Geodesic Problem

R. L. Herman

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Consider the line element

$$ds^2 = -\left[1 - \left(\frac{\omega r}{c}\right)^2\right] c^2\, dt^2 + dr^2 + r^2\, d\phi^2 - 2\omega r^2\, dt d\phi.$$

We want to find the geodesic equations and the Christoffel symbols.

1 Lagrangian Approach

Recall that for the line element $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = -c^2\, d\tau^2$ we seek to extremize

$$c\tau_{AB} = \int_0^1 \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}\, d\lambda.$$

Defining the Lagrangian

$$L(x^\gamma, \dot{x}^\gamma) = \sqrt{-g_{\alpha\beta}(x^\gamma) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}} = \frac{d\tau}{d\lambda},$$

the geodesic equations are found from the Euler-Lagrange Equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right) - \frac{\partial L}{\partial x^\gamma} = 0.$$

where $\gamma = 0, 1, 2, 3$.

For this problem the Lagrangian is given by

$$L = \sqrt{\left[1 - \left(\frac{\omega r}{c}\right)^2\right] c^2 \dot{t}^2 - \dot{r}^2 - r^2 \dot{\phi}^2 + 2\omega r^2 \dot{t}\dot{\phi}},$$

and we let $\gamma = t, \phi, r$.

We will write out the Euler-Lagrange equations for each $\gamma = t, \phi, r$, corresponding to the time, angle, and radial equations, respectively. From the geodesic equations in the form

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0$$

we can read off the Christoffel symbols.

Euler-Lagrange Equation for Time, t. Using $L\frac{d}{d\tau} = \frac{d}{d\lambda}$, we find

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) &= \frac{\partial L}{\partial t} \\
\frac{d}{d\lambda}\left(\frac{1}{2L}\left(2\left[1-\left(\frac{\omega r}{c}\right)^2\right]c^2\frac{dt}{d\lambda}+2\omega r^2\frac{d\phi}{d\lambda}\right)\right) &= 0 \\
L\frac{d}{d\tau}\left(\left[1-\left(\frac{\omega r}{c}\right)^2\right]c^2\frac{dt}{d\tau}+\omega r^2\frac{d\phi}{d\tau}\right) &= 0 \\
\left[c^2-\omega^2 r^2\right]\frac{d^2t}{d\tau^2}-2\omega^2 r\frac{dt}{d\tau}\frac{dr}{d\tau}+\omega r^2\frac{d^2\phi}{d\tau^2}+2\omega r\frac{d\phi}{d\tau}\frac{dr}{d\tau} &= 0. \qquad (1)
\end{aligned}$$

Euler-Lagrange Equation for Angle, ϕ.

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} \\
\frac{d}{d\lambda}\left(\frac{1}{2L}\left(-2r^2\frac{d\phi}{d\lambda}+2\omega r^2\frac{dt}{d\lambda}\right)\right) &= 0 \\
L\frac{d}{d\tau}\left(-r^2\frac{d\phi}{d\tau}+\omega r^2\frac{dt}{d\tau}\right) &= 0 \\
-r^2\frac{d^2\phi}{d\tau^2}-2r\frac{d\phi}{d\tau}\frac{dr}{d\tau}+\omega r^2\frac{d^2t}{d\tau^2}+2\omega r\frac{dt}{d\tau}\frac{dr}{d\tau} &= 0. \qquad (2)
\end{aligned}$$

Euler-Lagrange Equation for the Radial variable, r.

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\
\frac{d}{d\lambda}\left(\frac{1}{2L}\left(-2\frac{dr}{d\lambda}\right)\right) &= \frac{1}{2L}\left(-2\omega^2 r\left(\frac{dt}{d\lambda}\right)^2-2r\left(\frac{d\phi}{d\lambda}\right)^2+4\omega r\frac{dt}{d\lambda}\frac{d\phi}{d\lambda}\right) \\
L\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) &= L\left(2\omega^2 r\left(\frac{dt}{d\tau}\right)^2+2r\left(\frac{d\phi}{d\tau}\right)^2-4\omega r\frac{dt}{d\tau}\frac{d\phi}{d\tau}\right) \\
\frac{d^2r}{d\tau^2} &= \omega^2 r\left(\frac{dt}{d\tau}\right)^2+r\left(\frac{d\phi}{d\tau}\right)^2-2\omega r\frac{dt}{d\tau}\frac{d\phi}{d\tau}. \qquad (3)
\end{aligned}$$

The general geodesic equation in terms of the Christoffel symbols is

$$\frac{d^2x^\alpha}{d\tau^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}=0.$$

The radial equation is obtained for $x^\alpha = r$,

$$\frac{d^2r}{d\tau^2} = -\Gamma^r_{tt}\frac{dt}{d\tau}\frac{dt}{d\tau}-\Gamma^r_{\phi\phi}\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}-\Gamma^r_{t\phi}\frac{dt}{d\tau}\frac{d\phi}{d\tau}-\Gamma^r_{\phi t}\frac{d\phi}{d\tau}\frac{dt}{d\tau}. \qquad (4)$$

Comparing this to Equation (3), we have

$$\Gamma^r_{tt} = -\omega^2 r, \quad \Gamma^r_{\phi\phi} = -r, \quad \Gamma^r_{t\phi} = \Gamma^r_{\phi t} = \omega r.$$

Getting the other Christoffel symbols is trickier because Equations (1) and (2) both contain $\frac{d^2t}{d\tau^2}$ and $\frac{d^2\phi}{d\tau^2}$ terms. So, we need to solve these coupled equations for the desired terms. Noting from Equation (2) that

$$r^2\frac{d^2\phi}{d\tau^2} = -2r\frac{d\phi}{d\tau}\frac{dr}{d\tau} + \omega r^2\frac{d^2t}{d\tau^2} + 2\omega r\frac{dt}{d\tau}\frac{dr}{d\tau},$$

Equation (1) can be rewritten as

$$\begin{aligned}
\left[c^2 - \omega^2 r^2\right]\frac{d^2t}{d\tau^2} &= 2\omega^2 r\frac{dt}{d\tau}\frac{dr}{d\tau} - \omega r^2\frac{d^2\phi}{d\tau^2} - 2\omega r\frac{d\phi}{d\tau}\frac{dr}{d\tau} \\
&= 2\omega^2 r\frac{dt}{d\tau}\frac{dr}{d\tau} - 2\omega r\frac{d\phi}{d\tau}\frac{dr}{d\tau} \\
&\quad -\omega\left[-2r\frac{d\phi}{d\tau}\frac{dr}{d\tau} + \omega r^2\frac{d^2t}{d\tau^2} + 2\omega r\frac{dt}{d\tau}\frac{dr}{d\tau}\right] \\
\left[c^2 - \omega^2 r^2\right]\frac{d^2t}{d\tau^2} &= -\omega^2 r^2\frac{d^2t}{d\tau^2}.
\end{aligned}$$

Therefore,

$$\frac{d^2t}{d\tau^2} = 0$$

and

$$\Gamma^t_{\beta\gamma} = 0$$

for all β and γ.

This leaves the ϕ-equation as

$$\frac{d^2\phi}{d\tau^2} = -\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} + \frac{2\omega}{r}\frac{dt}{d\tau}\frac{dr}{d\tau}.$$

We can read off the Christoffel symbols as

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{tr} = \Gamma^\phi_{rt} = \frac{\omega}{r}.$$

2 Direct Computation of Christoffel Symbols

The Christoffel symbols can be computed directly from

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right]$$

where $\Gamma^\delta_{\beta\gamma} = \Gamma^\delta_{\gamma\beta}$.

The metric coefficients are found from the line element,

$$ds^2 = -\left[1 - \left(\frac{\omega r}{c}\right)^2\right]c^2\,dt^2 + dr^2 + r^2\,d\phi^2 - 2\omega r^2\,dtd\phi,$$

as

$$g_{tt} = \omega^2 r^2 - c^2, \quad g_{rr} = 1, \quad g_{\phi\phi} = r^2, \quad g_{t\phi} = g_{\phi t} = -\omega r^2.$$

We can determine the Christoffel symbols by inserting different values for α.

Time Equations For $\alpha = t$, we have

$$g_{t\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial x^{\gamma}} + \frac{\partial g_{t\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial t}\right].$$

The last term is zero. Since g_{tt} and $g_{t\phi}$ are not zero, we have

$$g_{tt}\Gamma^{t}_{\beta\gamma} + g_{t\phi}\Gamma^{\phi}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial x^{\gamma}} + \frac{\partial g_{t\gamma}}{\partial x^{\beta}}\right].$$

The metric coefficients are only functions of r, $\gamma = r$ or $\beta = r$. Noting that $g_{rr} = 1$ and using symmetry, $\Gamma^{t}_{\beta r} = \Gamma^{t}_{r\beta}$, we take $\gamma = r$ to find

$$g_{tt}\Gamma^{t}_{\beta r} + g_{t\phi}\Gamma^{\phi}_{\beta r} = \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial r} + \frac{\partial g_{tr}}{\partial x^{\beta}}\right] = \frac{1}{2}\frac{\partial g_{t\beta}}{\partial r}.$$

Now, $\beta = t$ or $\beta = \phi$. This gives

$$\begin{aligned} g_{tt}\Gamma^{t}_{tr} + g_{t\phi}\Gamma^{\phi}_{tr} &= \frac{1}{2}\frac{\partial g_{tt}}{\partial r} = \omega^2 r, \\ g_{tt}\Gamma^{t}_{\phi r} + g_{t\phi}\Gamma^{\phi}_{\phi r} &= \frac{1}{2}\frac{\partial g_{t\phi}}{\partial r} = -\omega r. \end{aligned}$$

We will solve these equations for the Christoffel symbols using the next results for $\alpha = \phi$.

Angle Equations For $\alpha = \phi$, we have

$$g_{\phi\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\phi\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\phi\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial \phi}\right].$$

The last term is zero. Since the metric coefficients are only functions of r, $\gamma = r$ or $\beta = r$. For $\gamma = r$ we have

$$g_{\phi\phi}\Gamma^{\phi}_{\beta r} + g_{\phi t}\Gamma^{t}_{\beta r} = \frac{1}{2}\left[\frac{\partial g_{\phi\beta}}{\partial r} + \frac{\partial g_{\phi r}}{\partial x^{\beta}}\right] = \frac{1}{2}\frac{\partial g_{\phi\beta}}{\partial r}.$$

Now, $\beta = t$ or $\beta = \phi$. This gives the equations

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{tr} + g_{\phi t}\Gamma^{t}_{tr} &= \frac{1}{2}\frac{\partial g_{\phi t}}{\partial r} = -\omega r \\ g_{\phi\phi}\Gamma^{\phi}_{\phi r} + g_{\phi t}\Gamma^{t}_{\phi r} &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r} = r. \end{aligned}$$

We will solve these equations for the Christoffel symbols later.

Radial Equations For $\alpha = r$, we have

$$g_{r\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial x^{\gamma}} + \frac{\partial g_{r\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial r}\right].$$

We see that $\delta = r$, giving

$$g_{rr}\Gamma^r_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial x^\gamma} + \frac{\partial g_{r\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial r}\right].$$

Using symmetry, $\Gamma^r_{\beta r} = \Gamma^r_{r\beta}$, we let $\gamma = t, \phi, r$:

$$\begin{aligned}
\Gamma^r_{\beta t} &= \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial t} + \frac{\partial g_{rt}}{\partial x^\beta} - \frac{\partial g_{\beta t}}{\partial r}\right] = -\frac{1}{2}\frac{\partial g_{\beta t}}{\partial r}. \\
\Gamma^r_{\beta\phi} &= \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial \phi} + \frac{\partial g_{r\phi}}{\partial x^\beta} - \frac{\partial g_{\beta\phi}}{\partial r}\right] = -\frac{1}{2}\frac{\partial g_{\beta\phi}}{\partial r}. \\
\Gamma^r_{\beta r} &= \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial r} + \frac{\partial g_{rr}}{\partial x^\beta} - \frac{\partial g_{\beta r}}{\partial r}\right] = 0.
\end{aligned}$$

We see from these results that

$$\begin{aligned}
\Gamma^r_{tt} &= -\frac{1}{2}\frac{\partial g_{tt}}{\partial r} = -\omega^2 r. \\
\Gamma^r_{\phi t} &= -\frac{1}{2}\frac{\partial g_{\phi t}}{\partial r} = \omega r. \\
\Gamma^r_{\phi\phi} &= -\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r} = -r.
\end{aligned}$$

We still need to extract the Christoffel symbols from the equations

$$\begin{aligned}
g_{tt}\Gamma^t_{tr} + g_{t\phi}\Gamma^\phi_{tr} &= \omega^2 r, \\
g_{tt}\Gamma^t_{\phi r} + g_{t\phi}\Gamma^\phi_{\phi r} &= -\omega r, \\
g_{\phi\phi}\Gamma^\phi_{tr} + g_{\phi t}\Gamma^t_{tr} &= -\omega r, \\
g_{\phi\phi}\Gamma^\phi_{\phi r} + g_{\phi t}\Gamma^t_{\phi r} &= r.
\end{aligned}$$

The first and third equations give

$$\begin{aligned}
(\omega^2 r^2 - c^2)\Gamma^t_{tr} - \omega r^2 \Gamma^\phi_{tr} &= \omega^2 r, \\
-\omega r^2 \Gamma^t_{tr} + r^2 \Gamma^\phi_{tr} &= -\omega r.
\end{aligned}$$

Multiply the second equation by ω,

$$\begin{aligned}
(\omega^2 r^2 - c^2)\Gamma^t_{tr} - \omega r^2 \Gamma^\phi_{tr} &= \omega^2 r, \\
-\omega^2 r^2 \Gamma^t_{tr} + \omega r^2 \Gamma^\phi_{tr} &= -\omega^2 r.
\end{aligned}$$

Adding, we have $-c^2\Gamma^t_{tr} = 0$. Therefore,

$$\Gamma^t_{tr} = 0, \quad \Gamma^\phi_{tr} = \Gamma^\phi_{rt} = \frac{\omega}{r}.$$

The second and fourth equations give

$$\begin{aligned}
(\omega^2 r^2 - c^2)\Gamma^t_{\phi r} - \omega r^2 \Gamma^\phi_{\phi r} &= -\omega r, \\
-\omega r^2 \Gamma^t_{\phi r} + r^2 \Gamma^\phi_{\phi r} &= r.
\end{aligned}$$

Multiply the second equation by ω,

$$\begin{aligned}(\omega^2 r^2 - c^2)\Gamma^t_{\phi r} - \omega r^2 \Gamma^\phi_{\phi r} &= -\omega r,\\ -\omega^2 r^2 \Gamma^t_{\phi r} + \omega r^2 \Gamma^\phi_{\phi r} &= \omega r.\end{aligned}$$

Adding, we have $-c^2\Gamma^t_{\phi r} = 0$. Therefore,

$$\Gamma^t_{\phi r} = 0, \quad \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}.$$

3 Summary

In summary, using both methods, we have the nonzero Christoffel symbols for this metric are given by

$$\Gamma^r_{tt} = -\omega^2 r, \quad \Gamma^r_{\phi\phi} = -r, \quad \Gamma^r_{t\phi} = \Gamma^r_{\phi t} = \omega r,$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \quad \Gamma^\phi_{tr} = \Gamma^\phi_{rt} = \frac{\omega}{r}.$$

The geodesic equations in standard form were found as

$$\begin{aligned}\frac{d^2 t}{d\tau^2} &= 0\\ \frac{d^2\phi}{d\tau^2} &= -\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau} + \frac{2\omega}{r}\frac{dt}{d\tau}\frac{dr}{d\tau}.\\ \frac{d^2 r}{d\tau^2} &= \omega^2 r\left(\frac{dt}{d\tau}\right)^2 + r\left(\frac{d\phi}{d\tau}\right)^2 - 2\omega r\frac{dt}{d\tau}\frac{d\phi}{d\tau}.\end{aligned}$$