Ch7 - Euler's Number / Euler's Characteristic: χ

Euler (1707-1783) in letter to Goldbach, 1950

3D Polyhedra (convex)

$\boxed{V - E + F = 2}$

greeks - Platonic Solids, Regular Polyhedra

congruent (identical in size, shape)

regular (angles, sides are equal)

polygonal faces

Same number of faces meeting each vertex.

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Vertices	4	8	6	12	20
Edges	6	12	12	30	30
Faces	4	6	8	20	12

self-dual   dual   dual

Verify Euler's Formula: $\boxed{\chi = V - E + F}$

$4 - 6 + 4 = 2$

$8 - 12 + 6 = 2$

$6 - 12 + 8 = 2$

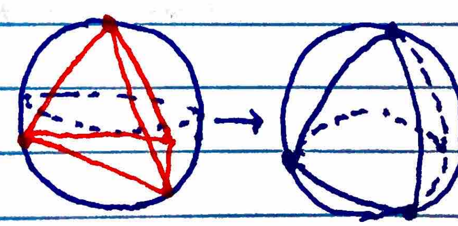
$12 - 30 + 20 = 2$

$20 - 30 + 12 = ?$

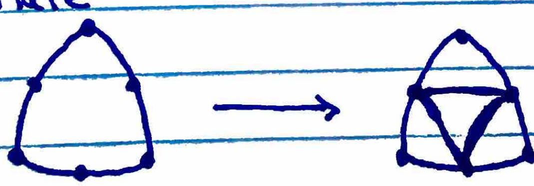
Circumscribe with sphere

project edges to surface

triangulate



$\chi = 2$ for sphere



add vertices - 3

gives edges - 3

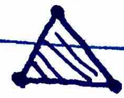
net 0

add lines - 3

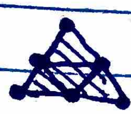
gives faces - 3

net 0

Planar Graphs



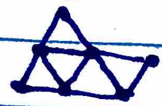
$V+E+F=3-3+1=1$



$V-E+F=6-9+4=1$

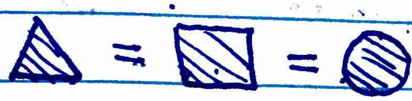
Subdivision, χ invariant

Add a triangle



$V-E+F=7-11+5=1$

These deformations preserve χ



$\chi=1$

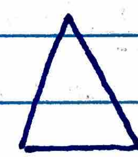
7.1 Simplicial Complexes

- built from simplices (pts, lines, triangles)

0-simplices points

1-simplices lines

2-simplices triangles



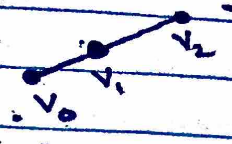
Simplicial circle

K-Simplex is a set of $K+1$ vertices in $\mathbb{R}^n$ and is the smallest convex subspace containing the vertices

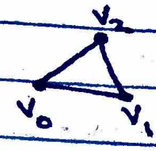
Ex **1-simplex** has 2 vertices + line connecting

v_0 v_1

Degeneracy 2 simplex 3 vertices



vs



Need to avoid first case!

Want $v_0, v_1, \ldots, v_k$ in general position, i.e.,

$v_1 - v_0, v_2 - v_1, \ldots, v_k - v_{k-1}$ have to be

linearly independent

Then, a K-simplex is the smallest convex subspace of $\mathbb{R}^n$ containing the set of $k+1$ vertices in general position.

Denote as $[v_0, \ldots, v_k]$ consisting of all linear combinations

$$t_0 v_0 + t_1 v_1 + \cdots + t_k v_k, \quad t_j \in [0,1], \; 0 \le j \le k.$$

Any nonempty subset of vertices $\rightarrow$ subsimplex

Its a face if omit only one vertex

K-simplex has $k+1$ faces

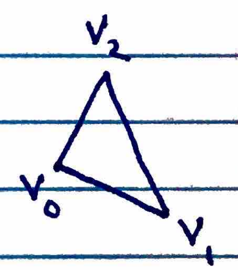
$2^{k+1} - 1$ subsimplices

Ex $[v_0, v_1, v_2]$

$[v_0]$ $[v_0, v_1]$ $[v_0, v_1, v_2]$

$[v_1]$ $[v_0, v_2]$

$[v_2]$ $[v_1, v_2]$



$k = 2$, $k+1 = 3$ faces, $2^3 - 1 = 7$ subsimplices

Union of faces = boundary

Barycentric Coordinates: $(1,0,0)$ $(1,1,0)$ $(1,1,1)$

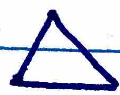
$(0,1,0)$ $(1,0,1)$

$(0,0,1)$ $(0,1,1)$

A simplicial complex $K \subset \mathbb{R}^n$ with list of simplices such that

1. Union of simplices $\subset K$ and each pt in K lies in the interior of only one simplex.
2. Every face of every simplex in list is also in list.

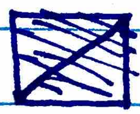
Ex Simplicial Circle – 1D



3 pts, 3 lines

$\chi = 3 - 3 = 0$

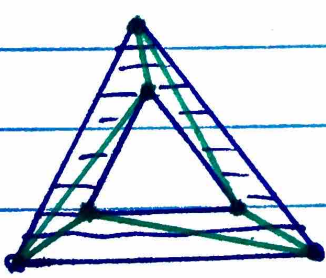
Ex Simplicial Square – 2D



4 pts, 5 lines, 2 2-simplices
0-simplices 1-simplices

$\chi = 4 - 5 + 2 = 1$

Ex Simplicial Annulus – 2D



6 0-simplices

12 1-simplices

6 2-simplices

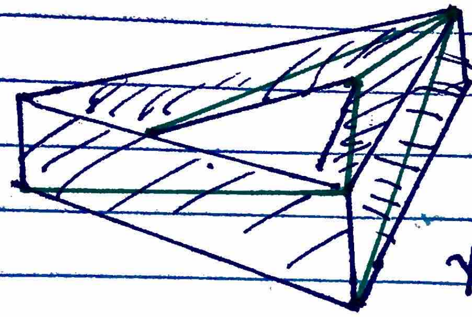
$\chi = 6 - 12 + 6$

Euler number T n-dim simplicial complex

i_k = # k-simplices in T

$$\chi(T) = i_0 - i_1 + i_2 - \cdots + (-1)^n i_n$$

Ex Simplicial Torus - 3D



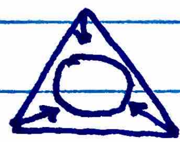
9 0-simplices

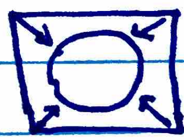
27 1-simplices

18 2-simplices

$\chi = 9 - 27 + 18 = 0$

Extend to topological spaces using homeomorphisms
- triangulations

Ex  $\chi(S^1) = \chi(T) = 0$



Other Spaces

sphere  $\chi = 2$

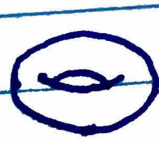
ball 

disk 

closed interval 

point 

$\chi = 1$ (ball, disk, closed interval, point)

Torus 

Double Torus