Time Dilation

There are two time dilation effects from the theories of relativity.

- Special Relativity: Moving clocks tick slower.
- General Relativity: Clocks in stronger gravitational fields tick slower.

1 Special Relativity

In Figure 1 are the light clocks used to derive the time dilation formula in class. Focusing on the right triangle, we identified the lengths of the sides and applied the Pythagorean Theorem. This gives

$$\left(\frac{ct}{2}\right)^2 = \left(\frac{vt}{2}\right)^2 + \left(\frac{c\tau}{2}\right)^2.$$

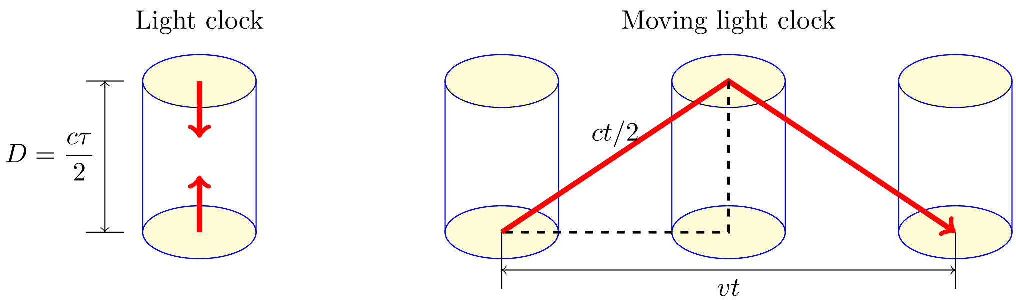

Figure 1: Stationary and moving clocks for time dilation computation.

We now solve for t in terms of τ. After canceling all of the 2's,

$$\begin{aligned}
c^2t^2 &= v^2t^2 + c^2\tau^2 \\
c^2t^2 - v^2t^2 &= c^2\tau^2 \\
(c^2 - v^2)t^2 &= c^2\tau^2 \\
\left(1 - \frac{v^2}{c^2}\right)t^2 &= \tau^2 \\
t &= \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (1)
\end{aligned}$$

This expression can now be used to compare Alice and Bob's clock readings in Figure 2.

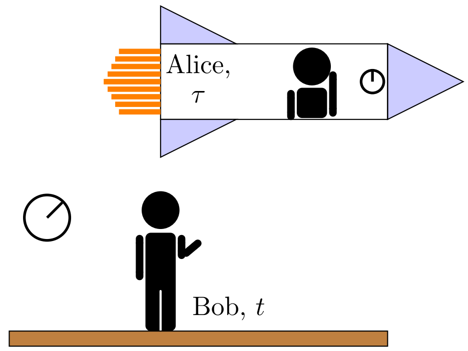

Figure 2: Alice's clock measures the *proper time,* τ. Bob's clock measures t.

Define

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Then, we can write $t = \gamma\tau$. Since $v < c$, then $v/c < 1$. So, $\sqrt{1 - \frac{v^2}{c^2}} < 1$ and thus, $\gamma > 1$. This means that $t \geq \tau$. We conclude that moving clocks tick slower.

An example I gave in class was a plane moving at 620 mph (277 m/s). What is γ? First compute

$$\frac{v}{c} = \frac{277}{3 \times 10^8} = 9.23 \times 10^{-7}.$$

Then,

$$1 - \left(\frac{v}{c}\right)^2 = 1 - \left(9.23 \times 10^{-7}\right)^2.$$

Putting this in a calculator, you are bound to get an answer of 1 and eventually you will not get to find out by how much the clocks differ.

So, we need an approximation to the difference between the times. Assume that an hour passes on the Earth (t) and we want to know by how much the time (τ) slows down on the plane. Since $\tau < t$, this difference is $t - \tau$. Such an approximation is known (binomial approximation) and we have

$$\begin{aligned} t - \tau &= t - t\sqrt{1 - \frac{v^2}{c^2}} \\ &\approx t - t\left[1 - \frac{v^2}{2c^2}\right] \\ &= \frac{v^2}{2c^2}t. \end{aligned} \tag{2}$$

So, now the time difference in the problem is found as

$$t - \tau = \frac{v^2}{2c^2}t \approx 4.26 \times 10^{-13}t.$$

For $t = 1$ hr $= 3600$ seconds, $t - \tau \approx 1.53$ nanoseconds.

2 General Relativity

The derivation of gravitational time dilation is a little more complicated. Clocks that are farther from massive bodies run more quickly than clocks close to massive bodies.

In the vicinity of a non-rotating massive spherically symmetric object the time dilation equation is

$$\Delta\tau = \sqrt{1 - \frac{2GM}{rc^2}}\Delta t \approx \left(1 - \frac{GM}{rc^2}\right)\Delta t,$$

where $\Delta\tau$ is the time interval measured by an observer close to a massive object and Δt the time interval measured by an observer far from the massive object. $G = 6.67 \times 10^{-11}$ $\mathrm{Nm^2/kg^2}$ is the universal gravitational constant, M is the mass of the massive object, and $c = 3.0 \times 10^8$ m/s is the speed of light in a vacuum.

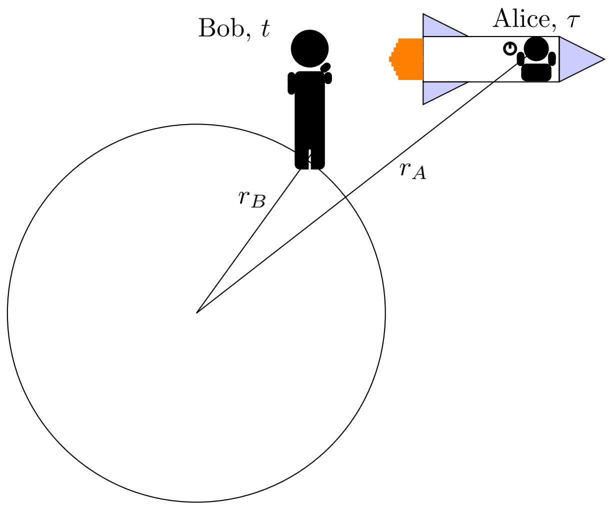

Figure 3: Gravitational dilation example.

As an example, let Bob be on the Earth and Alice fly off into space. Each measures a time (τ_A or τ_B) and the difference in their times due to gravitation is given by

$$\begin{aligned}\tau_A - \tau_B &\approx \left(1 - \frac{GM}{r_Ac^2}\right)\Delta t - \left(1 - \frac{GM}{r_Bc^2}\right)\Delta t \\ &= \left(\frac{GM}{r_Bc^2} - \frac{GM}{r_Ac^2}\right)\Delta t. \qquad (3)\end{aligned}$$