Find Euler characteristic for topological spaces.

Extend to topological spaces by <u>triangulation</u>

If one exists for top. space, it is <u>triangulable</u>.

A triangulation is a <u>simplicial complex K</u> and a <u>homeomorphism</u> $K \to T$.

Note: $\chi(S^2) = 2$ and $\chi(T^2) = 0 \Rightarrow S^2 \not\simeq T^2$.

<u>Thm</u> Two triangulable spaces that are homotopy equivalent have the same Euler characteristic. Pf - See Ch 10.

Therefore, since $S^2 \not\simeq T^2$, neither is contractible.

Note: $\chi(pt) = 1$.

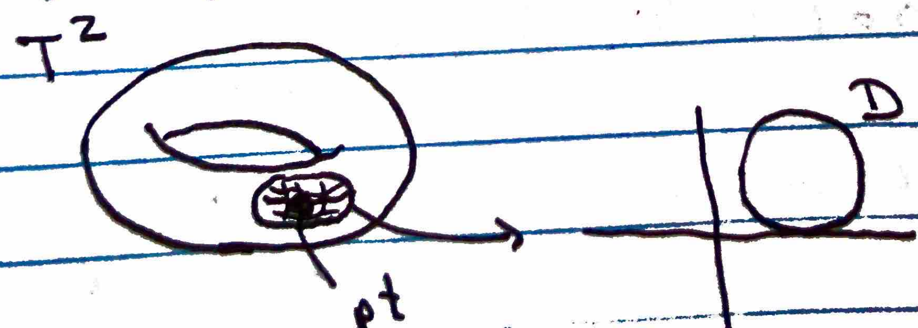
Many spaces that are not homeomorphic may have same χ.

[Recall 6.11 homeomorphic $\Rightarrow$ homotopy equivalent.]

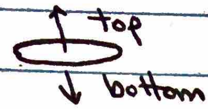
<u>Surfaces</u>

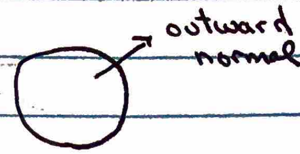
A <u>Surface</u> is a Hausdorff space such that around each pt there is a neighborhood homeomorphic to a disc in $\mathbb{R}^2$.

<u>Ex</u> $\mathbb{R}^2$, T^2, S^2, etc.



Orientable

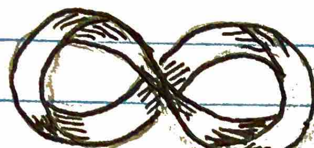
Ex D 

S^2 

inside vs outside

These surfaces are two-sided

Ex Mobius band – one-sided



not orientable

Classification Thm Two triangulable surfaces are triangulable iff they have the same Euler characteristic and orientability.

Properties Let Σ be a (oriented) surface. Then $\chi(\Sigma) \in \mathbb{Z}$.

if $\Sigma \simeq \Sigma'$, then $\chi(\Sigma) = \chi(\Sigma')$.

$\chi(\text{disc}) = 1$

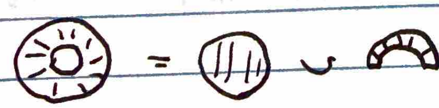
$\chi(\Sigma \amalg \Sigma') = \chi(\Sigma) + \chi(\Sigma')$

If $\Sigma' = \Sigma + \text{strip}$, $\chi(\Sigma') = \chi(\Sigma) - 1$.

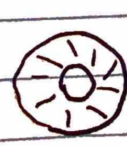
If $\Sigma' = \Sigma + \text{disc}$, $\chi(\Sigma') = \chi(\Sigma) + 1$

Ex 

So, $\chi = 1 + 1 = 2$

Ex  $\chi = 1 - 1 = 0$

Genus $$g = \frac{2 - \chi - c}{2}$$ $c = \#$ boundary components

Ex  $\chi = 0$, $c = 2 \Rightarrow g = 0$.

Closed surfaces, $c = 0$ (no boundary, S^2, T^2, G^2)

Ex T^2 $g = \frac{2 - 0}{2} = 1$ (= # holes)

There are only 5 regular polyhedra

- Consequence of Euler's formula

$$V - E + F = 2$$

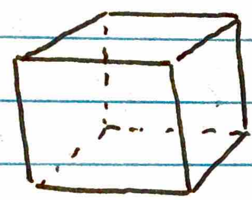
Pf/ n = # sides of each face

m = # faces meeting each vertex

Then

$$E = \frac{1}{2}Fn \quad \text{and} \quad V = \frac{1}{m}Fn$$

Ex Cube



$n = 4$

$m = 3$

$F = 6$

Then, $Fn = 24$ and

$$E = 12, \quad V = 8.$$

Pf (cont'd)

$$V - E + F = 2$$

$$\frac{1}{m}Fn - \frac{1}{2}Fn + F = 2$$

Solve for F:

$$F = \frac{4m}{2n - mn + 2m}$$

Then, $2n - mn + 2m \in \mathbb{Z}^+$ or

Only cases:

$n = 3 \Rightarrow m = 3, 4, 5$ – tetrahedron, octahedron, icosahedron

$n = 4 \Rightarrow m = 3$ – cube

$n = 5 \Rightarrow m = 3$ – dodecahedron

Ch 8 - Homotopy groups

Motivation: Set of homotopy classes of maps $S^1 \to S^1$
$= [S^1, S^1] = \mathbb{Z}$.

Note That $\mathbb{Z}$ is an Abelian group under addition.

groups

A group is a set G with a binary operation $\cdot$ satisfying

1) closure $\forall a,b \in G$, $a \cdot b \in G$.
2) associativity $\forall a,b,c \in G$, $(a\cdot b)\cdot c = a\cdot(b\cdot c)$.
3) identity $\exists e \in G \ni$. $a\cdot e = e\cdot a = a$, $\forall a \in G$.
4) inverse $\exists b \in G \ni$. $a \cdot b = b \cdot a = e$, $\forall a \in G$.

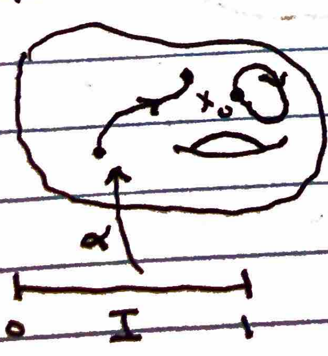
If $a\cdot b = b \cdot a$, $\forall a,b \in G$, Then The group is commutative, or Abelian.

Need an operation between homotopy classes.

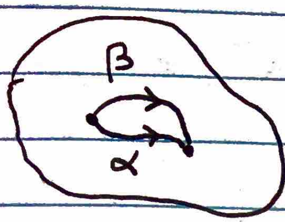
One approach - study loops based at x_0 on space M
= pointed space (M, x_0)
x_0 is called The base point.

Homotopic Paths

α = path in M $\alpha : I \to M$, continuous, from $\alpha(0)$ to $\alpha(1)$



$\leftarrow$ Loop if $\alpha(0) = \alpha(1) = x_0$



Define

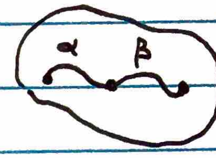
Homotopy $\overset{(t,s)}{H : I \times I} \to M$

where

$$H(t,0) = \alpha(t),\ H(t,1) = \beta(t),\ t \in I$$
$$H(0,s) = \alpha(0) = \beta(0)$$
$$H(1,s) = \alpha(1) = \beta(1)$$

"Multiplication" of paths

Consider paths α, β in $M \ni.$ $\alpha(1) = \beta(0)$

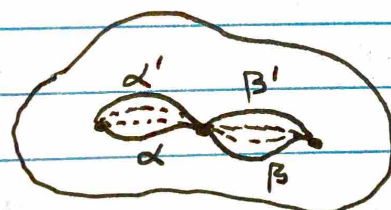


Define

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

So, $(\alpha*\beta)(0) = \alpha(0)$, $(\alpha*\beta)(1) = \beta(1)$

Lemma $\alpha \simeq \alpha'$, $\beta \simeq \beta' \Rightarrow \alpha * \beta \simeq \alpha' * \beta'$



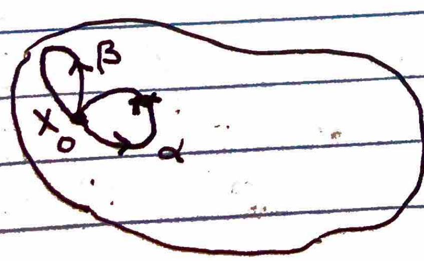
Extends to equivalence classes: $[\alpha] * [\beta] = [\alpha * \beta]$.

For loops on pointed spaces

$x_0 \in M$, base point

Consider loops based at x_0

Multiply loops based at x_0 and get loop based at x_0.

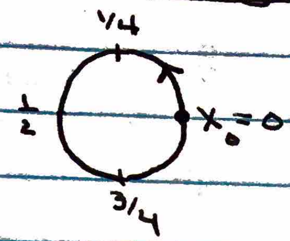


$$\alpha(0) = \alpha(1) = \beta(0) = \beta(1)$$

$$\Rightarrow (\alpha*\beta)(0) = (\alpha*\beta)(1)$$

This gives <u>Fundamental group</u> of M based at x_0: $\pi(M, x_0)$

<u>Ex</u> $M = S^1$



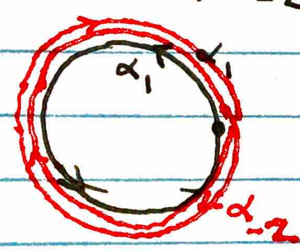
Loops $\alpha_0 = \text{const} = 0$

$\alpha_1(t) = t \pmod 1$

$\alpha_2(t) = 2t \pmod 1$, etc.

$\alpha_{-1}(t) = -t \pmod 1$, ...

Multiplication: $\alpha_1 * \alpha_{-2} \simeq \alpha_{-1}$



Then, we write $[\alpha_1] * [\alpha_{-2}] \simeq [\alpha_{-1}]$.

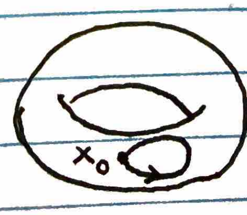
In general,

$$[\alpha_n] * [\alpha_m] = [\alpha_{n+m}]$$

So $\pi(S^1) = \mathbb{Z}$ under addition.

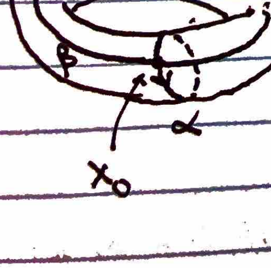
<u>Ex</u> T^2, torus

Types of loops



Trivial loop

Nontrivial loops



Every loop based at x_0 is of form $\alpha^n \beta^m$, $n, m \in \mathbb{Z}$

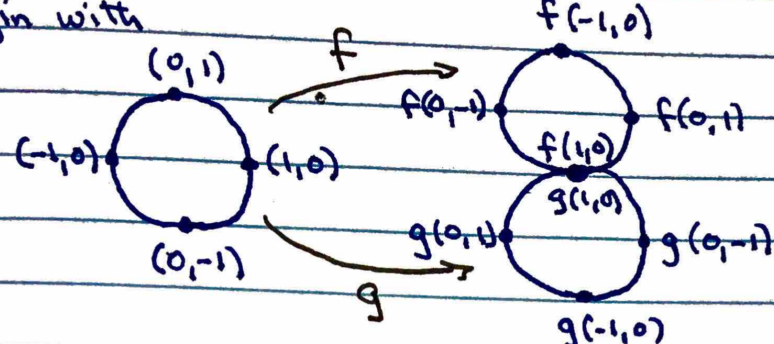
So

$$\pi(T^2) = \mathbb{Z} \times \mathbb{Z}.$$

Crossley Approach Pointed Maps $S^n \to X$

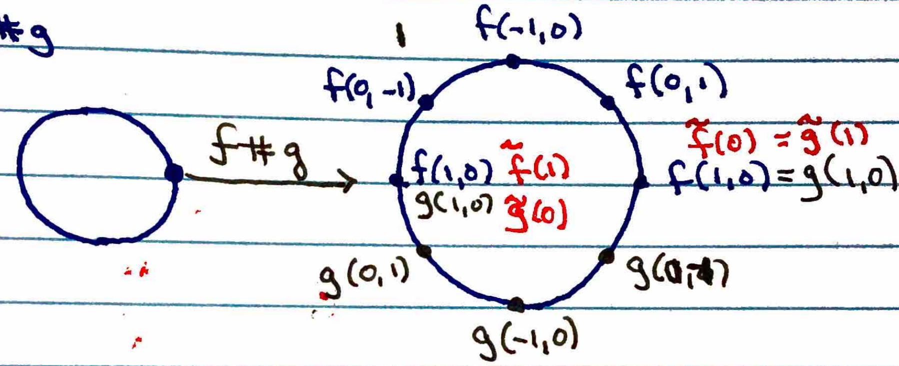
$\underline{n=1}$, $X = S^1$

Begin with



$f, g : S^1 \to S^1$

Define $f \# g$



Lifts

$$(\widetilde{f \# g})(t) = \begin{cases} \tilde{f}(2t), & t \le \frac{1}{2} \\ \tilde{g}(2t-1), & t \ge \frac{1}{2} \end{cases}$$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r] \arrow[dr] & S^1 \\ & S^1 \arrow[u, "f"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[ur, "\widetilde{f \# g}"] \arrow[r] \arrow[dr] & S^1 \\ & S^1 \arrow[u, "f \# g"'] \end{tikzcd}$$

$\tilde{f}(1) = \tilde{g}(0)$

$$\begin{aligned} \Rightarrow \deg(f \# g) &= (\widetilde{f \# g})(1) - (\widetilde{f \# g})(0) \\ &= \tilde{g}(1) - \tilde{f}(0) \\ &= \tilde{g}(1) - \tilde{g}(0) + \tilde{g}(0) - \tilde{f}(0) \\ &= \deg(g) + \tilde{f}(1) - \tilde{f}(0) \\ &= \deg(g) + \deg(f) \quad \in \mathbb{Z} \end{aligned}$$

This is addition in $\mathbb{Z}$ $\Rightarrow$ $\pi(S^1) = \mathbb{Z}$.

Next S^2.