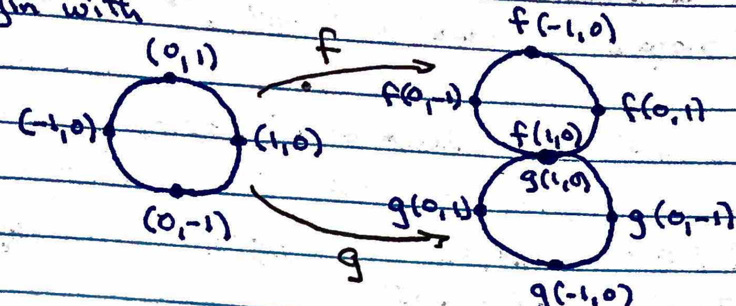
Crossley Approach Pointed Maps $S^n \longrightarrow X$

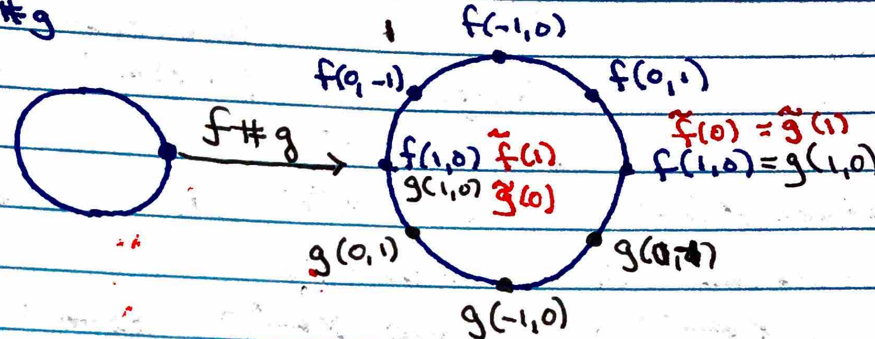
$n=1$, $X = S^1$

Begin with



$f, g : S^1 \to S^1$

Define $f \# g$



Lifts

$$(\widetilde{f \# g})(t) = \begin{cases} \tilde{f}(2t), & t \leq \frac{1}{2} \\ \tilde{g}(2t-1), & t \geq \frac{1}{2} \end{cases}$$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r] \arrow[dr] & S^1 \\ & S^1 \arrow[u, "f"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[ur, "\widetilde{f \# g}"] \arrow[r] \arrow[dr] & S^1 \\ & S^1 \arrow[u, "f \# g"'] \end{tikzcd}$$

$\tilde{f}(1) = \tilde{g}(0)$

$$\begin{aligned} \Rightarrow \quad \deg(f \# g) &= (\widetilde{f \# g})(1) - (\widetilde{f \# g})(0) \\ &= \tilde{g}(1) - \tilde{f}(0) \\ &= \tilde{g}(1) - \tilde{g}(0) + \tilde{g}(0) - \tilde{f}(0) \\ &= \deg(g) + \tilde{f}(1) - \tilde{f}(0) \\ &= \deg(g) + \deg(f) \quad \in \mathbb{Z} \end{aligned}$$

This is addition in $\mathbb{Z}$ $\Rightarrow$ $\pi(S^1) = \mathbb{Z}$.

Next S^2.

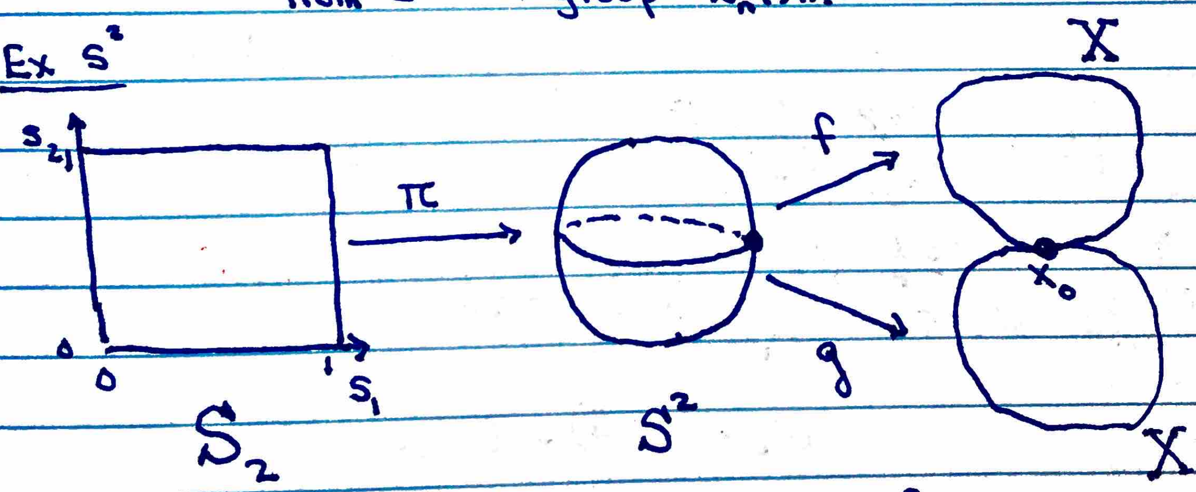
So far, we have addition on sets of maps $S^1 \to X$
For $X = S^1$, homotopy classes gives group $\mathbb{Z}$.

Use <u>pointed spaces</u>: Topological space with pt (X, x_0)
<u>pointed maps</u>: $f : (X, x_0) \to (Y, y_0)$, continuous & $f(x_0) = y_0$
<u>pointed homotopy</u> $F : (X, x_0) \times I \to (Y, y_0)$
$F(x_0, t) = y_0, \ \forall t \in [0,1]$

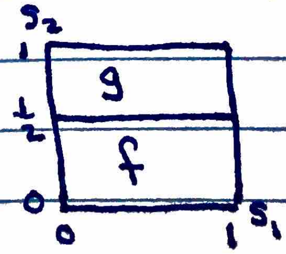
<u>Ex</u> S^1 with base pt $(1,0)$
$f, g : S^1 \to (X, x_0)$ $\quad f(1,0) = g(1,0) = x_0$
$\Rightarrow (f \# g)(1,0) = x_0$

<u>Ex</u> S^n From Ex 5.51 $\quad S^n \cong [0,1]^n/\sim$ (quotient by boundary)
i.e., identify pts with at least one coord 0, 1.
pointed maps $S^n \to X$ corresp to $[0,1]^n \to X$ with
boundary pts $\to x_0$.
Let $S_n = [0,1]^n$ Then consider
$f, g : (S_n, \partial S_n) \to (X, x_0)$
Define $f \# g$ and get addition on homotopy classes
from $S^n \to$ group $\pi_n(X)$.

<u>Ex</u> S^2



$$(f \# g)(s_1, s_2) = \begin{cases} f(s_1, 2s_2), & s_2 \le \frac{1}{2} \\ g(s_1, 2s_2 - 1), & s_2 \ge \frac{1}{2} \end{cases} \Rightarrow$$



$f(s_1, 1) = g(s_1, 0) = x_0$, $\qquad (f \# g)(s_1, \frac{1}{2}) = x_0$

Use pointed homotopies - show $\pi_2(X)$ is a group

1) $f \# g$ is a pointed map. (closure)

2) $f \# c \simeq f$, $c \# f \simeq f$ for c = constant maps, i.e., $g(s_1, s_2) = x_0$

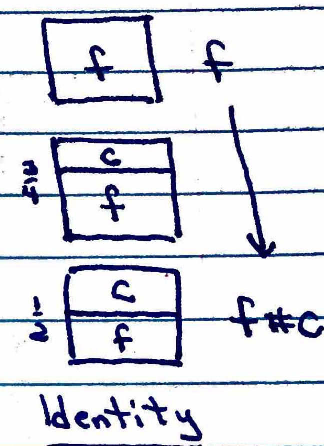
Let

$$H((s_1,s_2),t) = \begin{cases} f\left(s_1, \frac{2s_2}{2-t}\right), & s_2 \le 1 - \frac{t}{2} \\ x_0, & s_2 \ge 1 - \frac{t}{2} \end{cases}$$

$$H((s_1,s_2),0) = f(s_1,s_2) \quad s_2 \le 1$$

$$H((s_1,s_2),\tfrac{1}{2}) = \begin{cases} f(s_1, \frac{4}{3}s_2) & s_2 \le \frac{3}{4} \\ x_0 & s_2 \ge \frac{3}{4} \end{cases}$$

$$H((s_1,s_2),1) = \begin{cases} f(s_1,s_2) & s \le 1/2 \\ x_0 & s_2 \ge 1/2 \end{cases}$$



3) Inverse

Let $\bar{f}(s_1,s_2) = f(s_1, 1-s_2)$ and define

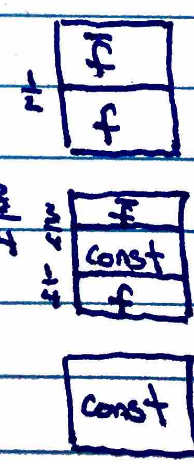
$$H((s_1,s_2),t) = \begin{cases} (f \# \bar{f})(s_1, \frac{1-t}{2}), & \frac{1-t}{2} \le s_2 \le \frac{1+t}{2} \\ (f \# \bar{f})(s_1,s_2), & \text{otherwise} \end{cases}$$

Evaluate at $t = 0, \frac{1}{2}, 1$

$$H((s_1,s_2),0) = \begin{cases} (f \# \bar{f})(s_1,\frac{1}{2}), & s_2 = 1/2 \\ (f \# \bar{f})(s_1,s_2), & s_2 \ne \frac{1}{2} \end{cases}$$

$$H((s_1,s_2),\tfrac{1}{2}) = \begin{matrix} (f \# \bar{f})(s_1,\frac{1}{4}), & \frac{1}{4} \le s_2 \le \frac{3}{4} \\ (f \# \bar{f})(s_1,s_2), & \text{otherwise} \end{matrix}$$

$$H((s_1,s_2),1) = (f \# \bar{f})(s_1,0)$$



So $f \# \bar{f} \simeq c$.

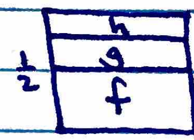
Similarly, $\bar{f} \# f \simeq c$.

4) <u>Associativity</u> $(f\#g)\#h \simeq f\#(g\#h)$

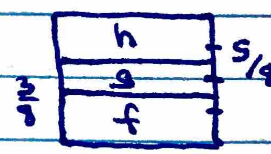
Consider

$$H((s_1,s_2),t) = \begin{cases} f\left(s_1, \frac{4s_2}{2-t}\right), & s_2 \le \frac{2-t}{4} \\ g(s_1, 4s_2+t-2), & \frac{2-t}{4} \le s_2 \le \frac{3-t}{4} \\ h\left(s_1, \frac{4s_2+t-3}{t+1}\right), & s_2 \ge \frac{3-t}{4} \end{cases}$$

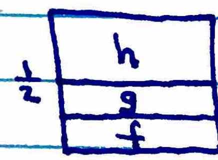
<u>at $t=0$</u> $= \begin{cases} f(s_1, 2s_2), & s_2 \le \frac{1}{2} \\ g(s_1, 4s_2-2), & \frac{1}{2} \le s_2 \le \frac{3}{4} \\ h(s_1, 4s_2-3), & s_2 \ge 3/4 \end{cases}$



at $t=\frac{1}{2}$ $= \begin{cases} f(s_1, \frac{8}{3}s_2), & s_2 \le \frac{3}{8} \\ g(s_1, 4s_2 - \frac{3}{2}), & \frac{3}{8} \le s_2 \le \frac{5}{8} \\ h\left(s_1, \frac{8s_2-5}{3}\right), & s_2 \ge \frac{5}{8} \end{cases}$



at $t=1$ $= \begin{cases} f(s_1, 4s_2), & s_2 \le \frac{1}{4} \\ g(s_1, 4s_2-1), & \frac{1}{4} \le s_2 \le \frac{1}{2} \\ h(s_1, 2s_2-1), & s_2 \ge \frac{1}{2} \end{cases}$



Let $n \in \mathbb{Z}^+$. The nth homotopy group of a pointed space is the group of homotopy classes of maps $S^n \to X$ with group operation $[f]+[g]=[f\#g]$.

<u>Ex</u> $X \subset \mathbb{R}^n$, convex $\quad f: S^n \to (X, x^0)$

Then, $f \simeq c$

Let $H(\vec{x},t) = tf(\vec{x}) + (1-t)x_0$, $\vec{x} \in S^n$

Since $H((1,0,\dots,0),t) = x_0$, $[f]=[c]$

or $\pi_n(X) = \{0\}$

Ex $X = S^1$

$\pi_1(S^1) = \mathbb{Z}$ (known already!)

$\pi_2(S^1) = 0$

$f: S^2 \to S^1 \Rightarrow (S_2, \partial S_2) \to (S^1, (1,0))$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ S_2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

$f(s_1, s_2) = (1,0)$, when $(s_1, s_2) \in \partial S_2$.

$\tilde{f}(s_1, s_2) \in \mathbb{Z}$ when $(s_1, s_2) \in \partial S_2$

Since ∂S_2 connected $\to$ $\mathbb{Z}$ discrete, $\tilde{f}$ is const. on ∂S_2.

Homotopy Lifting:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ S_2 \times [0,1] \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd}$$

$\tilde{F}(s_1, s_2, t) = t\,\tilde{f}(s_1, s_2) + (1-t)\tilde{f}(0,0)$, cont.

$\Rightarrow F = e \circ \tilde{F}$

$t=0$: $\cancel{\tilde{F}}\, F(s_1, s_2, 0) = f(0,0)$

$t=1$: $F(s_1, s_2, 1) = f(s_1, s_2)$

Then $f \simeq c$ on ∂S_2.

F is pointed homotopy $S_2 \times [0,1] \to S^1 \Rightarrow$ all maps $S_2 \to S^1$ are homotopic

Ex $\pi_n(S^1) = 0$, $n > 1$.

Leads to Eilenberg-MacLane Spaces used to classify fiber bundles

More generally

$\pi_i(S^n) = 0 \quad i < n$

$\pi_n(S^n) = \mathbb{Z}$

Hopf (1930's) $\pi_3(S^2)$ isomorphic to $\mathbb{Z}$

introduced Hopf map $\eta: S^3 \to S^2$

$\eta \not\simeq c$ (1931)

$\pi_3(S^2)$ = infinite cyclic group generated by η

How do you visualize S^3?

Hopf Fibration

What is fibration?

Product Spaces $\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$

$\mathbb{R}\times S^1$

$S^1\times S^1$

Projection $A\times B \to B$

pick $b\in B$: $A\times\{b\} \mapsto b$

$\uparrow$ fiber, B = base space

Ex

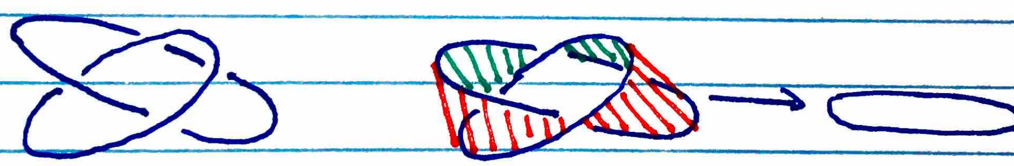

I fiber — $I\times S^1$ total space — S^1 base space

Total Space not always Cartesian product

Ex


Möbius band — S^1

Ex Hopf Band



Linked Circles — Hopf band — Center circle

Hopf Fibration of Spheres – total, base, fiber are all spheres

$S^0 \hookrightarrow S^1 \to S^1$

$S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2$ ← Hopf fibration

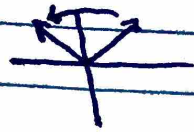
$S^3 \hookrightarrow S^7 \to S^4$

$S^7 \hookrightarrow S^{15} \to S^8$

embedding, projection

Hopf map - Need Quaternions

Rotation in $\mathbb{R}^2$



Use complex numbers

$z = a + bi = re^{i\theta}$, $a, b \in \mathbb{R}$, $i^2 = -1$

What algebraic structure for $\mathbb{R}^3$?

1843 William Rowan Hamilton

(while crossing Dublin bridge)

Quaternions $\mathbb{H}$ $a, b, c, d \in \mathbb{R}$, $a + bi + cj + dk$

where $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$

Note: $ik = i(ij) = -j$, etc.

$\mathbb{H}$ is a division algebra, a vector space with a bilinear product in which division is possible.