Simultaneity, Time Dilation, and Length Contraction Using Minkowski Diagrams and Lorentz Transformations

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Abstract

In these notes we present a simple introduction to the first consequences of special relativity (simultaneity, time dilation, and length contraction) as depicted using Lorentz transformations and the superimposed Minkowski diagrams for two observers.

Minkowski Diagrams

In this document we will consider the use of superimposed Minkowski diagrams displaying Lorentz boosts. We will first refer to Figure 1. There are two inertial reference frames, S and S'. The spacetime coordinates in S are given by (x, ct). Those in S' are given by (x', ct'). They are connected through a Lorentz transformation.

The Lorentz transformation in 1+1 dimensional spacetime is

Lorentz transformation.

$$\begin{aligned} x &= \gamma(x' + vt') = \gamma(x' + \beta ct'), && (1) \\ ct &= c\gamma(t' + \frac{vx'}{c^2}) = \gamma(ct' + \beta x'). && (2) \end{aligned}$$

Here $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$ is the Lorentz factor and $\beta = \frac{v}{c}$ is the dimensionless velocity.

The inverse transformation is

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(x - \beta ct), && (3) \\ ct' &= c\gamma(t - \frac{vx}{c^2}) = \gamma(ct - \beta x). && (4) \end{aligned}$$

As we proceed, it should be noted that distances in this spacetime, denoted by $(\Delta s)^2$ is not the Euclidean line element, but instead given by

$$(\Delta s)^2 = (\Delta x)^2 - (c\Delta t)^2. \qquad (5)$$

This can make some figures appear to have longer distances between events in spacetime than they actually are.

Under Lorentz transformations, $(\Delta s)^2$ is an invariant, i.e., $(\Delta s)^2 = (\Delta s')^2$. For easy reference, we note the form of the transformations on spatial and temporal increments:

$$\Delta x = \gamma(\Delta x' + \beta c \Delta t'), \quad (6)$$
$$c\Delta t = \gamma(c\Delta t' + \beta \Delta x'), \quad (7)$$
$$\Delta x' = \gamma(\Delta x - \beta c \Delta t), \quad (8)$$
$$c\Delta t' = \gamma(c\Delta t - \beta \Delta x). \quad (9)$$

We would like to describe the connections between the measurements of spatial and time intervals in the two frames of reference. (We will use one-dimensional spatial coordinates (x) and scaled time coordinates (ct). First, we describe how the two reference frames are related. In Figure 1 we see that the unprimed axes are orthogonal but the primed axes appear skewed. Let's determined how they were drawn.

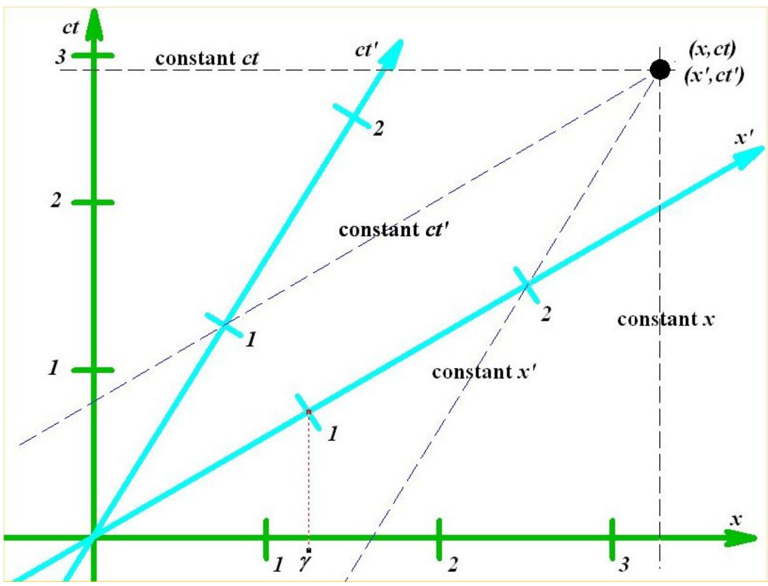

Figure 1: Diagram for investigating Lorentz boosts and changes in coordinates in 1+1 dimensions.

Consider Figure 1. A given point can be described in reference frame S with unprimed coordinates, (x, ct). As usual,we draw lines parallel to the axes to determine the values of the coordinates. In the same way, we can establish the primed coordinates. As seen in the figure, we can pick out the coordinates, (x', ct').

First, we can locate the primed axes with respect to the unprimed system using the equations for the Lorentz transformation. For the x'-axis, we set $x' = 1$ and $ct' = 0$ in Equations (1-2). Then we obtain $x = \gamma$ and $ct = \beta\gamma$. Thus, $ct = \beta x$. So, the x'-axis has slope $\beta = v/c$ with respect to the unprimed axes.

The ct' axes can be found in the same way. We set $x' = 0$ and $ct' = 1$. Then, $ct = \gamma$ and $x = \gamma\beta = \beta ct$. Thus, the slope of the ct'-axis is $1/\beta = c/v$.

In Figure 1 we show axes which have been drawn with $\beta = 0.6$. Thus, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{5}{4}$. Setting $x = \gamma$ and $ct = \beta\gamma$, we can locate the point $(1, 0)$ in the primed system. This is shown on the figure. Similarly, we can mark off unit "lengths" along the time axis. Just set $ct = \gamma$ and $x = \beta\gamma$ to get started. For the given value of $\beta = 0.6$, we just use multiples of $\gamma = 1.25$ to locate the integer markings on the primed axes.

Simultaneity

The first consequence of Einstein's theory of special relativity is simultaneity. In Figure 2 we show what two observers see when a light is turned on inside the train car. On the left the observer in the train sees two light rays leave the bulb, traveling at c. The two rays simultaneously hit opposite sides of the train car. On the right side we show what an outside observer at the train station sees. Again, each light ray moves at speed c, but the train is moving as well. Each ray travels the same distance from the starting point, denoted by the dashed line. However, the left ray strikes the train wall first. Therefore, this observe does not see the light rays simultaneously strike the wall of the train car.

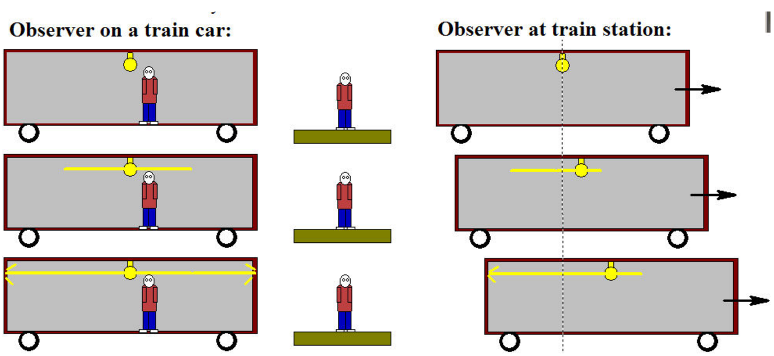

Figure 2: Comparing what different observers see when a light is turned on inside a moving train.

We can show that events simultaneous in system S will not be in system S' moving at speed v with respect to S using our Minkowski diagram. We first locate two simultaneous events in S, A and B, as shown in Figure 3. The horizontal dashed line indicates the common time in the S frame at which these two events take place. In order to determine the times recorded in the S' frame, we draw dashed lines through the events and parallel to the x'-axis. These lines intersect the ct'-axis at points C and D. It is obvious that an observer at rest with respect to the S' frame does not see events A and B as occurring at the same time.

Example This can be verified numerically using the Lorentz transformations. Let $\beta = 0.6$. From Figure 3 we have that $\Delta t = 0$ for two

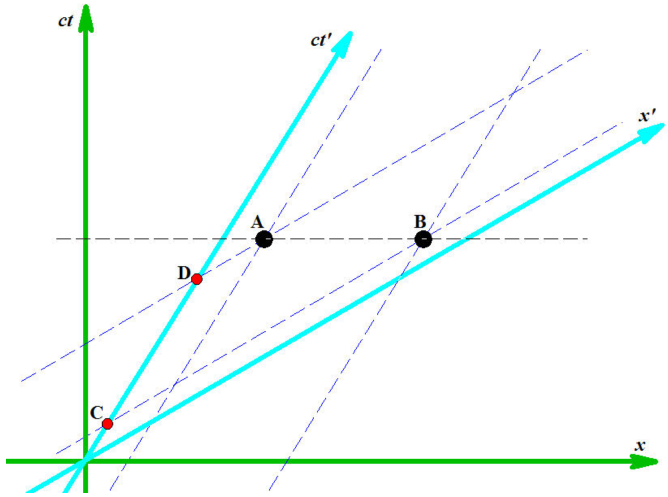

Figure 3: Diagram exhibiting simultaneous events.

simultaneous events in frame S. Let's say that event A occurs at position $x_1 = 2.0$ m, event B occurs at position 4.0 m, and they occur at $ct_1 = ct_2 = 3.0$cmin. (Note that a cmin = one $c\times$(one minute) is a unit of length!) Then from Equations 3-4 we have

$$x'_1 = \gamma(x_1 - \beta ct_1) = \frac{5}{4}(2.0 - 0.6(3.0)) = 0.25 \text{ m},$$

$$x'_2 = \gamma(x_2 - \beta ct_2) = \frac{5}{4}(4.0 - 0.6(3.0)) = 2.75 \text{ m},$$

$$ct'_1 = \gamma(ct_1 - \beta x_1) = \frac{5}{4}(3.0 - 0.6(2.0)) = 2.25 \text{ cmin},$$

$$ct'_2 = \gamma(ct_2 - \beta x_2) = \frac{5}{4}(3.0 - 0.6(4.0)) = 0.75 \text{ cmin},$$

Time Dilation

Next we take up time dilation. In order to measure time we need a simple clock with no moving parts, especially in the direction of motion of the clock. We introduce a time clock which consists of an enclosed contained in which a light ray moves perpendicular to the clock's velocity. At the top and bottom of the clock are two mirrors spaced a distance D apart as shown in Figure 4. In a frame at rest with the clock a light ray is seen simply to move from the lower mirror to the top mirror, reflect and return to the first mirror. This takes time $\Delta t_0 = 2D/c$. This gives the time for an observer at rest with respect to the clock. Note that we used a subscript to denote the *proper time*.

Now, consider that this observer is moving at speed v with the clock with respect to a stationary observer. What the stationary observer sees is shown in Figure 5. As the light ray leaves mirror one to

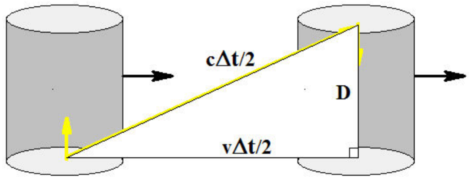

Figure 4: A light clock consists of two mirrors. In a frame at rest with the clock a light ray is seen simply to move from the lower mirror to the top mirror, reflect and return to the first mirror. This takes time $\Delta t_0 = 2D/c$.

mirror two and back, the clock moves forward. The second observers sees a triangular path traced as shown in the figure.

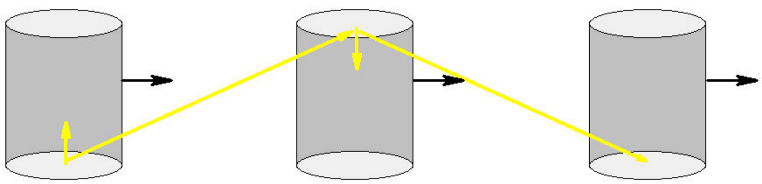

Figure 5: Depiction of moving light clock from the point of view of a stationary observer watching the light clock speed past. The light ray is seen to traverse a bent path.

We can relate the times measured by our two observers by referring to Figure 6. Let the time measured by the stationary observer be Δt for the round trip of the light ray as it travels from mirror one to two and back. Thus, the time to travel just between the two mirrors is $\Delta t/2$. The light ray travels at speed c and thus over distance $c\Delta t/2$ according to the second observer.

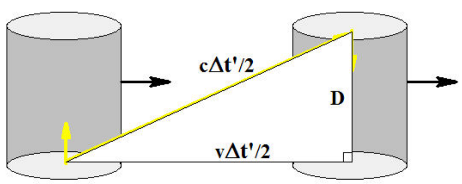

Figure 6: Diagram used for determining time dilation.

In the same time, the clock (and first observer) move forward a distance of $v\Delta t/2$. Using the Pythagorean Theorem, we have

$$D^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2.$$

Note that in the rest frame of the clock (observer one) we have $D = \frac{c\Delta t_0}{2}$. Thus,

$$\left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{v\Delta t}{2}\right)^2 = \left(\frac{c\Delta t}{2}\right)^2.$$

Solving for Δt, we find the time dilation equation

$$\Delta t = \gamma \Delta t_0, \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{10}$$

In this problem Δt_0 is the time measured by the moving clock and Δt is the time measured by the stationary observer. Since $\gamma \geq 1$, this indicates that *moving clocks tick slower.*

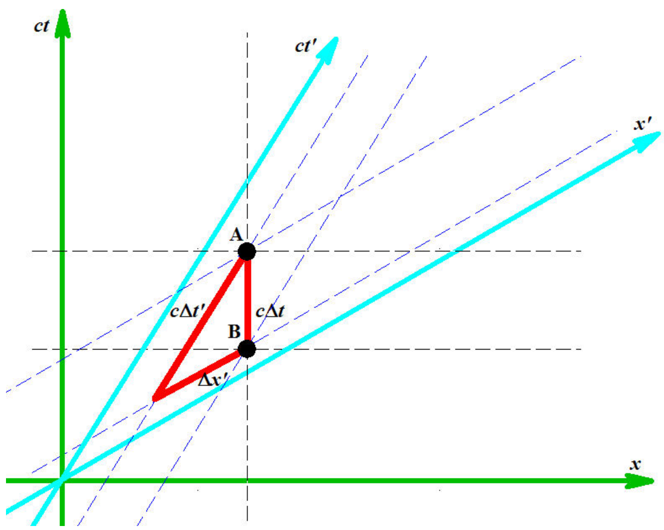

Figure 7: Diagram for showing time dilation for events located at a fixed point in frame S.

Now we want to show that the measurement of time intervals in the S frame are not the same as those in the S' frame using Minkowski diagrams. In Figure 7 we mark two events, A and B, located at the same point in space but different points in time, in the S frame. The horizontal (with respect to the x-axis) dashed lines mark off the times along the ct-axis. Drawing lines parallel to the x'-axis shows intersections with the ct'-axis. The respective time intervals are marked as $c\Delta t$ and $c\Delta t'$, respectively. How are these time intervals related?

We can use the Lorentz transformations to find this out. Note that $\Delta x = 0$. Using $c\Delta t' = \gamma(c\Delta t - \beta \Delta x)$, we have

$$\Delta t' = \gamma \Delta t.$$

Notice that events A and B are on the world line of a particle at rest on the dashed line connecting these events. Therefore, Δt measure the proper time. Thus, we have that the proper time in S is less than the time measured in S'.

This is not the Minkowski diagram that one would use to describe the moving light clock earlier in the section. The diagram we would need is in Figure 8. In this case the events A and B correspond to when the light ray leaves mirror one (A) and returns to mirror 2 (B). In frame S' this occurs at a fixed position, so $\Delta x' = 0$. Using $c\Delta t = \gamma(c\Delta t' + \beta \Delta x',)$ we recover the time dilation equation $\Delta t = \gamma \Delta t'$, where $\Delta t'$ is the proper time in this example. This is the same result as in Equation 10.

As a further note, we can derive this result using the invariance

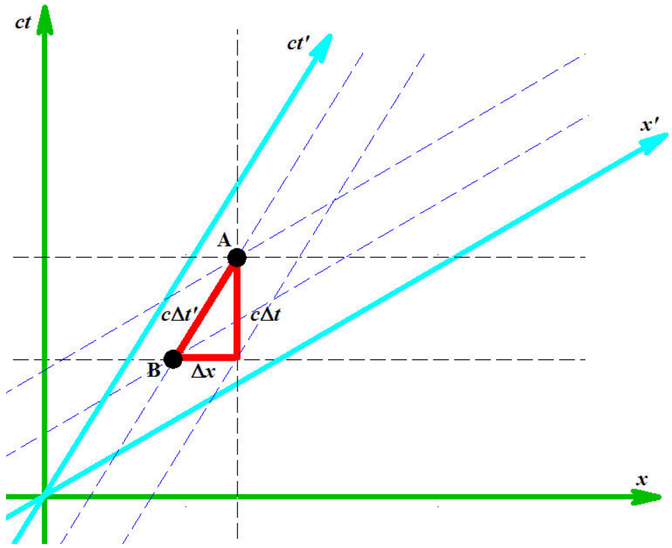

Figure 8: Diagram for showing time dilation for events located at a fixed point in frame S'.

relation

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 = -(c\Delta t')^2 + (\Delta x')^2.$$

Setting $\Delta x' = 0$ we have

$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2.$$

Notice that the sides of the triangle in Figure 8 do not satisfy the Pythagorean relation from Euclidean geometry!

We would like to relate the time increments. So, we have to eliminate (Δx). Recall that the slope of the hypotenuse on the triangle is

$$\frac{c\Delta t}{\Delta x} = \frac{1}{\beta}.$$

Thus, $\Delta x = \beta c\Delta t$. This yields

$$\begin{aligned} (c\Delta t)^2 - (\Delta x)^2 &= (c\Delta t')^2 \\ (c\Delta t)^2 - (\beta c\Delta t)^2 &= (c\Delta t')^2 \\ (1-\beta^2)(c\Delta t)^2 &= (c\Delta t')^2. \end{aligned} \tag{11}$$

Therefore, we once again obtain $\Delta t = \gamma\Delta t'$.

Length Contraction

Finally, we want to look at the idea of length contraction. This is depicted in Figure 9. We begin with a rod (or, measuring stick) whose length is L_0 as measured by an observer in the rest frame of the rod, which will be S'. We now need to determine how one measures the rod when the rod is moving at speed v past a second observer.

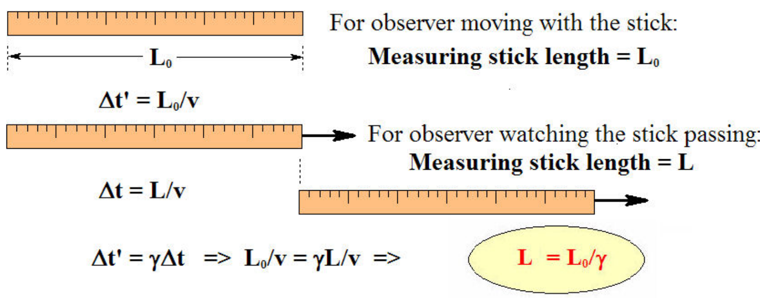

Figure 9: Diagram for determining length contraction for a moving meter stick.

As the rod moves, we have a hard time lining up a meter stick next to the rod to make any measurements. Instead, we watch as the rod passes a fixed point and record the time interval from the time the first end passes the point to the time the back end does. The time obtained is

$$\Delta t = \frac{L}{v},$$

where L is the length of the moving rod as recorded by the stationary observer. The observer in the rest frame of the rod would record a time of

$$\Delta t' = \frac{L_0}{v}.$$

However, we can use time dilation to relate the times. The time measured by the stationary observer is measured by focussing on a fixed point in space. So, Δt is the proper time in system S. It is shorter than that measured in S'. Thus,

$$\Delta t' = \gamma \Delta t.$$

Note that the time measured in frame S' cannot be a proper time as the observer has to measure two different times in two different locations, as we will see using the Minkowski diagrams.

Continuing with the computation, we have so far $\Delta t' = \gamma \Delta t$, $\Delta t = \frac{L}{v}$, and $\Delta t' = \frac{L_0}{v}$. Eliminating the time variables, we are left with the length contraction equation:

$$L = \frac{L_0}{\gamma}.$$

This indicates that the proper length is larger than the length measured in other inertial frames. So, *moving rods contract*.

We now return to Minkowski diagrams. We will determine which system records shorter lengths in space in two cases. In Figure 11 the rod is at rest in reference frame S and in Figure 10 the rod is at rest in the S' frame.

The earlier example is depicted by the diagram in Figure 10. The observer in frame S' initially places the rod along the x'-axis. As time

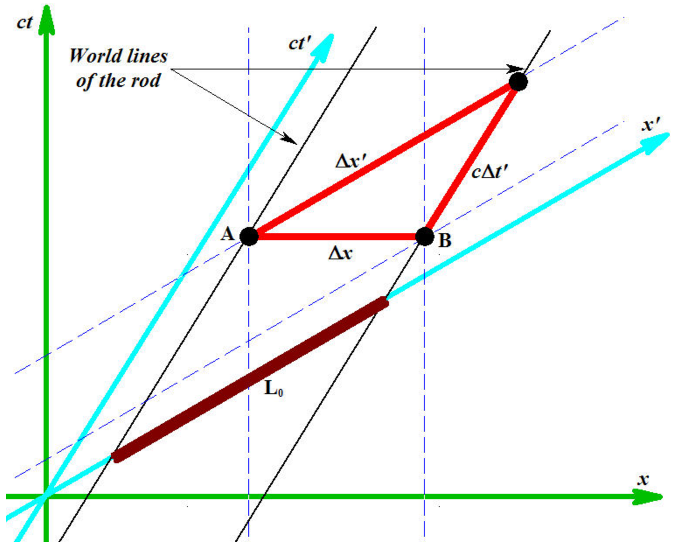

Figure 10: Diagram for determining length contraction.

evolves, the world lines traced out by the ends of the rod trace out the two parallel black solid lines shown. During the time interval $\Delta t'$ the observer measures the rod length as $\Delta x'$ as indicated. The observer at rest with respect to reference frame S measures the ends of the rod at a fixed time and finds that the length of the moving rod is Δx. From the Lorentz transformations with $\Delta t = 0$, we have

$$\Delta x' = \gamma \Delta x.$$

This is the length contraction equation.

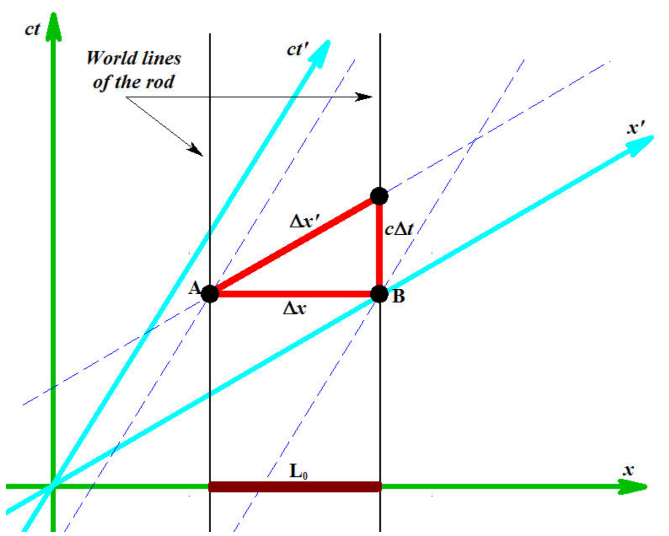

Figure 11: Diagram for determining length contraction.

In a similar manner, a rod at rest with respect to frame S is depicted in Figure 11. An typical example would be the situation where one person would stand on a train platform and the second stands in the moving train and makes a measurement of the length of the platform. The platform is initially aligned with the x-axis. The world lines for the ends of the platform are shown as two black parallel lines. The observer on the train measures the length at a fixed time, so $\Delta t' = 0$. The Lorentz transformation gives

$$\Delta x = \gamma \Delta x'.$$

Again, the apparently moving platform yields a shorter length according to the observer on the train.

Notice that in both cases the Euclidean lengths of the $\Delta x'$ sides of each triangle appear longer than side Δx between points A and B. How can this be? Remember, these increments in spacetime are given by the invariant Δs in both systems. First consider the situation in Figure 10. We have that $(\Delta s)^2$ between A and B is given by

$$(\Delta x)^2 = (\Delta x')^2 - (-c\Delta t')^2.$$

From the Lorentz transformations, setting $\Delta t = 0$, we also have

$$c\Delta t' = -\beta\gamma\Delta x.$$

So,

$$\begin{aligned} (\Delta x)^2 &= (\Delta x')^2 - (-c\Delta t')^2 \\ &= (\Delta x')^2 - (\beta\gamma\Delta x)^2 \end{aligned} \tag{12}$$

Rearranging,

$$(\Delta x')^2 = (1 + \beta^2\gamma^2)(\Delta x)^2 = \gamma^2(\Delta x)^2,$$

where we have used

$$1 + \beta^2\gamma^2 = 1 + \frac{\beta^2}{1-\beta^2} = \frac{1}{1-\beta^2} = \gamma^2.$$

The final result is that $\Delta x' = \gamma\Delta x$. This indicates that $\Delta x' > \Delta x$.

In Figure 11 the computation is simpler. We have

$$(\Delta x')^2 = (\Delta x)^2 - (c\Delta t)^2.$$

From the slope of the segment $\Delta x'$ is given as

$$\frac{c\Delta t}{\Delta x} = \beta.$$

So,

$$(\Delta x')^2 = (\Delta x)^2 - (\beta\Delta x)^2 = \gamma^{-2}(\Delta x)^2.$$

This leads to the relation $\Delta x = \gamma\Delta x'$, showing $\Delta x > \Delta x'$ in Figure 11.

Summary

We have presented an introduction to some of the consequences of special relativity (simultaneity, time dilation, and length contraction) as depicted using Lorentz transformations and the superimposed Minkowski diagrams for two observers. We gave simple derivations of the time dilation and length contraction equations, derived them from the Lorentz transformations, Minkowski diagrams and the 1+1 dimensional increment form of the Minkowski line element.

Denoting the proper time interval by Δt_0 and the proper length by L_0, the time dilation and length contraction equations can be written as

$$\Delta t = \gamma \Delta t_0, \tag{13}$$

$$L = \frac{L_0}{\gamma}, \tag{14}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = v/c$.