.]

(while crossing Dublin bridge)

Quaternions $\mathbb{H}$ $\quad a,b,c,d \in \mathbb{R}$, $a+bi+cj+dk$

where $i^2=j^2=k^2=-1$, $ij=k$, $jk=i$, $ki=j$

Note: $ik = i(ij) = -j$, etc.

$\mathbb{H}$ is a division algebra, a vector space with a bilinear product in which division is possible.

$\boxed{i^2=j^2=k^2=ijk=-1}$ Carved on bridge, Oct 16, 1843

- Competed with vector calculus (debate 1890-1894)

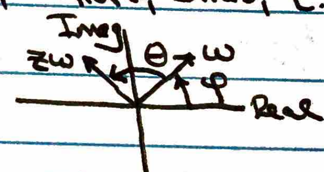
Hamilton vs Gibbs and their followers

[later - Grassman, Clifford, Dirac, E. Cartan]

$\mathbb{C}$ - Rotation in the plane, $\mathbb{R}^2$

$z = e^{i\theta}$, $\theta \in [0, 2\pi]$

$|z|^2 = z\bar{z} = 1$, $\bar{z} = a - ib$



$\bar{z}$ - complex conjugate

$|z|$ - complex modulus

$zw = R_z(w)$ = rotate w by θ

Let $w = re^{i\varphi}$. Then $zw = e^{i\theta}re^{i\varphi} = re^{i(\varphi+\theta)}$

[Extend to $\mathbb{R}^3$ - Hamilton tried $a+bi+cj$. $j^2=-1$

Ex: $(a_1+b_1i+c_1j)(a_2+b_2i+c_2j) = ?$

$ij = ?$ - Let $ij \equiv k$ Eureka moment!

Only normed division algebras [$|zw| = |z||w|$]

have dim $= 1, 2, 4, 8$

$\hookrightarrow$ octonions

Quaternions

Representations: $q = v_0 + v_1 i + v_2 j + v_3 k$
$$\equiv (v_0, \vec{v})$$
v_0 – scalar part; $\vec{v}$ – vector part

Operations:

addition – add components

multiplication

$$(v_0, \vec{v})(w_0, \vec{w}) = (v_0 w_0 - \vec{v}\cdot\vec{w},\ v_0\vec{w} + w_0\vec{v} + \vec{v}\times\vec{w})$$

conjugate: $\bar{q} = (v_0, -\vec{v})$

norm: $\|q\|^2 = q\bar{q} = v_0^2 + v_1^2 + v_2^2 + v_3^2$

Then, $\|pq\| = \|p\|\,\|q\|$

inverse: $q^{-1} = \bar{q}/\|q\|^2 \Rightarrow qq^{-1} = \dfrac{q\bar{q}}{\|q\|^2} = 1$, $\|q\| \neq 0$.

Unit quaternions:

$$\|q\|^2 = v_0^2 + v_1^2 + v_2^2 + v_3^2 = 1$$

Set of unit quaternions $= S^3 \subset \mathbb{R}^4$ $(\simeq \mathbb{C}^2)$

Similar to unit complex #s $= S^1 \subset \mathbb{R}^2$ $(\simeq \mathbb{C})$

Rotations in 3D

$\mathbb{R}^3$ – represent by pure quaternion $(v_0 = 0)$
$$\{x\hat{i} + y\hat{j} + z\hat{k} \mid x, y, z \in \mathbb{R}\}$$

For unit vector $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$, consider
$$q = \cos\varphi + \vec{u}\sin\varphi \equiv e^{u\varphi}$$

For any vector $\vec{v}$, $e^{u\varphi}\vec{v}e^{-u\varphi}$ gives rotation of $\vec{v}$ about $\vec{u}$ by 2φ.

Can show S^3 + quaternion multiplication gives group $SO(3)$.

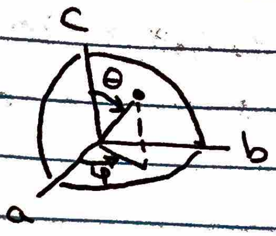
__Hopf map__ $\eta: S^3 \to S^2$

Let $(a,b,c) \in S^2$

$a = \sin\Theta \cos\varphi$

$b = \sin\Theta \sin\varphi$

$c = \cos\Theta$



Then, $a^2+b^2+c^2=1$.

Let $(x,y,w,z) \in S^3$. Define

$x = \beta\cos\Phi \qquad w = \alpha\cos\Theta$

$y = \beta\sin\Phi \qquad z = \alpha\sin\Theta$

Then, $x^2+y^2+w^2+z^2 = \beta^2+\alpha^2 = 1$

__Hopf map__ $\eta(x,y,w,z) = (2(xz+yw), 2(yz-xw), -x^2+y^2+w^2+z^2) \in \mathbb{R}^3$

$\stackrel{?}{=} (a,b,c)$

$c = -x^2+y^2+w^2+z^2 = -\beta^2+\alpha^2 \quad \& \quad \alpha^2+\beta^2=1$

$\Rightarrow \quad \beta = \sqrt{(1-c)/2}, \; \alpha = \sqrt{(1+c)/2}, \; \alpha\beta = \frac{\sqrt{1-c^2}}{\sqrt{4}}$

$xz+yw = \alpha\beta(\cos\Phi\sin\Theta + \sin\Phi\cos\Theta) = \alpha\beta\sin(\Theta+\Phi)$

$yz-xw = \alpha\beta(\sin\Phi\sin\Theta - \cos\Phi\cos\Theta) = -\alpha\beta\cos(\Theta+\Phi)$

$a = \sqrt{1-c^2}\sin\psi \qquad$ where $\psi = \Theta+\Phi$.

$b = -\sqrt{1-c^2}\cos\psi \qquad$ and $\tan\psi = -a/b$

__Hopf map from unit quaternions__

$q = w + x\hat{i} + y\hat{j} + z\hat{k}, \qquad w^2+x^2+y^2+z^2=1$

$q^{-1} = w - x\hat{i} - y\hat{j} - z\hat{k}$

$\eta: S^3 \to S^2: \qquad \eta(q) = qkq^{-1}$

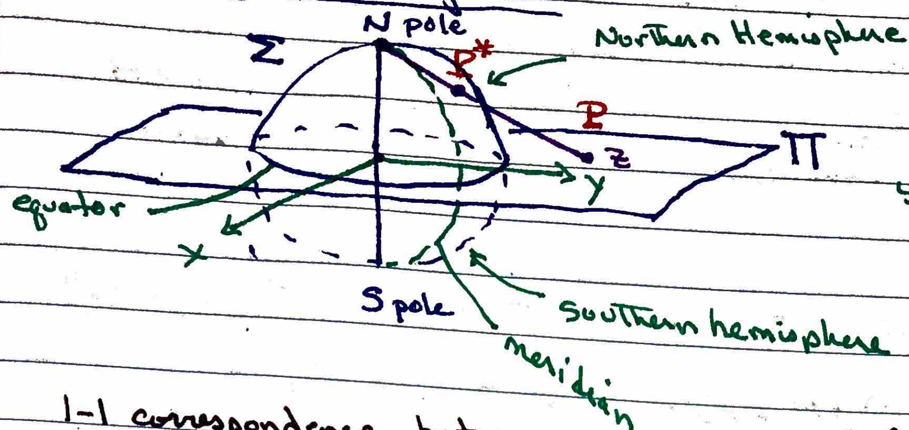
$= (0, 2wy+2xz, 2yz-2wx, w^2+z^2-x^2-y^2) \in \mathbb{R}^3$

and $\|\eta(q)\|=1 \Rightarrow \eta(q) \in S^2$

[Note: Above $c \to -c$]
(fixed in red)

Notes from complex analysis course.

21. Stereographic Projection



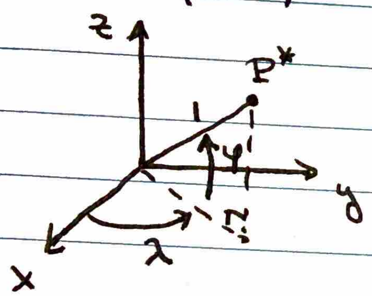
y > 0 Eastern

1-1 correspondence between points $\Sigma - \{N\}$ and $\Pi \equiv \mathbb{C}$

$P : (x, y, 0) \in \Pi$

$P^* : (\xi, \eta, \zeta) \in \Sigma$ $\longrightarrow$ Use spherical coordinates

$\xi^2 + \eta^2 + \zeta^2 = 1$
(unit sphere)

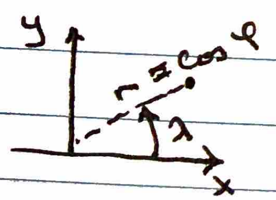


Longitude λ, $-\pi < \lambda \leq \pi$

Latitude φ, $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$

$\arg z = \lambda$

$|z| = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$



$z = \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)(\cos\lambda + i\sin\lambda)$

and

$\lambda = \arg z$

$\varphi = 2\tan^{-1}|z| - \frac{\pi}{2}$, $z \neq 0$

Parametrization:

$$\xi = \cos\varphi\cos\lambda$$
$$\eta = \cos\varphi\sin\lambda$$
$$\zeta = \sin\varphi$$

Then

$$\xi = \frac{2x}{x^2+y^2+1}$$

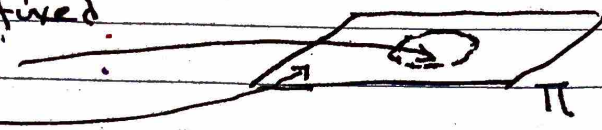
$$\eta = \frac{2y}{x^2+y^2+1} \quad \text{and} \quad x = \frac{\xi}{1-\zeta}$$

$$\zeta = \frac{x^2+y^2-1}{x^2+y^2+1} \qquad y = \frac{\eta}{1-\zeta}$$

Circle-preserving property   circles $\in \Sigma \to$ circles/lines $\in \Pi$

- Continuous curves in Π map to cont. curves $\in \Sigma$

Pts on Equator are fixed
Lower hemisphere
Upper hemisphere



Circle on Σ $\Sigma \cap$ plane : $A\xi + B\eta + C\zeta + D = 0$

$$\Rightarrow \quad Ax + By + (C+D)(x^2+y^2) + (D-C) = 0 \in \Pi$$

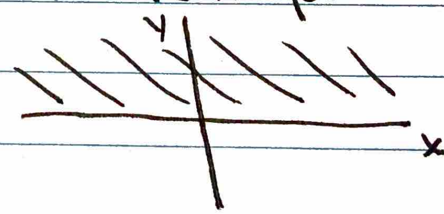
if $C+D \neq 0 \Rightarrow$ circles
if $C+D = 0 \Rightarrow$ lines

Where is ∞?

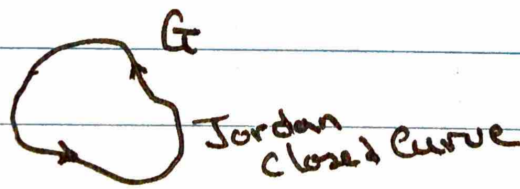
Extended Complex Plane $\mathbb{C} \cup \{\infty\}$

Compact - every sequence has a limit pt.
If G is an unbounded set, Then ∞ can be a boundary point or an interior point of G.

Ex Upper half plane



Ex



Functions - Extend to include ∞.

$$\lim_{z \to \infty} f(z) = c \quad \Rightarrow \quad f(\infty) = c$$

$$\lim_{z \to z_0} f(z) = \infty \quad \Rightarrow \quad f(z_0) = \infty$$

Ex Rational function $\dfrac{a_0 + a_1 z + \dots + a_m z^m}{b_0 + b_1 z + \dots + b_n z^n} \xrightarrow{z \to \infty} \dfrac{a_m z^m}{b_n z^n}$

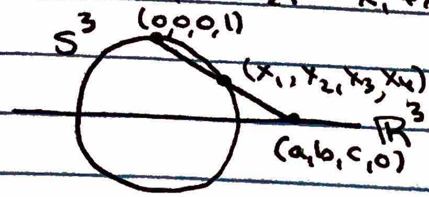
$a_m, b_n \neq 0$

Extend Stereographic Projection

View $\mathbb{C}^2$ as $\mathbb{R}^4$ $\qquad (z_1, z_2) \longrightarrow (x_1, x_2, x_3, x_4)$

where $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$

$1 = |z_1|^2 + |z_2|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$



Equation of Line

$$\frac{x_1 - 0}{a - 0} = \frac{x_2 - 0}{b - 0} = \frac{x_3 - 0}{c - 0} = \frac{x_4 - 1}{0 - 1} = \lambda$$

or $\quad x_1 = a\lambda,\ x_2 = b\lambda,\ x_3 = c\lambda,\ x_4 = 1 - \lambda$

So,

$$\begin{aligned} 1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2 \\ &= (a^2 + b^2 + c^2)\lambda^2 + (1-\lambda)^2 \\ &= (a^2 + b^2 + c^2 + 1)\lambda^2 + 1 - 2\lambda \end{aligned} \qquad \text{Let } x = a^2 + b^2 + c^2$$

Then $\quad \lambda[(x^2+1)\lambda - 2] = 0$

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{2}{x^2+1}$$

So, $\quad x_1 = \frac{2a}{x^2+1},\ x_2 = \frac{2b}{x^2+1},\ x_3 = \frac{2c}{x^2+1},\ x_4 = \frac{x^2-1}{x^2+1}$

This gives map $\mathbb{R}^3 \to S^3$.

For $S^3 \to \mathbb{R}^3$

$$x_1 = a\lambda,\ x_2 = b\lambda,\ x_3 = c\lambda,\ x_4 = 1 - \lambda$$

For $\lambda = 1 - x_4$,

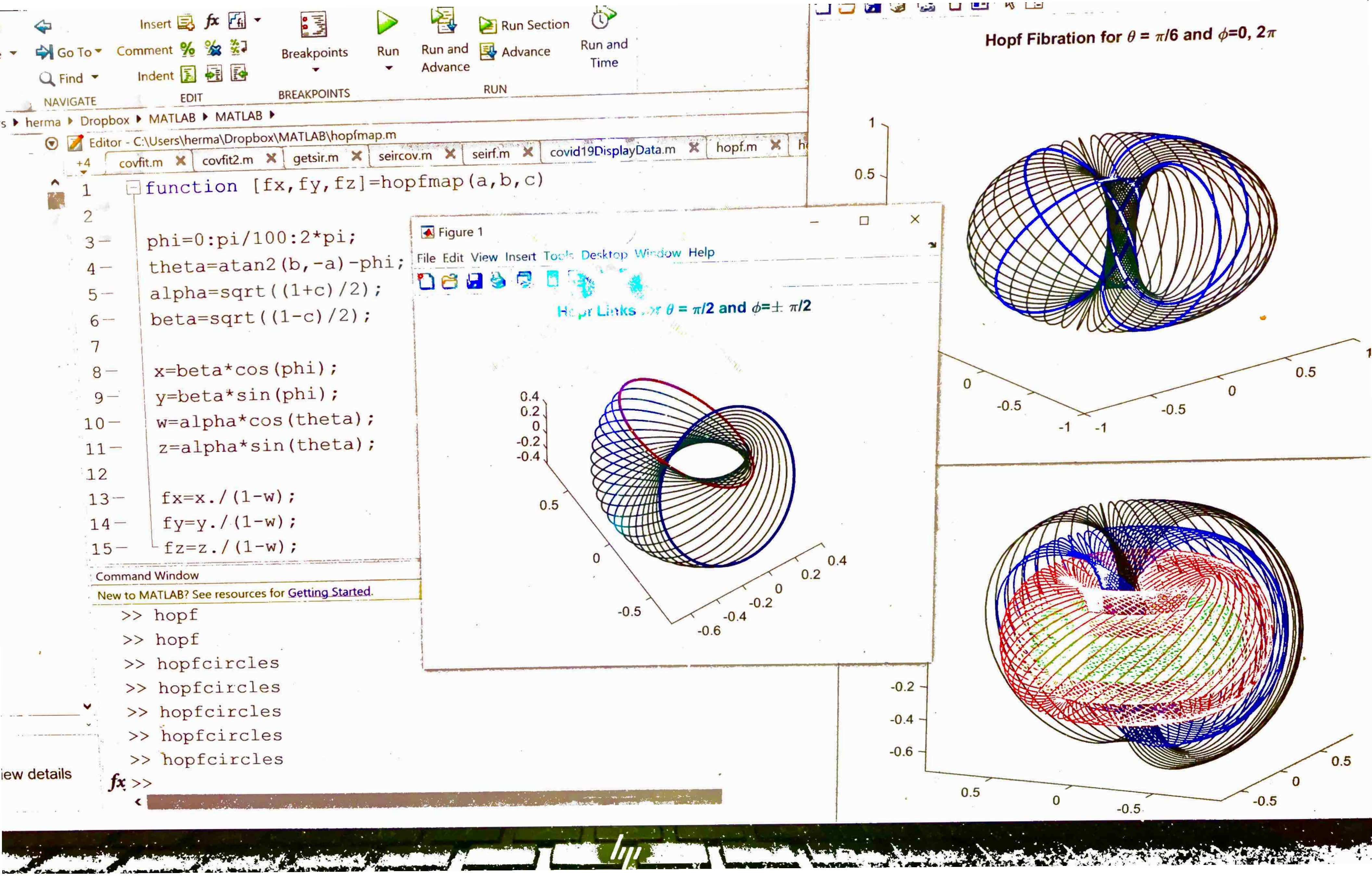
$$a = \frac{x_1}{1-x_4},\ b = \frac{x_2}{1-x_4},\ c = \frac{x_3}{1-x_4},\quad x_4 \neq 1.$$

Notes: Circles $\to$ Circles

Images of circles w/same $r \to T^2 \subset \mathbb{R}^3$

Vary r and get nested tori

The circles are <u>Villarceau circles</u>



Insert
Go To
Comment
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NAVIGATE
EDIT
Breakpoints
BREAKPOINTS
Run
Run and Advance
Run Section
Advance
Run and Time
RUN
herma ▸ Dropbox ▸ MATLAB ▸ MATLAB ▸
Editor - C:\Users\herma\Dropbox\MATLAB\hopfmap.m
covfit.m
covfit2.m
getsir.m
seircov.m
seirf.m
covid19DisplayData.m
hopf.m
function [fx,fy,fz]=hopfmap(a,b,c)
phi=0:pi/100:2*pi;
theta=atan2(b,-a)-phi;
alpha=sqrt((1+c)/2);
beta=sqrt((1-c)/2);
x=beta*cos(phi);
y=beta*sin(phi);
w=alpha*cos(theta);
z=alpha*sin(theta);
fx=x./(1-w);
fy=y./(1-w);
fz=z./(1-w);
Command Window
New to MATLAB? See resources for Getting Started.
>> hopf
>> hopf
>> hopfcircles
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>> hopfcircles
fx >>
Figure 1
File Edit View Insert Tools Desktop Window Help
Hopf Links for θ = π/2 and ϕ=± π/2
Hopf Fibration for θ = π/6 and ϕ=0, 2π