Topics for Midterm Exam – MAT 516

I. Basics
   a. Know how to use polar forms $z = re^{i\theta}$, $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}, \tan\theta = \frac{y}{x}$
   b. $e^{i\pi} = -1, e^{2\pi ik} = 1$ for k an integer
   c. Complex Modulus and complex conjugate
   d. nth roots $z^{1/n} = r^{1/n}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n})$ for $k = 0,1,\ldots,n-1$
   e. Logarithms $\ln z = \ln r + i(\theta + 2n\pi)$, n = integer
   f. Cauchy-Riemann Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$
   g. Harmonic Functions: CR => $\nabla^2 u = 0$
   h. Holomorphic, Analytic, Entire, Meromorphic.

II. Complex Functions
   a. Determine real and imaginary parts of functions: $f(z) = u(x,y) + iv(x,y)$
   b. Map points, lines, circles, curves, regions under a given function.
   c. Multivaluedness, principal values, etc: logarithms and other transcendental functions, branch points.
   d. Linear fractional transformations – basic ideas and determination given three points in each plane, cross ratio, $\frac{w - w_1}{w - w_2} : \frac{w_3 - w_1}{w_3 - w_2} = \frac{z - z_1}{z - z_2} : \frac{z_3 - z_1}{z_3 - z_2}$.
   e. Specific functions:
      $\frac{1}{z}, az + c, (z-a)^n, P_n(z), \frac{P_m(z)}{Q_n(z)}, e^z, \cos z, \cosh z, \ldots, z^{1/n}, \ln(z)$
      $\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{az+b}{cz+d}, ad - bc \neq 0.$
   f. Basic ideas about Stereographic projection (labels), extended complex plane, point at infinity. $\xi = \frac{2x}{x^2+y^2+1}, \eta = \frac{2y}{x^2+y^2+1}, \zeta = \xi = \frac{x^2+y^2-1}{x^2+y^2+1}$, $x = \frac{\xi}{1-\zeta}, x = \frac{\eta}{1-\zeta}$.
   g. Conformal mappings, meaning of $\mathrm{Arg}(f'(z_0)), |f'(z_0)|$

III. Series Expansions
   a. Power series, Laurent series, Taylor series
   b. Circle/radius of convergence
   c. Geometric series
      i. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1,$
      ii. $\frac{1}{1-z} = \frac{-1}{z} \times \frac{1}{1 - \frac{1}{z}} = -\sum_{n=1}^{\infty} z^{-n}, |z| > 1$
   d. Convergence Tests – Ratio and Root tests for $\sum_n a_n (z - z_0)^n$
   e. Known series expansions: $e^z, \sin z, \cos z, \ln z, (1-z)^{-1}, \sinh z, \cosh z$.

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f. Differentiation and integration of series.
g. Finding Laurent series expansions throughout the plane.

IV. Integration
a. Complex Path Integrals – parametrized over line segment, arcs, etc.
b. Cauchy's Theorem
c. Cauchy Integral Theorem
d. Singularities – Removable, Essential and Poles
e. Computing Residues
i. $\text{Res}[f(z); z = z_0] = \lim_{z \to z_0} (z - z_0) f(z)$ - simple poles
ii. $\text{Res}[f(z); z = z_0] = \lim_{z \to z_0} \frac{1}{k!} \frac{d^{k-1}}{dz^{k-1}} [(z - z_0)^k f(z)]$ - poles of order k
f. Residue Theorem $\int_C f(z)\,dz = 2\pi i \sum_{\text{Poles inside C}} \text{Residues}$
g. Integrands of the form $f(\cos\theta, \sin\theta)$: $\cos\theta = \frac{z + z^{-1}}{2}$, $\sin\theta = \frac{z - z^{-1}}{2i}$
h. Infinite (Cauchy Principal Value)
i. Jordan's Lemma
ii. Poles on Contour
iii. Branch Cuts

V. Miscellaneous Topics
a. Laplace's Equation
b. Fluid Flow $f(z) = \phi(x, y) + i\psi(x, y)$ - potential and stream functions
c. Conformal Mappings
d. Bilinear Transformations
e. Schwarz-Christoffel Transformations