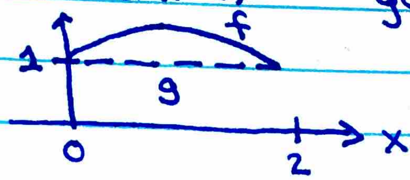
Ch 6 - Crossley - Homotopy Review March 24th

Topology is about continuous maps between topological spaces.
When are functions equivalent?
When one can be deformed into the other.

Ex $f:[0,2]\to\mathbb{R}$ $g:[0,2]\to\mathbb{R}$
$f(x)=1+x^2(x-2)^2$ $g(x)=1$



Deform $f(x)$ to $g(x)$ using $H:[0,2]\times[0,1]\to\mathbb{R}$, continuous, such that
$H(x,0)=f(x)$,
$H(x,1)=g(x)$.

Let $H(x,t)=1+(1-t)\,x^2(x-2)^2$.

Def The maps $f,g:S\to T$ are homotopic if there exist a continous $F:S\times[0,1]\to T$ $\ni$
$F(s,0)=f(s)$, $\forall s\in S$ and
$F(s,1)=g(s)$, $\forall s\in S$.
F is called a homotopy between f and g. $f\simeq g$

Ex $f:S^1\to\mathbb{R}^2$, $f(x,y)=(x,y)$ inclusion
$g:S^1\to\mathbb{R}^2$, $g(x,y)=(0,0)$ constant
$F((x,y),t)=(1-t)f(x,y)$

Ex $f, g : \mathbb{R} \to \mathbb{R}$, continuous

Define $F : \mathbb{R} \times I \to \mathbb{R}$ by $F(x,t) = (1-t)f(x) + t g(x)$

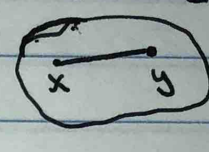
where $I = [0,1]$.

Prop. T convex, S top. space. For $f, g : S \to T$, $f \simeq g$.

Def. $T \subset \mathbb{R}^n$ is convex if given $x, y \in T$, there is a line from x to y in T. $[tx + (1-t)y \in T,\ t \in I.]$

Pf/ Define $F : S \times I \to T$

by $F(x,t) = t f(x) + (1-t) g(x)$.



Equivalence Classes

1) Reflexive $f \simeq f$.

2) Symmetric $f \simeq g \Rightarrow g \simeq f$.

3) Transitive $f \simeq g,\ g \simeq h \Rightarrow f \simeq h$.

Proofs in Lemmas 6.6, 6.7, 6.8

Homotopy Classes $[S, T]$ = set of classes of maps $S \to T$ that are homotopic

$[\mathbb{R}, \mathbb{R}]$ one element

$[S^1, S^1]$ one element for each integer

Homotopy Equivalent

S, T are homotopy equivalent if there exists continuous maps f, g,

$f : S \to T$

$g : T \to S$

$S \xrightarrow{f} T \xrightarrow{g} S$

such that $g \circ f \simeq id_S \equiv 1_S$ and $f \circ g \simeq id_T$.

Then we write $S \simeq T$.

<u>Lemma</u> Let $S \simeq T$ and Q any topological space.
Then $[S,Q] = [T,Q]$
$[Q,S] = [Q,T]$

Pf/ $S \simeq T \Rightarrow \exists\ f: S \to T,\ g: T \to S$ whose compositions are homotopic to identies

Let $h: S \to Q$ $\qquad$ $j: T \to Q$

$$\begin{tikzcd} & Q & \\ & S \arrow[u, "h"] & T \arrow[l, "g"] \arrow[ul, "h\circ g"'] \end{tikzcd} \qquad \begin{tikzcd} & Q & \\ & S \arrow[u, "j\circ f"] \arrow[r, "f"'] & T \arrow[ul, "j"'] \end{tikzcd}$$

$(h \circ g) \circ f = h \circ (g \circ f) = h \circ 1_S = h$
$(j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_T = j$

<u>Lemma</u> If S is homeomorphic to T,
The $S \simeq T$.

<u>Ex</u> S = single point. Then, $S \simeq \mathbb{R}$.
Let $f: \mathbb{R} \to S$ be the constant function
and $g: S \to \mathbb{R}$ given by $g(x) = 0$.

Then, $f \circ g: S \to S$ is 1_S
$g \circ f: \mathbb{R} \to \mathbb{R}$ $\quad (g \circ f)(x) = 0$
is constant function

By Ex 6.4 all f^n's $: \mathbb{R} \to \mathbb{R}$ are homotopic
Thus $g \circ f \simeq 1_{\mathbb{R}}$
Then, $[\mathbb{R}, \mathbb{R}] = [\{0\}, \{0\}]$
Can deform $\mathbb{R}$ to a point.
Spaces deformable to a point are called <u>contractible</u>

Ex $[0,1] \simeq \{0\}$

$f: [0,1] \to \{0\}$ $\quad f(x) = 0,\ x \in [0,1]$

$g: \{0\} \to [0,1]$ $\quad g(0) = 0$

$f \circ g: \{0\} \to \{0\}$ is $1_{\{0\}}$

$(g \circ f)(x) = 0,\ \forall x \in [0,1]$ is homotopic to $1_{[0,1]}$

Ex $(0,1) \simeq \{0\}$ Let $g(0) = \frac{1}{2}$

Use homotopy $\quad H(x,t) = \frac{1-t}{2} + tx$

$H(x,t) \subset (0,1)$ for $(x,t) \in (0,1) \times I$

and $H(x,t)$ is continuous.

In general $(a,b) \simeq \{0\}$.

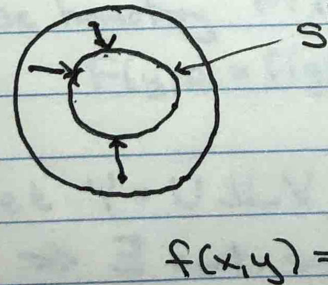
Prop Let S be contractible, T a top. space

Then, any continuous functions $f, g: T \to S$ are homotopic.

So, any continuous function to a contractible space is homotopic to a constant function.

Ex **Annulus** $\quad A = \{(x,y) \in \mathbb{R}^2 : 1 \leq \sqrt{x^2+y^2} \leq 2\}$

$A \simeq S^1$



Define $f: S^1 \to A$ inclusion map

$g: A \to S^1$ inward projection

$f(x,y) = (x,y)$ $\qquad S^1 \xrightarrow{f} A \xrightarrow{g} S^1$

$g(x,y) = \frac{(x,y)}{\sqrt{x^2+y^2}}$

$g \circ f = 1_{S^1}$ and $(f \circ g)(x,y) = \frac{(x,y)}{\sqrt{x^2+y^2}} \simeq 1_A$

$F((x,y),t) = t(x,y) + \frac{1-t}{\sqrt{x^2+y^2}}(x,y)$

1) cont.
2) $F(-,0) = f \circ g$
3) $F(-,1) = (x,y)$

Homotopy Equivalence Review

$$S \simeq T \quad \begin{cases} \text{For continuous } f, g: & \text{such that} \\ S \xrightarrow{f} T \xrightarrow{g} S & g \circ f \simeq 1_S \\ T \xrightarrow{g} S \xrightarrow{f} T & f \circ g \simeq 1_T \end{cases}$$

f, g homotopy equivalences

6.13 $[0,1] \simeq \{0\}$

6.14 $(0,1) \simeq \{0\}$

6.15 $(a,b) \simeq \{0\}$

6.16 $f, g: T \to S$ (contractible) and continuous $\Rightarrow f \simeq g \simeq$ const. map

6.17 $A \simeq S^1$, $A =$ annulus

6.18 $\mathbb{C}^\times = \mathbb{R}^2 - \{(0,0)\} \simeq S^1$

6.19 S^0 not contractible

6.20 X (connected) $\not\simeq$ Y (disconnected)

6.21 $S^1 \not\simeq S^0$

6.20 - Proof

Suppose $X \simeq Y$ with $f: X \to Y$, $g: Y \to X$

and $f \circ g \simeq 1_Y$, $g \circ f \simeq 1_X$

Use homotopy $F: Y \times I \to Y$ where

$$F(y,0) = f(g(y)), \quad F(y,1) = y, \quad \forall y \in Y.$$

Let $Y = U \amalg V$, nonempty, open U, V

$\Rightarrow \exists$ cont. surjection $p: Y \to S^0$

$p(y) = 1,\ y \in U$

$p(y) = -1,\ y \in V$

X connected $\Rightarrow$ $\mathrm{Im} f \subset U$ or $\mathrm{Im} f \subset V$

If $\mathrm{Im} f \subset U$, $\exists\, v \in V$ and define map

$h: [0,1] \to S^0$ by $h(t) = p(F(v,t))$

where $h(0) = 1$ and $h(1) = -1$.

But h is a continuous surjection. By Lemma 4.3 there is no continuous surjection $T \to S^0$ for T connected! So, $X \not\simeq Y$.

Ex $X = [0,1]$, $Y = S^0$

Ex $X = S^1$, $Y = S^0$

6.22 $S = \{1,2\}$ + indiscrete topology (not Hausdorff)

$T = \{0\}$ (Hausdorff)

$f: T \to S$ $\quad f(0) = 1$ cont.

$g: S \to T$ $\quad g(s) = 0$ cont.

Then, $g \circ f = 1_T$ $\quad (g \circ f: T \to T)$

$(f \circ g)(s) = 1$ $\quad (f \circ g: S \to S)$

constant map $\simeq 1_S$ [below]

So, $S \simeq T$.

Note: Homotopy $F: S \times I \to S$

S topology indiscrete $\Rightarrow$ F continuous

Define

$$F(s,t) = \begin{cases} s, & t \leq 1/2 \\ 1, & t \geq 1/2 \end{cases}$$

where $F(s,0) = s = 1_S$

$F(s,1) = 1$ const. map.

The Circle

S^1 is not contractible
+ all homotopy classes $f: S^1 \to S^1$

Open circle and consider $[0,1] \to S^1$

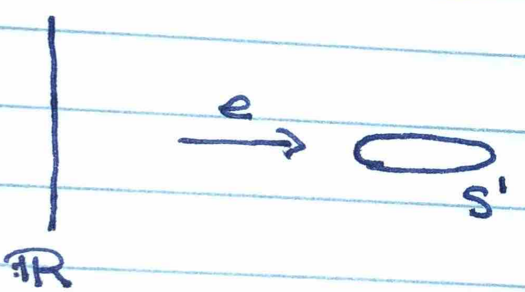
Since for $X = [0,1]$, $A = \partial X \Rightarrow X/A \approx S^1$

Lift maps $g: [0,1] \to S^1$ to $\tilde{g}: [0,1] \to \mathbb{R}$

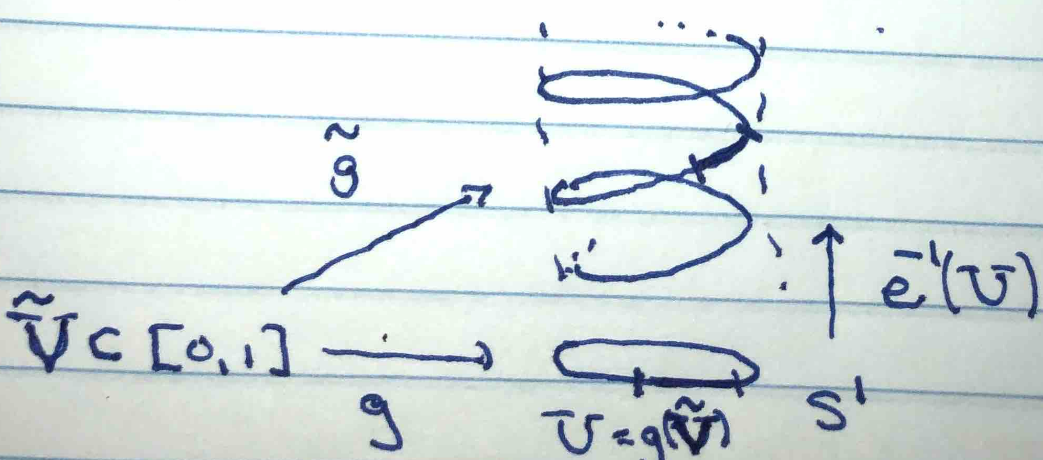
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e"] \\ {[0,1]} \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & S^1 \end{tikzcd}$$

Path Lifting

Recall Exponential Map pg 32-33 $\quad e: \mathbb{R} \to S^1$



$e(x) = (\cos 2\pi x, \sin 2\pi x)$ coils $\mathbb{R}$ around circle



$\tilde{V} = [\delta_1, \delta_2]$