Sphere Eversions
- a journey into differential topology

MAT 451/551, Spring 2020

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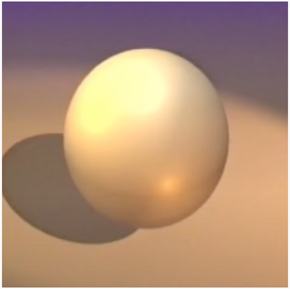
Outline

The Problem

How Do You Turn a Sphere Inside-out?

Sphere Eversion - a continuous deformation, allowing the surface to pass through itself, without puncturing, ripping, creasing, or pinching.



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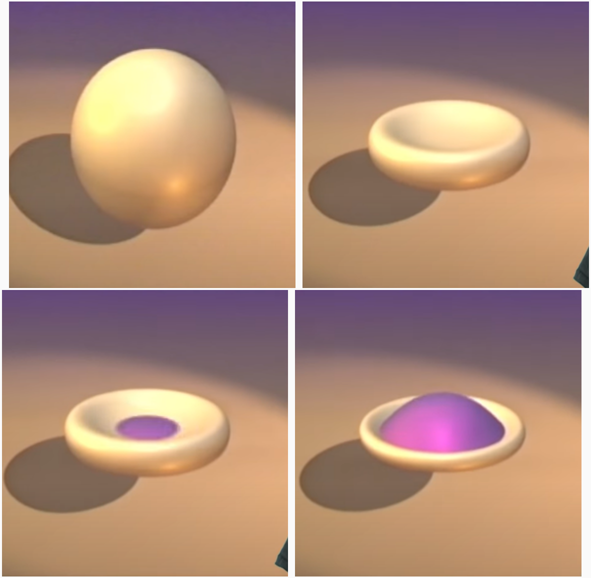
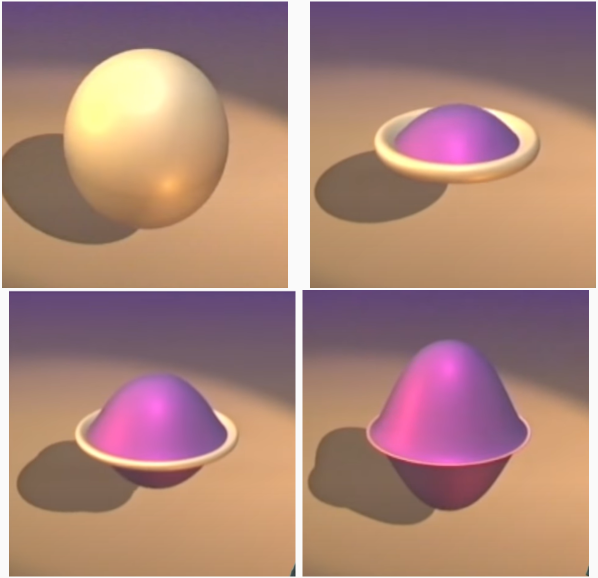


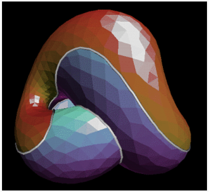
How Do You Turn a Sphere Inside-out?

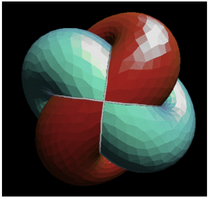


History and Definitions

The Players

- Stephen Smale
- Arnold Shapiro, Tony Phillips, and Bernard Morin
- Charles Pugh, and Nelson Max
- Bill Thurston
- George K. Francis, Stuart Levy and John M. Sullivan
- Arnaud Chéritat





Shapiro's Ideas in Francis and Morin

Definitions:

An **immersion** $f : M^2 \to \mathbb{R}^3$ of a closed surface in space is a smooth map whose differential is everywhere of maximal rank.

A one-parameter deformation, $f_t : M^2 \to \mathbb{R}^3, t \in [0,1]$, is a **regular homotopy** if $t \mapsto f_t$ is continuous in the C^1-topology on the space of immersions.

If f is one-to-one, it is called an **embedding** and a regular homotopy of embeddings is called an **isotopy**.

An **eversion of the sphere** is a regular homotopy from the standard embedding $f_0 : S^2 \to \mathbb{R}^3$ to $f_1(x,y,z) = f_0(-x,-y,-z)$.

Francis and Morin's History

Werner Boy (1879-1914), Hilbert's student, dissertation of 1901. The tangent winding number classifies immersions of the circle in the plane up to regular homotopy.

Whitney and Graustein (1937) found an elegant proof of this theorem.

Smale (1950's), 2D problem. All immersions of the sphere in space are regularly homotopic; therefore, there must exist an eversion of the sphere.

Bott asked for an explicit geometrical construction.

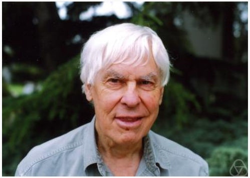
H. Hopf suggested and Kuiper sketched using Boy's surface, an immersed real projective plane.

Arnold Shapiro (1960) explained such a motion to Morin.

Turning a Surface Inside Out - Anthony Phillips

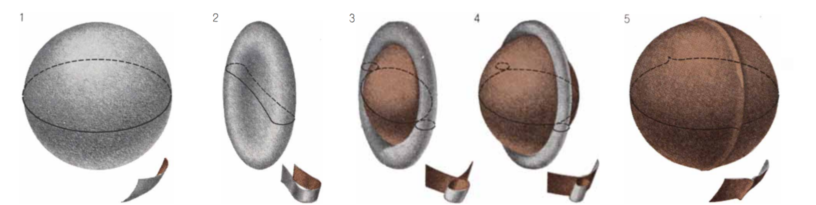
Scientific American, p 112-120, 1966.

- Hilbert - “mathematical theory not perfect unless it can be explained clearly to a layman.”
- Stephen Smale (1930-)
  - Fields Medal, 1966.
  - Wolf Prize, 2007, and others.
  - 1959 - Any two maps from S^2 into $\mathbb{R}^3$ are regularly homotopic.
  - Differential topology.
  - No pictures.
- Intuitively wrong
  - Graduate Advisor, Bott.
- Nobody knew how to do it! Until ...



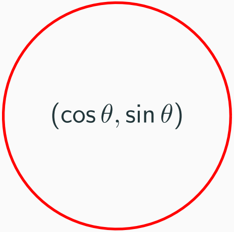
First Visualizations

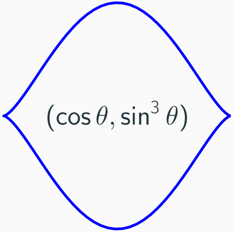
- Nicolaas Kuiper (1920-1994), Dutch
- Arnold Shapiro (1921-1962)
- Told Bernard Morin (1931-2018), 1960.

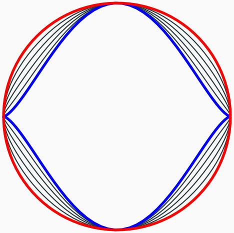


INADMISSIBLE PROCEDURE for turning a sphere inside out entails pushing regions on opposite sides toward the center (*2*) and through each other. The original interior (*color*) begins to protrude on two sides (*3*); these two sides are pulled out to form a sphere (*4 and 5*). When the looped portion of the original surface is pulled through itself, a "crease" is introduced in the surface; this violates a law of differential topology, a discipline of mathematics that is concerned only with smooth surfaces. In this discipline moving a surface through itself is permissible. The ribbons at bottom depict a section of the surface during stages of deformation.

Homotopy







$$(t\cos\theta + (1-t)\cos\theta, t\sin\theta + (1-t)\sin^3\theta)$$

Regular Homotopic

Regular homotopy is a smooth deformation without cuts, tears, or creases.

Hass and Hughes (1985) - orientable 2-manifold of genus g immersed in R^3 has 4^g regular homotopy classes.

Ex - Sphere, $g = 0$ so only one class - Smale.

- Morin's led to Nelson Max's first computer graphics.
- Optimized in *Optiverse* - minimized bending energy.
- Dirac's Belt Trick - `http://virtualmathmuseum.org/Surface/dirac-belt/DiracBelt.html` used in *Outside-In*
- For a torus expect 4 classes.

Shapiro’s Eversion

Shapiro's Eversion - 1966 Science American

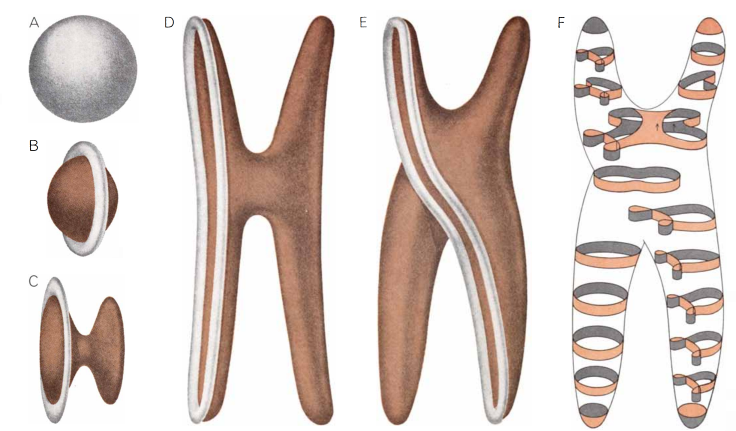
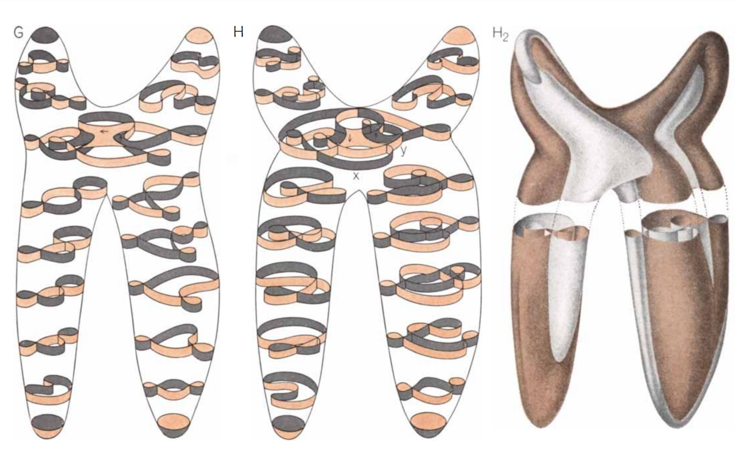


Figure 1: 1966 Turning a Surface Inside Out.

Shapiro's Eversion - 1966 Science American



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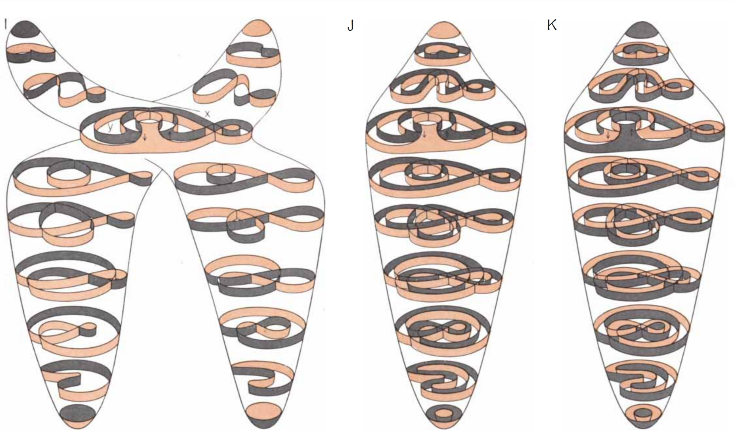

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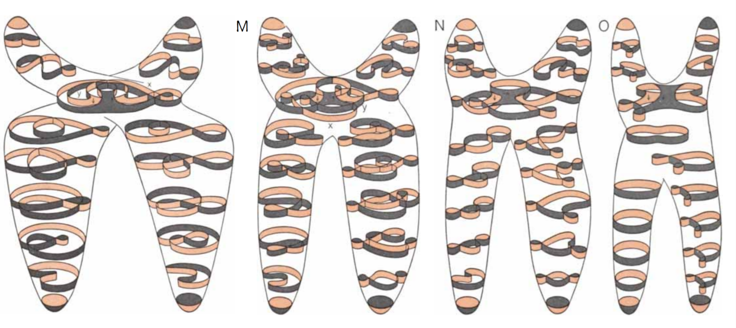

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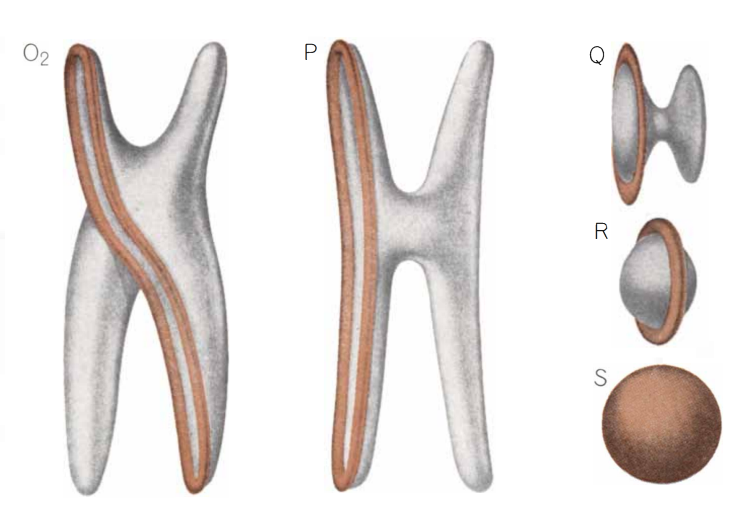

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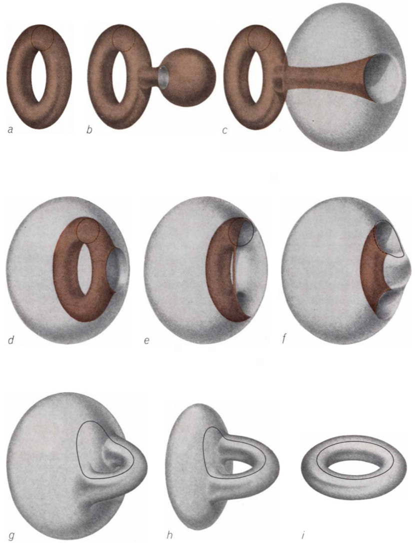
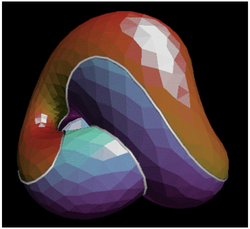


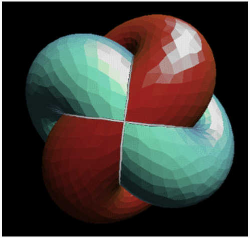
TURNING A TORUS INSIDE OUT involves the regular homotopy for everting the sphere. The torus is depicted with a circle designating its meridian (*a*). A small sphere is extruded from the torus (*b*) and everted (*c*). Then the inside-out sphere is enlarged until it engulfs the torus (*d*). Next the tube leading from the outside of the everted sphere to the inside of the torus is enlarged and the torus is pulled through it (“*e*,” “*f*” *and* “*g*”). Finally sphere is shrunk (“*h*” *and* “*i*”). In the process the meridian has become a latitude.

Boy’s and Morin’s Surfaces

Halfway Models

- Spherical surface, half inside-out,
- Simplify the halfway-model to a sphere,
- Get eversion by performing this simplification first backwards, then forwards after applying symmetry.
- Shapiro and Phillips used Boy’s surface (top figure).
- Pugh and Max used a Morin surface.
- It has four lobes: two from inside, two from outside (bottom figure).
- A 90^o rotation: switches the sides.





Boy’s Surface

```
In[•]:= w = r * Exp[I * t];
        g1 = -3 / 2 * Im[w * (1 - w^4) / (w^6 + Sqrt[5] * w^3 - 1)];
        g2 = -3 / 2 * Re[w * (1 + w^4) / (w^6 + Sqrt[5] * w^3 - 1)];
        g3 = Im[(1 + w^6) / (w^6 + Sqrt[5] * w^3 - 1)] - 1 / 2;

In[•]:= x = g1 / (g1^2 + g2^2 + g3^2);
        y = g2 / (g1^2 + g2^2 + g3^2);
        z = g3 / (g1^2 + g2^2 + g3^2);

In[•]:= ParametricPlot3D[{x, y, z}, {t, 0, 2 π}, {r, 0, 1}]

Out[•]=
```

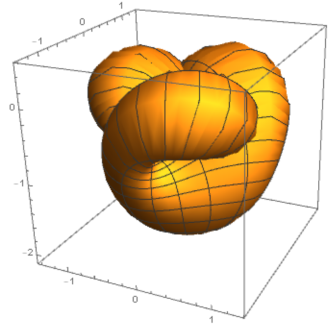


Figure 2: Boy’s Surface Parametrization in Mathematica

Boys.nb * - Wolfram Mathematica 12.0

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```
In[1]:= n = 2;
        k = 1;
        K = Cos[u] / (Sqrt[2] - k * Sin[2 * u] * Sin[n * v]);
        x = K * (2 / (n - 1) * Cos[u] * Cos[(n - 1) * v] + Sqrt[2] * Sin[u] * Cos[v])
        y = K * (2 / (n - 1) * Cos[u] * Sin[(n - 1) * v] - Sqrt[2] * Sin[u] * Sin[v])
        z = K * Cos[u]
```

Out[4]= $\dfrac{\mathrm{Cos}[u]\left(2\,\mathrm{Cos}[u]\,\mathrm{Cos}[v]+\sqrt{2}\,\mathrm{Cos}[v]\,\mathrm{Sin}[u]\right)}{\sqrt{2}-\mathrm{Sin}[2u]\,\mathrm{Sin}[2v]}$

Out[5]= $\dfrac{\mathrm{Cos}[u]\left(2\,\mathrm{Cos}[u]\,\mathrm{Sin}[v]-\sqrt{2}\,\mathrm{Sin}[u]\,\mathrm{Sin}[v]\right)}{\sqrt{2}-\mathrm{Sin}[2u]\,\mathrm{Sin}[2v]}$

Out[6]= $\dfrac{\mathrm{Cos}[u]^2}{\sqrt{2}-\mathrm{Sin}[2u]\,\mathrm{Sin}[2v]}$

```
In[7]:= ParametricPlot3D[{x, y, z}, {u, 0, 2 * π}, {v, 0, 2 * π}]
```

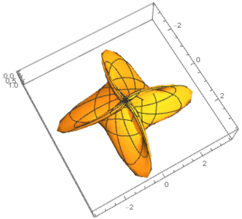


Figure 3: Morin’s Surface Parametrization in Mathematica

First Visualization

Turning a Sphere Inside Out

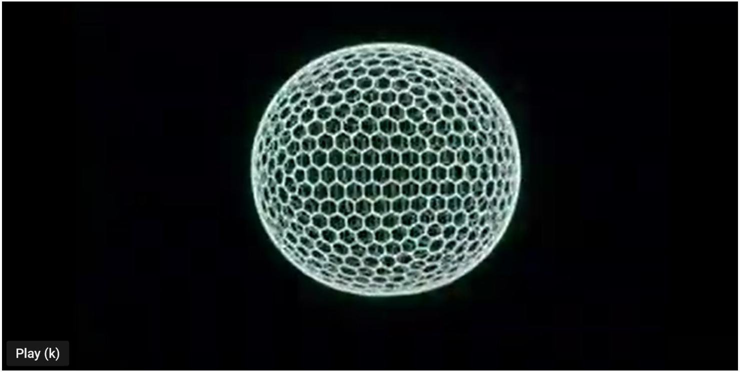
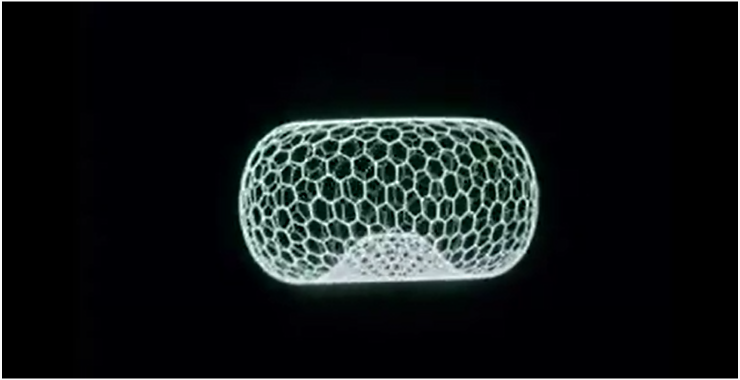


Figure 4: 1975-7 Nelson Max and Bernard Morin
https://archive.org/details/turningasphereinsideout

Turning a Sphere Inside Out



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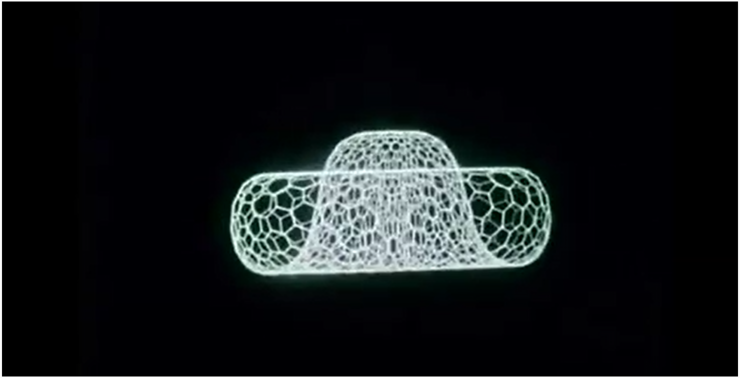

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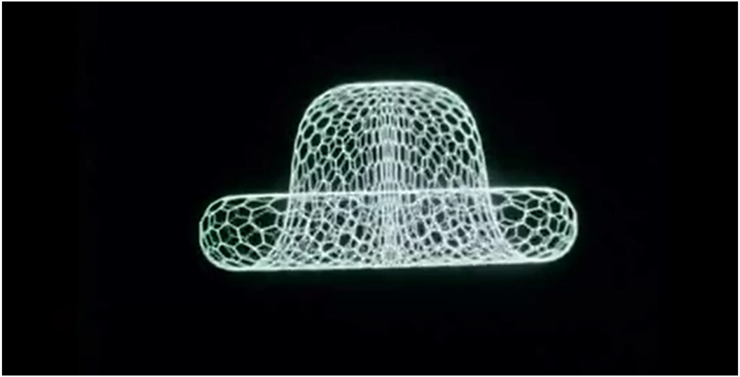

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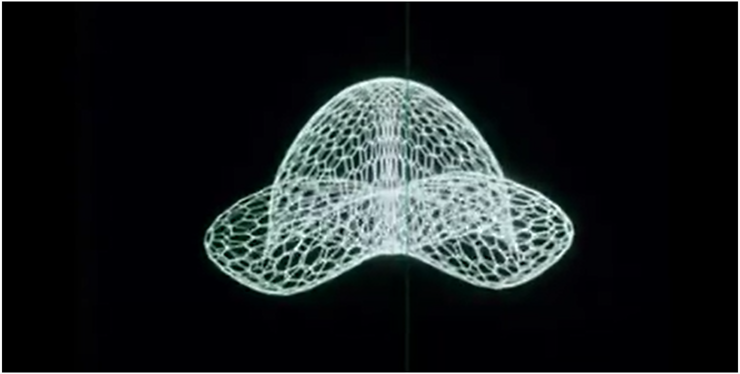

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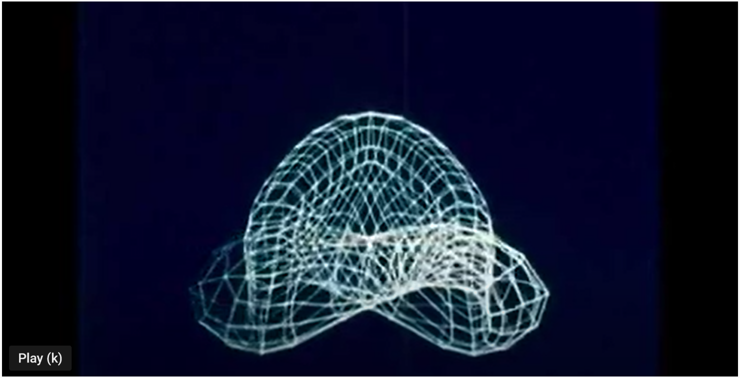

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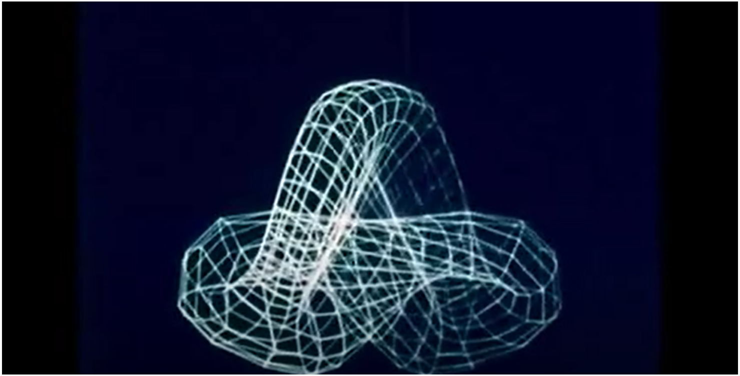

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Other Eversions

Other Sphere Eversions

- Bill Thurston gave new proof of Smale's original theorem.
  - Take any homotopy between two surfaces, and make it regular (take out pinching or creasing).
  - Add corrugations to make surface more flexible.
  - Demonstrated in 1994, "Outside In," Geometry Center.
- Minimax - process of energy minimization.
  - Sullivan, Levy, and Francis, *The Optiverse* 1998.

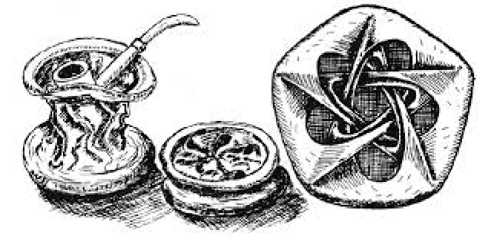
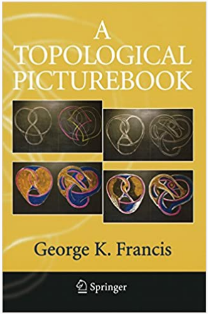


Figure 5: Tobacco pouch inversions, G. Francis.

Video Links

- Turning the Sphere Inside Out (1976) - First polygonal physical model animation, Nelson Max.
- Outside In, Thurston, The Geometry Center.
- The Optiverse, (1998) John Sullivan, George Francis, and Stuart Levy.



Torus Eversion

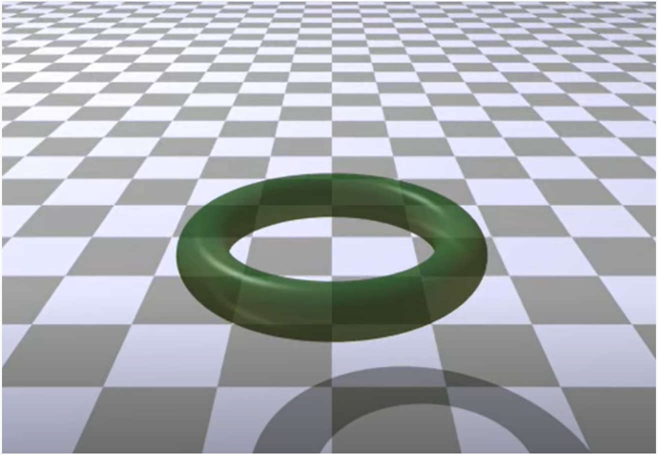


Figure 6: Start with a Torus.

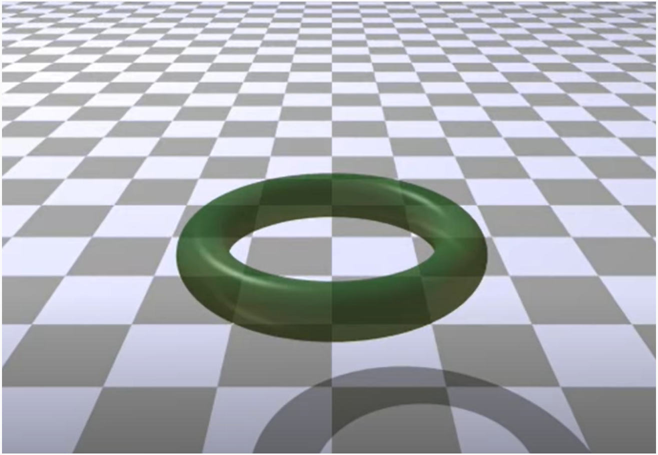
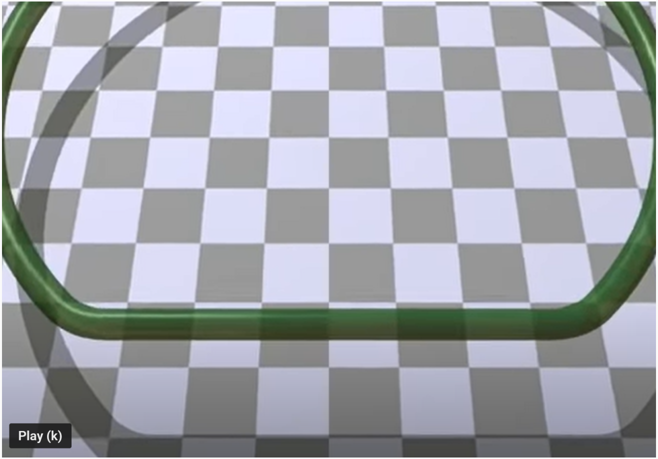


Figure 7: *`https://www.youtube.com/watch?v=kQcy5DvpvlM`*.

Torus Eversion - Arnaud Chéritat



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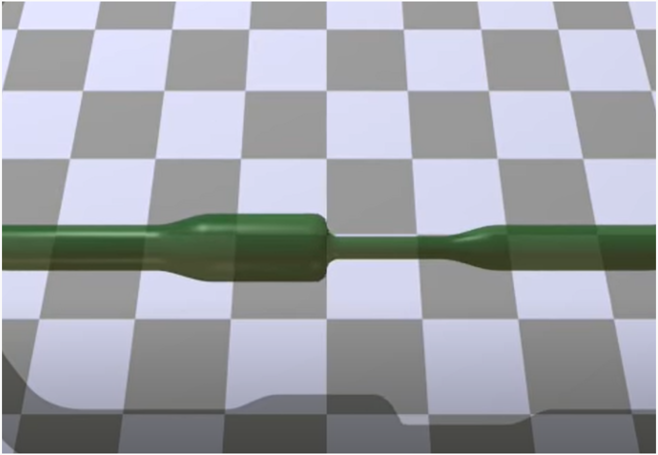

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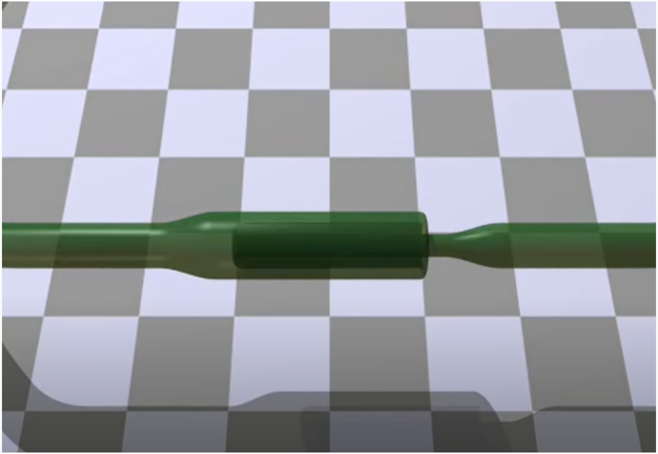

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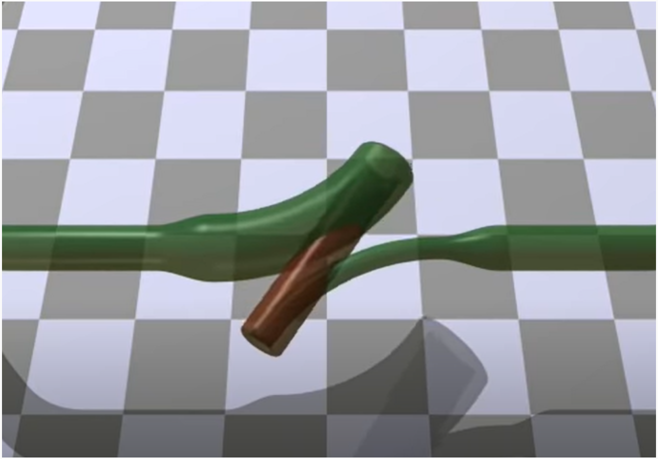

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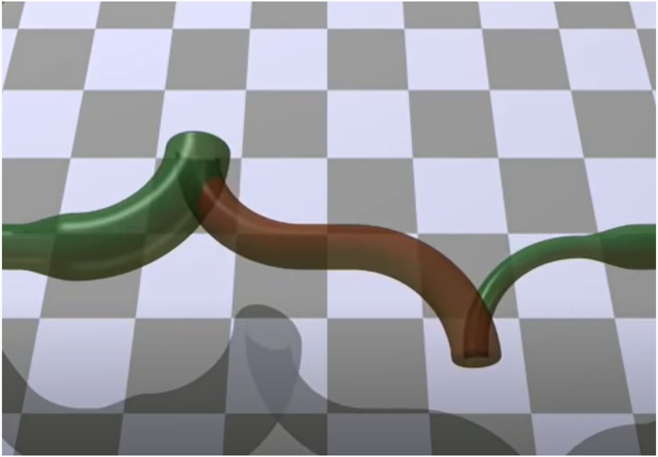

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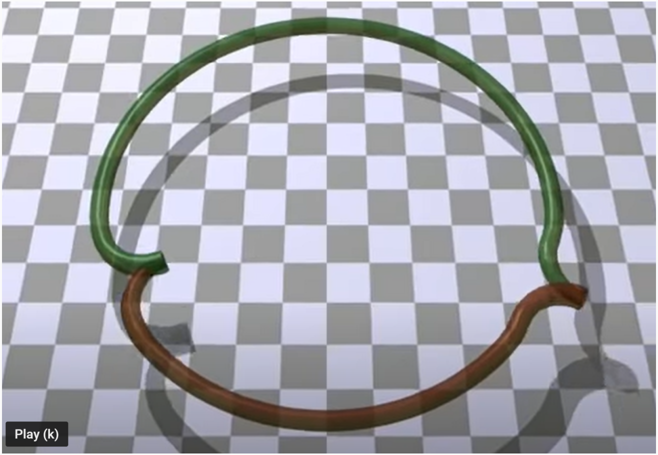

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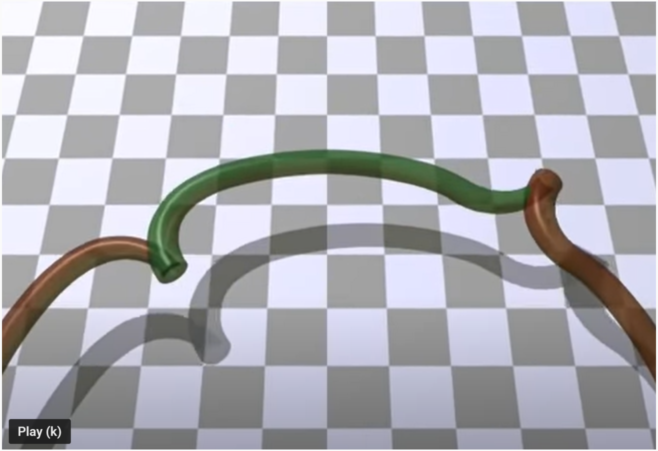

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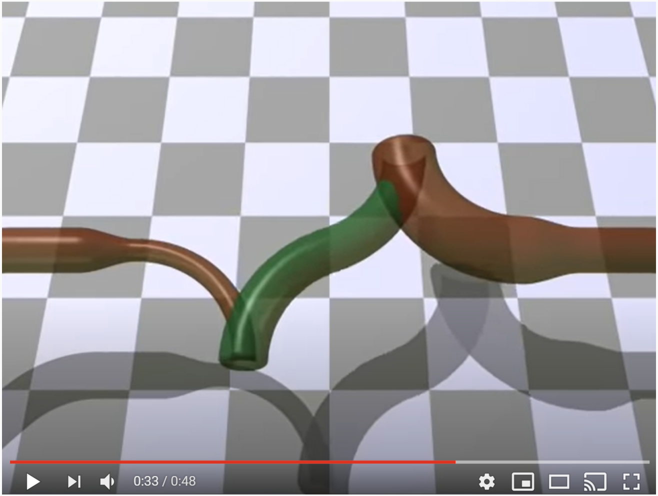

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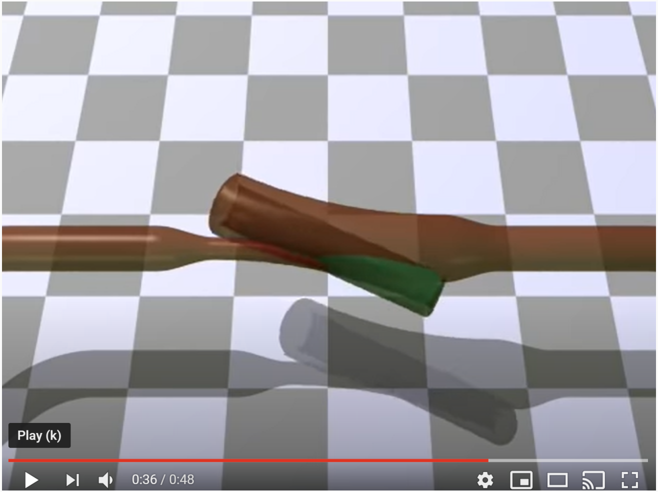

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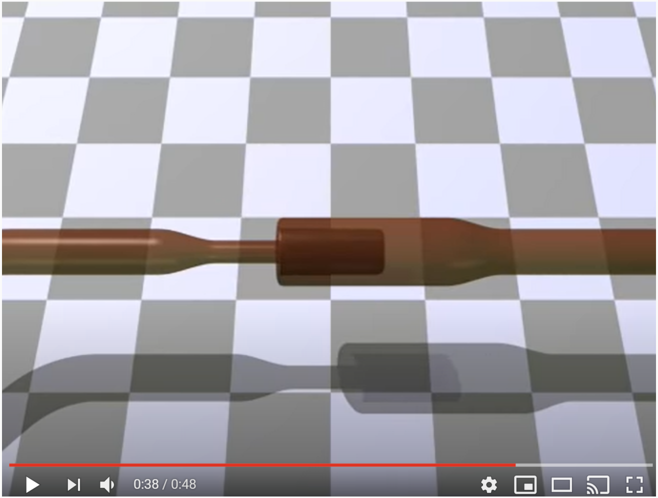

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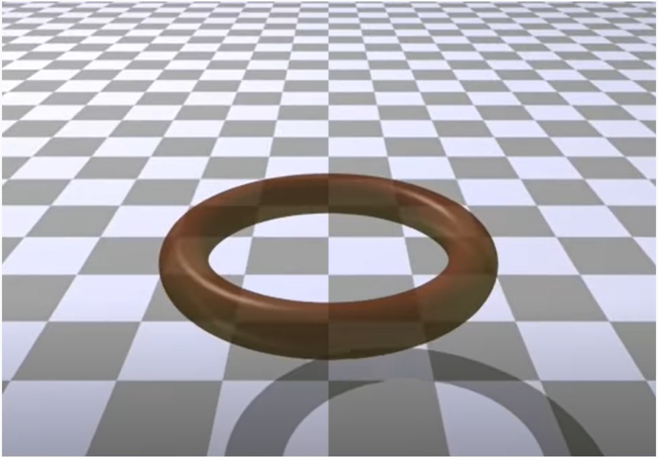

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Double Klein Bottle

Explained by the Mathologer

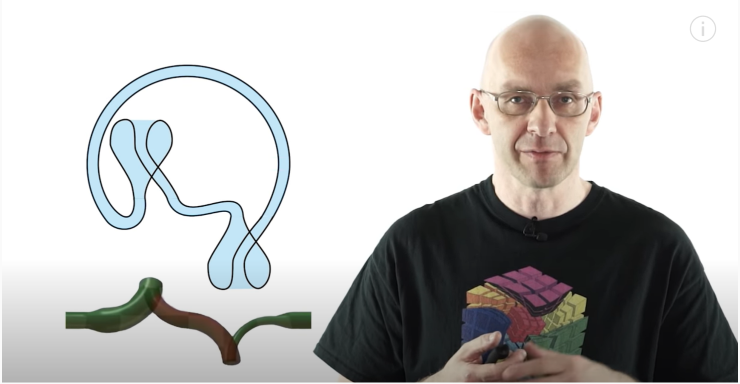

Figure 8: Double Klein Bottle Another Version.

Schematics

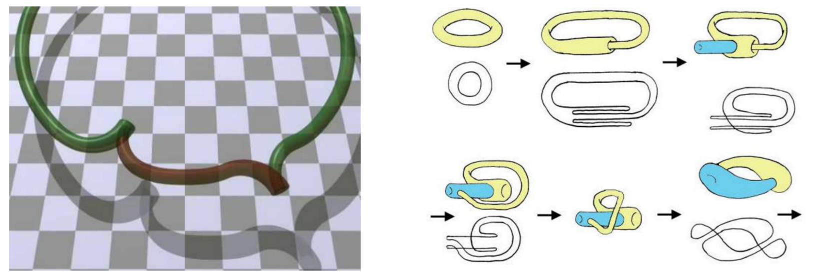

Figure 9: Torus Eversion Schematics.

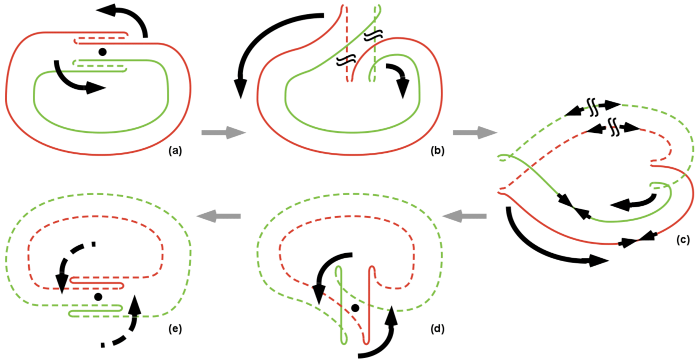

Figure 9: *Turning a torus inside out: (a) → (e) schematic view of two parallels (equatorials).*

Figure 9: Torus Eversion Schematics.

Summary

References

- Apéry, F., 1992, An Algebraic Halfway Model For The Eversion Of The Sphere, *Tóhoku Math. J.* **44**, 103-150.
- Bednorz, A. and Bednorz, W., 2019, Analytic sphere eversion using ruled surfaces, *Differential Geometry and its Applications* **64**, 59-79.
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