Degree of $f: S^1 \to S^1$

Start with $\pi: [0,1] \to S^1$

Form composition $g = f \circ \pi$ noting $g(0) = g(1)$

$$\begin{tikzcd} & S^1 \\ {[0,1]} \arrow[ur, "g"] \arrow[r, "\pi"'] & S^1 \arrow[u, "f"'] \end{tikzcd}$$

π is continuous surjection

Lift g to $\tilde{g}$:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, "\tilde{g}"] \arrow[r, "g"] \arrow[dr, "\pi"'] & S^1 \\ & S^1 \arrow[u, "f"'] \end{tikzcd}$$

Note:
$e\tilde{g}(0) = e\tilde{g}(1)$

Recall, $e(x) = (\cos 2\pi x, \sin 2\pi x)$

So $\quad e(t) = e(s)$ iff $t - s \in \mathbb{Z}$

Therefore, $\quad \tilde{g}(1) - \tilde{g}(0) \in \mathbb{Z}$

For each f, $\quad \deg(f) = \tilde{g}(1) - \tilde{g}(0) \in \mathbb{Z}$ (also winding number)

Since $\tilde{g}$ is unique depending only on $\tilde{g}(0)$,

$e\tilde{g}(0) = g(0) = e\bar{g}(0) \qquad$ [$\bar{g}$ another lift]

$\Rightarrow \tilde{g}(0) = \bar{g}(0) + c, \quad c \in \mathbb{Z}$

$x \mapsto \bar{g}(x) + c$ lift of g, agrees w/ $\tilde{g}$ at 0.

Uniqueness $\Rightarrow \tilde{g} = \bar{g} + c$ and

$\tilde{g}(1) - \tilde{g}(0) = \bar{g}(1) - \bar{g}(0) \Rightarrow$ same degree of f
indep of lift.

6.24 <u>Constant Function</u> $f: S^1 \to S^1 \Rightarrow \deg(f) = 0$

Let $g = \pi \circ f = \text{const.}$

Pick $\tilde{g} = \text{const}$

If $x \in \mathbb{R}$, $e(x) = g(0)$

Let $\tilde{g}(t) = x$, $\forall t \in [0,1]$

Then $e\tilde{g}(t) = e(x) = g(t)$

So, $eg(1) - eg(0) = g(1) - g(0) = \underline{0 = \deg f}$

6.25 <u>Identity</u> $f: S^1 \to S^1$

$g(t) = (\cos 2\pi t, \sin 2\pi t)$

$\tilde{g}(t) = t$

$e\tilde{g}(t) = e = g(t)$

$\tilde{g}(1) - \tilde{g}(0) = 1 - 0 = \underline{1 = \deg f}$

<u>Goal – Homotopic maps have the same degree</u>

6.28 <u>Homotopy Lifting</u> Consider $f, g: [0,1] \to S^1$, cont.

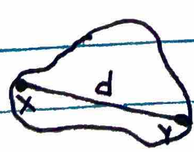
w/ Homotopy $F: [0,1] \times [0,1] \to S^1$

$\exists!\ \tilde{F}: [0,1]^2 \to \mathbb{R}$, cont. where $e\tilde{F}(s,t) = F(s,t)$, $\forall s,t \in [0,1]$

and $\tilde{F}(0,0) = x$.

$$\begin{tikzcd}
 & R \arrow[d, "e"] \\
{[0,1]^2} \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & S^1
\end{tikzcd}$$

Proof similar to before – need domain splitting and Lebesgue Lemma



diameter of set $d(x,y) < d$

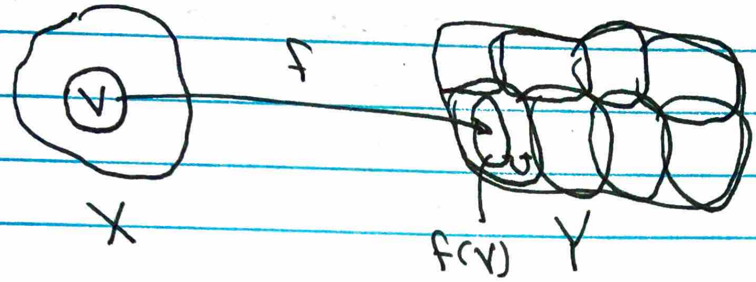
Domain Splitting

$f: X \to Y$, X (compact) $\subset \mathbb{R}^n$

$\mathcal{U}$ - open cover for Y

Then $\exists\, \delta > 0 \ni.$ whenever $V \subset X$ w/ diam $< \delta$

$f(V) \subset U \in \mathcal{U}$.



Premages of U's $\longrightarrow$ open cover for X since f cont.

V is in one of these. $\Rightarrow$ $f(V) \subset U \in \mathcal{U}$

Find δ from Lebesgue Lemma

Lebesgue Lemma

given compact subspace $X \subset \mathbb{R}^n$ and open cover $\mathcal{U}$ of X $\exists\, \delta > 0 \ni.$ any $U \in$ ~~$\mathcal{U}$~~ with diameter $< \delta$ is in a set in $\mathcal{U}$.

Pf/ X compact $\Rightarrow$ can refine $\mathcal{U}$: $\{U_1, \ldots, U_n\}$

Define $f_i : X \to \mathbb{R}$ $\quad f(x_i) = \begin{cases} \text{largest } r \ni. B_r(x) \subset U_i \\ 0, \quad x \notin U_i \end{cases}$

$f(x) = \max\{f_i(x) \mid 1 \leq i \leq n\}$ continuous

gives largest $r \ni.$ $B_r(x) \subset U_j$ for some j

If $\exists\, \delta > 0 \ni.$ $f(x) \geq \delta, \forall x$ Then every open ball with $r < \delta$ is in some U_i.

Since X compact, $f(x) > 0\ \forall x$ Prop 4.7 $\Rightarrow$ Im f Compact Heine-Borel $\Rightarrow$ Im f closed & it complement is open and contains 0. So, $(-\delta, \delta)$ is complement and $f(x) \geq \delta, \forall x$.

Return to 6.28 – Homotopy Lifting

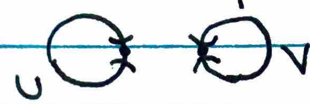
Pf/ Recall $F : [0,1]^2 \to S^1$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]^2} \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & S^1 \end{tikzcd}$$

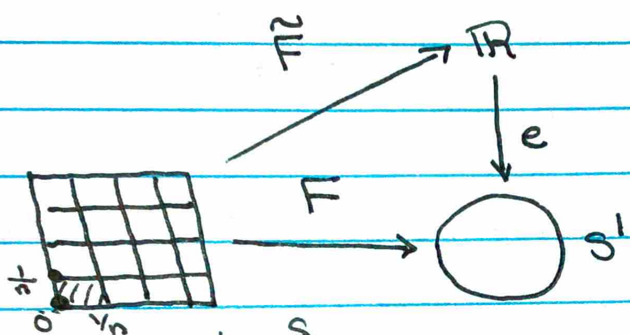
Cover S^1 with U, V:

$U = S^1 - \{(1,0)\}$

$V = S^1 - \{(0,1)\}$



Find $\delta > 0 \ni$ subsets $\subset [0,1] \times [0,1]$ with diameter $< \delta$ are mapped to U or V.



$n \times n$ grid with $\frac{1}{n} < \frac{\delta}{\sqrt{2}} < \delta$

$\tilde{F}(0,0) = x \in \mathbb{R}$ determines component $e^{-1}(U)$ or $e^{-1}(V)$

Define $\tilde{F}$ on $[0, \frac{1}{n}] \times [0, \frac{1}{n}]$ — homeomorphism w/ U or V

giving $\tilde{F}(0, \frac{1}{n})$

Do same for $[0, \frac{1}{n}] \times [\frac{1}{n}, \frac{2}{n}]$

gives $\tilde{F}$ on path $[0, \frac{1}{n}] \times \frac{1}{n}$ based on $(0, \frac{1}{n})$

but this was also done in last step.

Uniqueness of path lifting they agree.

Continue process up 1st strip and then rest of grid.

Can show $\tilde{F}$ is unique.

6.31 $f,g: S^1 \to S^1$ homotopic $\Rightarrow \deg(f) = \deg(g)$

Pf/ Let $H: S^1 \times [0,1] \longrightarrow S^1$ be homotopy between f, g

Lift to $\tilde{H}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]\times[0,1]} \arrow[ur, "\tilde{H}"] \arrow[r, "H\circ\pi"] \arrow[dr, "\pi"'] & S^1 \\ & S^1\times[0,1] \arrow[u, "H"'] \end{tikzcd}$$

Restrict $\tilde{H}$ to $[0,1]\times\{0\}$ gives lift for f

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, "\tilde{f}"] \arrow[r, "f\circ\pi"] \arrow[dr, "\pi"'] & S^1 \\ & S^1 \arrow[u, "f"'] \end{tikzcd}$$

So, $\quad \deg(f) = \tilde{H}(1,0) - \tilde{H}(0,0)$

Similarly $\deg(g) = \tilde{H}(1,1) - \tilde{H}(0,1)$

Consider $D(t) = \tilde{H}(1,t) - \tilde{H}(0,t)$, continuous

$D: [0,1] \longrightarrow \mathbb{Z}$

So, $\deg(f) = D(0)$, $\quad \deg(g) = D(1)$

By Lemma 4.18 [D: Connected $\to$ Discrete, cont. $\Rightarrow D = $ const.]

D is constant map. So, $\deg(f) = \deg(g)$.

Cor 6.32 S^1 is not contractible

Pf/ Let $f: S^1 \to \{0\}$
$g: \{0\} \to S^1$ be homotopy equivalences

Then $(g \circ f)(x,y) = g(0), \ \forall x,y \in S^1$
this is constant function, $\deg = 0$

To be homotopy equiv., $g \circ f = 1_{S^1}$
but identity has $\deg = 1$

If $\deg(g \circ f) \neq \deg(1_{S^1})$ then $g \circ f \not\simeq 1_{S^1}$ by Cor 6.31

6.33 If $f,g: S^1 \to S^1$ have $\deg(f) = \deg(g)$, Then f,g homotopic.
[Converse to 6.31]

Let $(f \circ \pi)(0) = (g \circ \pi)(0)$ [otherwise need lemma 6.34]
Can lift to $\tilde{f}, \tilde{g}$ such that $\tilde{f}(0) = \tilde{g}(0)$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ {[0,1]} \arrow[ur, "\tilde{f},\tilde{g}"] \arrow[r, "f\circ\pi"', "g\circ\pi"] \arrow[dr, "\pi"'] & S^1 \\ & S^1 \arrow[u, "f,g"'] \end{tikzcd} \quad \xrightarrow[\text{Lift}]{\text{Homotopy}} \quad \begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]^2} \arrow[ur, "\tilde{H}"] \arrow[r, "H"'] & S^1 \end{tikzcd}$$

$\deg f = \tilde{f}(1) - \tilde{f}(0)$
$\Rightarrow \tilde{f}(1) = \deg f + \tilde{f}(0)$
$= \deg g + \tilde{g}(0)$
$= \tilde{g}(1)$

Define
$\tilde{H}(s,t) = t\tilde{f}(s) + (1-t)\tilde{g}(s)$
$\tilde{H}(1,t) - \tilde{H}(0,t) = \deg f \in \mathbb{Z}$

$e \circ \tilde{H} = H$ homotopy between f,g

Cor 6.35 $[S^1, S^1] = \mathbb{Z}$

$$\left.\begin{array}{l} \text{Every map has integer degree} \\ \text{Homotopic maps have same degree} \\ \text{Nonhomotopic maps have diff. degrees} \end{array}\right\} \Rightarrow [S^1, S^1] \subset \mathbb{Z}$$

But - For any $n \in \mathbb{Z}$, $z \mapsto z^n$, $(z \in \mathbb{C}, |z|=1)$ has degree n.
So, $[S^1, S^1] = \mathbb{Z}$.