Elliptic functions and Elliptic Integrals

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Nonlinear Pendulum

We motivate the need for elliptic integrals by looking for the solution of the nonlinear pendulum equation,

$$\ddot{\theta} + \omega^2 \sin\theta = 0. \tag{1}$$

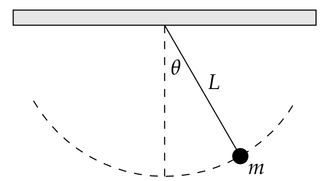

Figure 1: A simple pendulum consists of a point mass m attached to a string of length L. It is released from an angle θ_0.

This models a mass m attached to a string of length L undergoing periodic motion. Pulling the mass to an angle of θ_0 and releasing it, what is the resulting motion?

We employ a technique that is useful for equations of the form

$$\ddot{\theta} + F(\theta) = 0$$

when it is easy to integrate the function $F(\theta)$. Namely, we note that

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 + \int^{\theta(t)} F(\phi)\, d\phi\right] = (\ddot{\theta} + F(\theta))\dot{\theta}.$$

For the nonlinear pendulum problem, we multiply Equation (1) by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} + \omega^2 \sin\theta\dot{\theta} = 0$$

and note that the left side of this equation is a perfect derivative. Thus,

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta\right] = 0.$$

Therefore, the quantity in the brackets is a constant. So, we can write

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = c. \tag{2}$$

The constant in Equation (2) can be found using the initial conditions, $\theta(0) = \theta_0$, $\dot{\theta}(0) = 0$. Evaluating Equation (2) at $t = 0$, we have

$$c = -\omega^2\cos\theta_0.$$

Solving for $\dot{\theta}$, we obtain

$$\frac{d\theta}{dt} = \omega\sqrt{2(\cos\theta - \cos\theta_0)}.$$

This equation is a separable first order equation and we can rearrange and integrate the terms to find that

$$\frac{1}{2}\dot{\theta}^2 - \omega^2\cos\theta = -\omega^2\cos\theta_0. \tag{3}$$

We can solve for $\dot{\theta}$ and integrate the differential equation to obtain

$$t = \int dt = \int \frac{d\theta}{\omega\sqrt{2(\cos\theta - \cos\theta_0)}}.$$

At this point one says that the problem has been solved by quadratures.Namely, the solution is given in terms of some integral. We will proceed to rewrite this integral in the standard form of an elliptic integral.

Using the half angle formula,

$$\sin^2\frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta),$$

we can rewrite the argument in the radical as

$$\cos\theta - \cos\theta_0 = 2\left[\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right].$$

Noting that a motion from $\theta = 0$ to $\theta = \theta_0$ is a quarter of a cycle, we have that

$$T = \frac{2}{\omega}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}. \tag{4}$$

This result can now be transformed into an elliptic integral.[1] We define

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

and

$$k = \sin\frac{\theta_0}{2}.$$

Then, Equation (4) becomes

$$T = \frac{4}{\omega}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{5}$$

This is done by noting that $dz = \frac{1}{2k}\cos\frac{\theta}{2}\,d\theta = \frac{1}{2k}(1-k^2z^2)^{1/2}\,d\theta$ and that $\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = k^2(1-z^2)$. The integral in this result is called the complete elliptic integral of the first kind.

[1] Elliptic integrals were first studied by Leonhard Euler and Giulio Carlo de' Toschi di Fagnano (1682-1766) , who studied the lengths of curves such as the ellipse and the lemniscate,

$$(x^2+y^2)^2 = x^2 - y^2.$$

The complete elliptic integral of the first kind.

Elliptic Integrals of First and Second Kind

There are several elliptic integrals. They are defined as

$$F(\phi,k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{6}$$

$$= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{7}$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{8}$$

$$= \int_0^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{9}$$

$$E(\phi,k) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta \tag{10}$$

$$= \int_0^{\sin\phi} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{11}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta \tag{12}$$

$$\tag{13}$$

$$= \int_0^{1} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}\, dt \tag{14}$$

Elliptic Functions

Elliptic functions result from the inversion of elliptic integrals. Consider

$$u(\sin\phi,k) = F(\phi,k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{15}$$

$$= \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{16}$$

Note:$F(\phi,0) = \phi$ and $F(\phi,1) = \ln(\sec\phi + \tan\phi)$. In these cases F is obviously monotone increasing and thus there must be an inverse.

The inverse of $F(u,k)$ is $\operatorname{sn}(u,k) = \sin\phi = \sin \operatorname{am} u$, where

$$\operatorname{am}(u,k) = \phi = F^{-1}(u,k)$$

am is called the amplitude. Note that $\operatorname{sn}(u,0) = \sin u$ and $\operatorname{sn}(u,1) = \tanh u$.

Similarly, we have

$$u = \int_0^{\operatorname{cn}(u,k)} \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}. \tag{17}$$

$$u = \int_0^{\operatorname{dn}(u,k)} \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}. \tag{18}$$

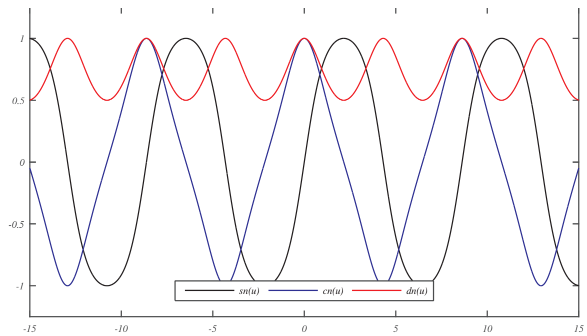

Figure 2: Plots of the Jacobi elliptic functions for $m = 0.75$.

The Jacobi elliptic functions for $m = 0.75$ are shown in Figure 2. We note that these functions are periodic. The Jacobi elliptic functions are related by

$$\begin{aligned}
\sin\phi &= \operatorname{sn}(u,k) && (19)\\
\cos\phi &= \operatorname{sn}(u,k) && (20)\\
\sqrt{1-k^2\sin^2\phi} &= \operatorname{dn}(u,k) && (21)\\
& && (22)
\end{aligned}$$

Furthermore, we have the identities

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \quad k^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1.$$

Derivatives Derivatives of the Jacobi elliptic functions are easily found. First, we note that

$$\frac{d(\operatorname{sn} u)}{du} = \frac{d(\operatorname{sn} u)}{d\phi}\frac{d\phi}{du} = \operatorname{cn} u\sqrt{1-k^2\sin^2\phi} = \operatorname{cn} u \operatorname{dn} u,$$

where

$$\frac{du}{d\phi} = \frac{1}{\sqrt{1-k^2\sin^2\phi}}$$

results from integrating $F(\phi, k)$.

Similarly, we have $\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u$, and $\frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u$.

Differential Equations

Let $y = \operatorname{sn} u$. Using

$$\frac{d(\operatorname{sn} u)}{du} = \operatorname{cn} u \operatorname{dn} u,$$

we have

$$\frac{dy}{du} = \sqrt{1-y^2}\sqrt{1-k^2y^2},$$

or

$$\left(\frac{dy}{du}\right)^2 = (1-y^2)(1-k^2y^2).$$

Differentiating with respect to u again, we have the nonlinear second order differential equation

$$y'' = -(1+k^2)y + 2k^2y^3.$$

We note that this differential equation is amenable to solution using Simulink. Such a model is shown in Figure 3.

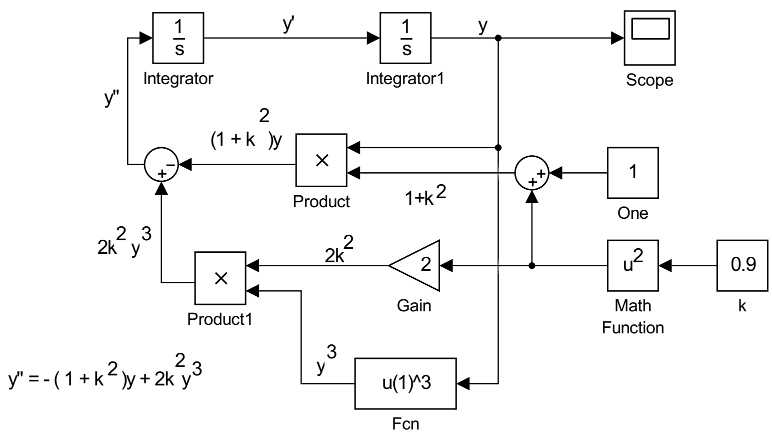

Figure 3: Simulink model for solving $y'' = -(1+k^2)y + 2k^2y^3$.

Periodicity

Consider

$$\begin{aligned} F(\phi + 2\pi, k) &= \int_0^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \\ &= \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} + \int_{\phi}^{\phi+2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= F(\phi,k) + \int_0^{2\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= F(\phi,k) + 4K(k). \end{aligned} \tag{23}$$

Since $F(\phi + 2\pi, k) = u + 4K$, we have

$$\operatorname{sn}(u+4K) = \sin(\operatorname{am}(u+4K)) = \sin(\operatorname{am}(u)+2\pi) = \sin\operatorname{am}(u) = \operatorname{sn} u.$$

In general, we have

$$\operatorname{sn}(u+2K,k) = -\operatorname{sn}(u,k) \tag{24}$$

$$\operatorname{cn}(u+2K,k) = -\operatorname{cn}(u,k) \tag{25}$$

$$\operatorname{dn}(u+2K,k) = \operatorname{dn}(u,k). \tag{26}$$

The plots of $\operatorname{sn}(u)$, $\operatorname{cn}(u)$, and $dn(u)$, are shown in Figures 4-6.

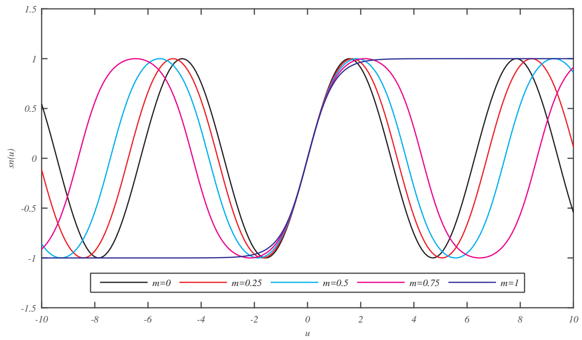

Figure 4: Plots of $\mathrm{sn}\,(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

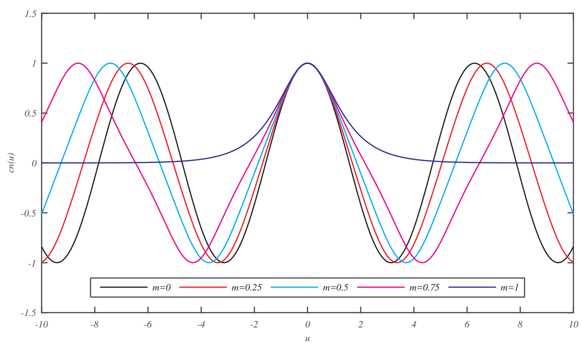

Figure 5: Plots of $\mathrm{cn}\,(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

Complex Arguments

Values of the Jacobi elliptic functions for complex arguments can be found using Jacobi's imaginary transformations,

$$\mathrm{sn}\,(iu,k) = i\,\mathrm{sc}\,(u,k') \tag{27}$$

$$\mathrm{cn}\,(iu,k) = \mathrm{nc}\,(u,k') \tag{28}$$

$$\mathrm{dn}\,(iu,k) = \mathrm{dc}\,(u,k'). \tag{29}$$

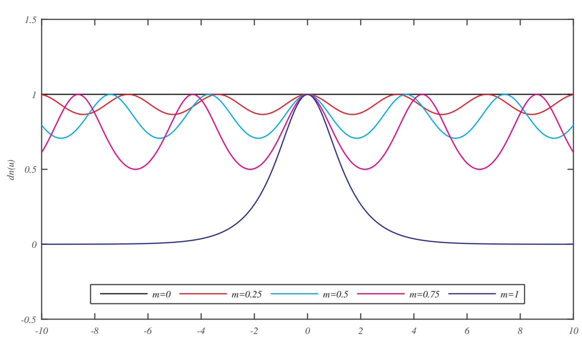

Figure 6: Plots of $\mathrm{dn}\,(u,k)$ for $m = 0, 0.25, 0.50, 0.75, 1.00$.

These results are found by rewriting the elliptic integral. We show this for the first result by considering $u = F(\phi, k)$ in the form

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

We introduce the transformation

$$\begin{aligned} \sin\theta &= \frac{2t}{1+t^2}, \\ \cos\theta &= \sqrt{1 - \left(\frac{2t}{1+t^2}\right)^2} \\ &= \frac{1-t^2}{1+t^2}. \end{aligned} \tag{30}$$

This gives

$$\cos\theta\, d\theta = \frac{2(1+t^2) - 4t^2}{(1+t^2)^2} dt = \frac{2(1-t^2)}{(1+t^2)^2}\, dt,$$

or $d\theta = \frac{2}{1+t^2} dt$

Applying this variable substitution to the elliptic integral, we have

$$\begin{aligned} u &= \int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= 2\int_0^s \frac{dt}{(1+t^2)\sqrt{1 - k^2\left(\frac{2t}{1+t^2}\right)^2}} \\ &= 2\int_0^s \frac{dt}{\sqrt{(1+t^2)^2 - 4k^2t^2}} \\ &= 2\int_0^s \frac{dt}{\sqrt{1 + 2(1-2k^2)t^2 + t^4}}. \end{aligned} \tag{31}$$

Inserting $t = ix$, and noting that the integrand is an even function of x, we obtain

$$\begin{aligned} u &= i\int_0^{-is} \frac{dx}{\sqrt{1 - 2(1-2k^2)x^2 + x^4}}. \\ &= -i\int_0^{is} \frac{dx}{\sqrt{1 - 2(1-2k^2)x^2 + x^4}}. \end{aligned} \tag{32}$$

Introducing $k^2 = 1 - k'^2$, leads to

$$\begin{aligned} u &= -i\int_0^{is} \frac{dx}{\sqrt{1 - 2(1-2(1-k'^2))x^2 + x^4}} \\ &= -i\int_0^{is} \frac{dx}{\sqrt{1 - 2(-1+k'^2)x^2 + x^4}} \\ iu &= \int_0^{is} \frac{dx}{\sqrt{1 + 2(1-k'^2)x^2 + x^4}}. \end{aligned} \tag{33}$$

Therefore, we have Equation (33) is the same as Equation (31) and the inverse function is $\operatorname{sn}(iu, k')$.

Using the transformation, we find that $\operatorname{sn}(iu, k')$ is pure imaginary:

$$\begin{aligned} \operatorname{sn}(iu, k') &= \frac{2is}{1-s^2} \\ &= i\frac{\sin\phi}{\cos\phi} \\ &= i\frac{\operatorname{sn}(u,k)}{\operatorname{cn}(u,k)} \\ &= i\operatorname{sc}(u,k). \end{aligned} \tag{34}$$

We can exchange k with k' to obtain the final result $\operatorname{sn}(iu, k) = i\operatorname{sc}(u, k')$.

There is a problem when $\operatorname{cn}(u, k') = 0$. Noting that

$$\operatorname{sn}(0,k) = 0, \quad \operatorname{cn}(0,k) = 1, \quad \operatorname{dn}(0,k) = 1,$$

and

$$\operatorname{sn}(K,k) = 1, \quad \operatorname{cn}(K,k) = 0, \quad \operatorname{dn}(K,k) = k',$$

and that $\operatorname{cn}(u,k)$ has period $4K$, then $\operatorname{cn}(u,k') = 0$ for $u = (2n+1)K'$. Thus, $\operatorname{sn}(iu,k)$ has imaginary period of $2iK'$.

Plots of the Jacobi elliptic functions in the complex plane using domain coloring for $k = 0.7$ are shown in Figures 7-9. In this case we have $K(.7) = 1.8457$ and $K'(.7) = K(\sqrt{1-.7^2}) = 1.8626$. This gives the periods for $sn(u)$ as 7.3828 and $3.7253i$, which can be seen in Figure 7.

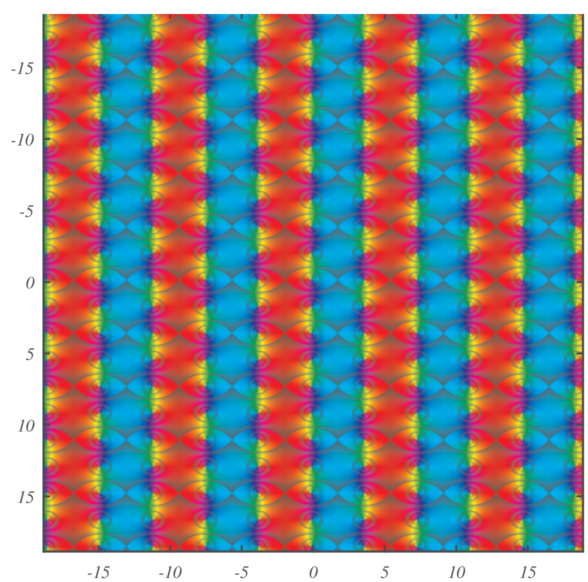

Figure 7: Domain coloring plot of $\operatorname{sn}(u,k)$ for $u = x + iy$ and $k = 0.7$.

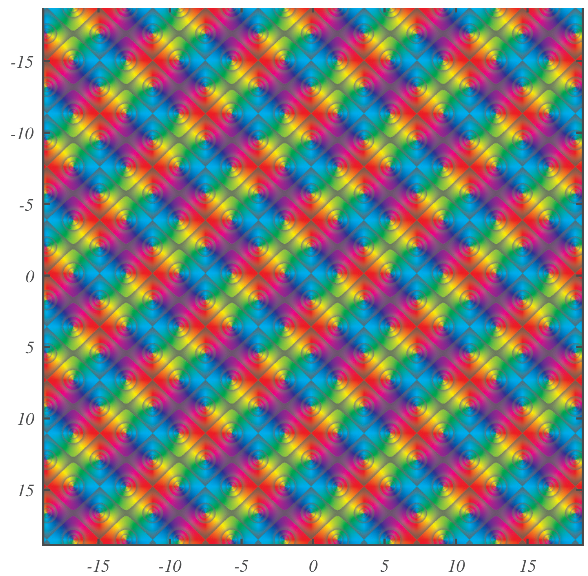

Figure 8: Domain coloring plot of $\mathrm{cn}\,(u, k)$ for $u = x + iy$ and $k = 0.7$.

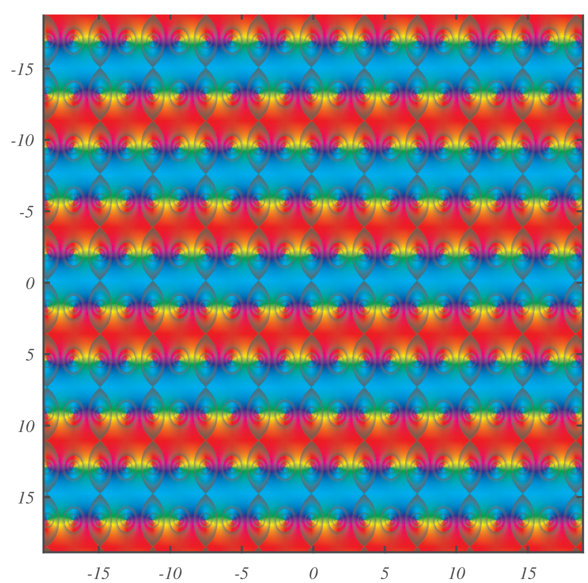

Figure 9: Domain coloring plot of $\mathrm{dn}\,(u, k)$ for $u = x + iy$ and $k = 0.7$.

Addition Formulae Letting $s_i = \operatorname{sn}(u_i)$, for $i = 1, 2$, etc., we have

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}. \tag{35}$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{sn} u \operatorname{sn} v \operatorname{dn} u \operatorname{dn} v}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}. \tag{36}$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{cn} u \operatorname{cn} v}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 y}. \tag{37}$$

From these formulae and the Jacobi imaginary transformation, one can derive formula for complex arguments.

Arithmetic-Geometric Mean

The Arithmetic-Geometric Mean (AGM) iteration of Gauss is given by a two-term recursion

$$a_{n+1} = \frac{a_n + b_n}{2},$$

$$b_{n+1} = \sqrt{a_n b_n}. \tag{38}$$

These sequences converge to a common limit,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a_0, b_0).$$

In 1799 Gauss saw that

$$\frac{1}{M(1, \sqrt{2})} \approx \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{1 - t^2}}$$

up to eleven decimal places. This is an example of

$$\frac{1}{M(1, x)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - x^2)\sin^2\theta}}.$$

Letting $x = \sin\alpha$, we can write

$$K(\cos\alpha) = \frac{\pi}{2} \frac{1}{M(1, \sin\alpha)}.$$