Geodesic Equations for the Wormhole Metric

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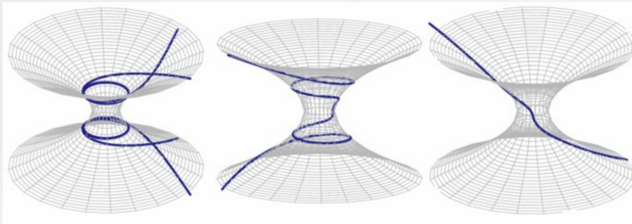
The Wormhole Metric

Morris and Thorne wormhole metric: [M. S. Morris, K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, *Am. J. Phys.* **56**, 395-412, 1988.]

$$\boxed{ds^2 = -c^2dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)}$$



Embedding $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2)$

Consider 2D slices (t =const, $\theta = \pi/2$). Then, ($c = 1$) becomes

$$d\Sigma^2 = dr^2 + (b^2 + r^2)\, d\phi^2.$$

Compare to cylindrical coordinate line element $(\rho(r), \psi, z(r))$

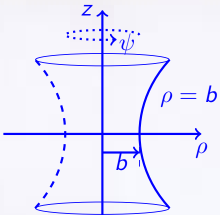
$$dS^2 = d\rho^2 + \rho^2\, d\psi^2 + dz^2 = \left[\left(\frac{dz}{dr}\right)^2 + \left(\frac{d\rho}{dr}\right)^2\right] dr^2 + \rho^2(r)\, d\phi^2.$$

Then, $\rho^2 = r^2 + b^2$ and

$$\left(\frac{dz}{dr}\right)^2 = \frac{b^2}{b^2 + r^2}.$$

Integrating, $z = b\sinh^{-1}\frac{r}{b}$, or

$$\rho = b\cosh\frac{z}{b}.$$



Lagrangian

Begin with metric $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. Then, ($c = 1$)

$$\tau_{AB} = \int_0^1 \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}\, d\lambda.$$

Euler-Lagrange Equations $\Rightarrow$ Geodesic Equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\gamma}\right) - \frac{\partial L}{\partial x^\gamma} = 0.$$

where $\gamma = 0, 1, 2, 3$, $\dot{x}^\gamma = \dfrac{dx^\gamma}{d\lambda}$, and

$$L(x^\gamma, \dot{x}^\gamma) = \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} = \frac{d\tau}{d\lambda}.$$

Geodesic Equations

The Key Equations

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0,$$

or in terms of the four-velocity:

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0.$$

Here the Christoffel Symbols are given by

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right]$$

where $\Gamma^\delta_{\beta\gamma} = \Gamma^\delta_{\gamma\beta}$.

Wormhole Geodesics via Lagrangian

Begin with the proper time,

$$d\tau^2 = -ds^2 = dt^2 - dr^2 - (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2),$$

Write the Lagrangian,

$$L = \sqrt{\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - (b^2 + r^2)\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)},$$

Apply the Euler-Lagrange equation for each variable: t, r, θ, ϕ.

Example - time variable t, $\quad \dot{t} \equiv \frac{dt}{d\lambda}$:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0.$$

Time Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for t: [Recall that $L\frac{d}{d\tau} = \frac{d}{d\lambda}$]

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) &= \frac{\partial L}{\partial t} \\
\frac{d}{d\lambda}\left(\frac{\cancel{2}}{\cancel{2}L}\frac{dt}{d\lambda}\right) &= 0 \\
L\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right) &= 0
\end{aligned}$$

$$\boxed{\frac{d^2 t}{d\tau^2} = 0.}$$

Radial Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for r:

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) &= \frac{\partial L}{\partial r} \\
L\frac{d}{d\tau}\left(-\frac{1}{L}\frac{dr}{d\lambda}\right) &= -\frac{1}{2L}(2r)\left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]
\end{aligned}$$

$$\boxed{\frac{d^2 r}{d\tau^2} = r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right].}$$

The θ-Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for θ:

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{\partial L}{\partial \theta} \\
L\frac{d}{d\tau}\left(-\frac{b^2 + r^2}{L}\frac{d\theta}{d\lambda}\right) &= -\frac{1}{2L}(b^2 + r^2)\left[2\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]
\end{aligned}$$

$$\boxed{\frac{d}{d\tau}\left((b^2 + r^2)\frac{d\theta}{d\tau}\right) = (b^2 + r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2.}$$

The ϕ-Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right]^{1/2}$$

Geodesic Equation for ϕ:

$$\begin{aligned}
\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) &= \frac{\partial L}{\partial \phi} \\
L\frac{d}{d\tau}\left(-\frac{b^2 + r^2}{L}\sin^2\theta\frac{d\phi}{d\lambda}\right) &= 0
\end{aligned}$$

$$\boxed{\frac{d}{d\tau}\left((b^2 + r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) = 0.}$$

Set of Geodesic Equations

$$\begin{aligned}
\frac{d^2 t}{d\tau^2} &= 0 \\
\frac{d^2 r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \\
\frac{d}{d\tau}\left((b^2 + r^2)\frac{d\theta}{d\tau}\right) &= (b^2 + r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \\
\frac{d}{d\tau}\left((b^2 + r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) &= 0.
\end{aligned}$$

- Solve for geodesics $(t(\tau), r(\tau), \theta(\tau), \phi(\tau))$.
- Read off Christoffel Symbols, $\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}$

Christoffel Symbols

Start with general Geodesic Equation:

$$\boxed{\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$

$$\begin{aligned}
\frac{d^2 t}{d\tau^2} &= 0 \\
\frac{d^2 r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]
\end{aligned}$$

Read off the coefficients:

- $\Gamma^t_{\beta\gamma} = 0$.
- $\Gamma^r_{\theta\theta} = -r$, $\Gamma^r_{\phi\phi} = -r\sin^2\theta$.

Christoffel Symbols (cont'd)

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$

$$\begin{aligned}
\frac{d}{d\tau}\left((b^2+r^2)\frac{d\theta}{d\tau}\right) &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 \\
(b^2+r^2)\frac{d^2\theta}{d\tau^2} + 2r\frac{dr}{d\tau}\frac{d\theta}{d\tau} &= (b^2+r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 \\
\frac{d^2\theta}{d\tau^2} &= -\frac{2r}{b^2+r^2}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2.
\end{aligned}$$

- $\Gamma^\theta_{\theta r} = \frac{r}{b^2+r^2} = \Gamma^\theta_{r\theta}$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.
- Note: $\Gamma^\theta_{\theta r}$ and $\Gamma^\theta_{r\theta}$ contribute equally, thus there is no 2.

Christoffel Symbols (cont'd)

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}}$$

$$\begin{aligned}
\frac{d}{d\tau}\left((b^2+r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) &= 0. \\
(b^2+r^2)\sin^2\theta\frac{d^2\phi}{d\tau} &= -2r\sin^2\theta\frac{dr}{d\tau}\frac{d\phi}{d\tau} \\
&\quad -2(b^2+r^2)\sin\theta\cos\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}. \\
\frac{d^2\phi}{d\tau} &= -\frac{2r}{b^2+r^2}\frac{dr}{d\tau}\frac{d\phi}{d\tau} - 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau}.
\end{aligned}$$

- $\Gamma^\phi_{\phi r} = \frac{r}{b^2+r^2} = \Gamma^\phi_{r\phi}$, $\Gamma^\phi_{\phi\theta} = \cot\theta$.

Christoffel Symbols from Metric

The Christoffel symbols are defined by

$$\boxed{g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right].}$$

For the wormhole metric,

$$ds^2 = -dt^2 + dr^2 + (b^2+r^2)(d\theta^2 + \sin^2\theta\, d\phi^2).$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b^2+r^2 & 0 \\ 0 & 0 & 0 & (b^2+r^2)\sin^2\theta \end{pmatrix},$$

or, $g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2+r^2$, $g_{\phi\phi} = (b^2+r^2)\sin^2\theta$.

Christoffel Symbols $\Gamma^t_{\beta\gamma}$

The nonzero metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2+r^2$, $g_{\phi\phi} = (b^2+r^2)\sin^2\theta$.
Let $\alpha = t$ and $x^\alpha = t$, then

$$g_{t\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial x^\gamma} + \frac{\partial g_{t\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial t}\right].$$

Since the $g_{t\mu}$ is nonzero and constant for $\mu = t$,

$$\begin{aligned}
g_{tt}\Gamma^t_{\beta\gamma} &= \frac{1}{2}\left[\frac{\partial g_{t\beta}}{\partial x^\gamma} + \frac{\partial g_{t\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial t}\right] \\
g_{tt}\Gamma^t_{tt} &= \frac{1}{2}\left[\frac{\partial g_{tt}}{\partial t} + \frac{\partial g_{tt}}{\partial t} - \frac{\partial g_{tt}}{\partial t}\right] = 0. \qquad (1)
\end{aligned}$$

So, $\Gamma^t_{\alpha\beta} = 0$ for all α and β.

Christoffel Symbols $\Gamma^r_{\alpha\beta}$

The metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2+r^2$, $g_{\phi\phi} = (b^2+r^2)\sin^2\theta$.
Let $\alpha = r$ and $x^\alpha = r$, then

$$g_{r\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{r\beta}}{\partial x^\gamma} + \frac{\partial g_{r\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial r}\right].$$

Thus, $\delta = r$ and either $\beta = \gamma = \theta$ or $\beta = \gamma = \phi$. So, we have

$$g_{rr}\Gamma^r_{\theta\theta} = \frac{1}{2}\left[\frac{\partial g_{r\theta}}{\partial\theta} + \frac{\partial g_{r\theta}}{\partial\theta} - \frac{\partial g_{\theta\theta}}{\partial r}\right].$$

$$g_{rr}\Gamma^r_{\phi\phi} = \frac{1}{2}\left[\frac{\partial g_{r\phi}}{\partial\phi} + \frac{\partial g_{r\phi}}{\partial\phi} - \frac{\partial g_{\phi\phi}}{\partial r}\right].$$

Therefore, since $g_{rr} = 1$,

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\phi\phi} = -r\sin^2\theta.$$

Christoffel Symbols $\Gamma^\theta_{\alpha\beta}$

The metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2+r^2$, $g_{\phi\phi} = (b^2+r^2)\sin^2\theta$.
Let $\alpha = \theta$ and $x^\alpha = \theta$, then

$$g_{\theta\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\theta\beta}}{\partial x^\gamma} + \frac{\partial g_{\theta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial\theta}\right].$$

Thus, $\delta = \theta$. We take $\beta = \theta$ or $\beta = \phi$ due to symmetry. So, we have

$$g_{\theta\theta}\Gamma^\theta_{\theta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\theta\theta}}{\partial x^\gamma} + \frac{\partial g_{\theta\gamma}}{\partial\theta} - \frac{\partial g_{\theta\gamma}}{\partial\theta}\right].$$

$$g_{\theta\theta}\Gamma^\theta_{\phi\gamma} = \frac{1}{2}\left[\frac{\partial g_{\theta\phi}}{\partial x^\gamma} + \frac{\partial g_{\theta\gamma}}{\partial\phi} - \frac{\partial g_{\phi\gamma}}{\partial\theta}\right].$$

Nonzero terms occur for $\gamma = r$ in first and $\gamma = \phi$ in second equation.

Christoffel Symbols $\Gamma^{\theta}_{\alpha\beta}$ (cont'd)

Since $g_{\theta\theta} = b^2 + r^2$ and $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$, we have

$$\begin{aligned} g_{\theta\theta}\Gamma^{\theta}_{\theta r} &= \frac{1}{2}\left[\frac{\partial g_{\theta\theta}}{\partial r} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta r}}{\partial \theta}\right]. \\ (b^2 + r^2)\Gamma^{\theta}_{\theta r} &= \frac{1}{2}\frac{\partial g_{\theta\theta}}{\partial r} = r \\ \Gamma^{\theta}_{\theta r} &= \frac{r}{b^2 + r^2} = \Gamma^{\theta}_{r\theta}. \end{aligned}$$

and

$$\begin{aligned} g_{\theta\theta}\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2}\left[\frac{\partial g_{\theta\phi}}{\partial \phi} + \frac{\partial g_{\theta\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial \theta}\right]. \\ (b^2 + r^2)\Gamma^{\theta}_{\phi\phi} &= -\frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial \theta} = -(b^2 + r^2)\sin\theta\cos\theta \\ \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta. \end{aligned}$$

Christoffel Symbols $\Gamma^{\phi}_{\alpha\beta}$

The metric elements are
$g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$.
Let $\alpha = \phi$ and $x^{\alpha} = \phi$, then

$$g_{\phi\delta}\Gamma^{\delta}_{\beta\gamma} = \frac{1}{2}\left[\frac{\partial g_{\phi\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\phi\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial \phi}\right].$$

Thus, $\delta = \phi$ and we take $\beta = \phi$ due to symmetry. So, we have

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{\phi\gamma} &= \frac{1}{2}\left[\frac{\partial g_{\phi\phi}}{\partial x^{\gamma}} + \frac{\partial g_{\phi\gamma}}{\partial \phi} - \frac{\partial g_{\phi\gamma}}{\partial \phi}\right] \\ &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial x^{\gamma}}. \end{aligned}$$

Since $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$, then $\gamma = r$ or $\gamma = \theta$.

Christoffel Symbols $\Gamma^{\phi}_{\alpha\beta}$ (cont'd)

Since $g_{\phi\phi} = (b^2 + r^2)\sin^2\theta$, we have

$$\begin{aligned} g_{\phi\phi}\Gamma^{\phi}_{\phi r} &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial r}. \\ (b^2 + r^2)\sin^2\theta\,\Gamma^{\phi}_{\phi r} &= r\sin^2\theta \\ g_{\phi\phi}\Gamma^{\phi}_{\phi\theta} &= \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial \theta}. \\ (b^2 + r^2)\sin^2\theta\,\Gamma^{\phi}_{\phi\theta} &= (b^2 + r^2)\sin\theta\cos\theta \end{aligned}$$

Therefore, we have

$$\Gamma^{\phi}_{\phi r} = \frac{r}{b^2 + r^2} = \Gamma^{\phi}_{r\phi}, \quad \Gamma^{\phi}_{\phi\theta} = \cot\theta = \Gamma^{\phi}_{\theta\phi}.$$

Wormhole Metric and Geodesic Equations

$$\boxed{ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2\theta\, d\phi^2).}$$

$$\begin{aligned} \frac{d^2 t}{d\tau^2} &= 0 \\ \frac{d^2 r}{d\tau^2} &= r\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \\ \frac{d}{d\tau}\left((b^2 + r^2)\frac{d\theta}{d\tau}\right) &= (b^2 + r^2)\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \\ \frac{d}{d\tau}\left((b^2 + r^2)\sin^2\theta\frac{d\phi}{d\tau}\right) &= 0. \end{aligned}$$

Christoffel Symbols $\Gamma^{r}_{\theta\theta} = -r$, $\Gamma^{r}_{\phi\phi} = -r\sin^2\theta$, $\Gamma^{\theta}_{\theta r} = \frac{r}{b^2+r^2} = \Gamma^{\theta}_{r\theta}$, $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^{\phi}_{\phi r} = \frac{r}{b^2+r^2} = \Gamma^{\phi}_{r\phi}$, $\Gamma^{\phi}_{\phi\theta} = \cot\theta = \Gamma^{\phi}_{\theta\phi}$.

Maple Routine

```
> restart: with( tensor ):
Declare coordinates in desired order.
> coord := [t, r, theta, phi]:
Enter metric components to produce g.
> gg:=array(symmetric,sparse,1..4,1..4):
  gg[1,1] := -1:  gg[2,2] := 1:     gg[3,3] := r^2+b^2:     gg[4,4] := (r^2+b^2)*sin(theta)^2:
> g := create( [-1,-1], eval(gg));
```

$$g := \text{table}([compts = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2+b^2 & 0 \\ 0 & 0 & 0 & (r^2+b^2)\sin(\theta)^2 \end{bmatrix}, index\_char = [-1,-1]])$$

```
Run main routine and display Chritoffel symbols (of second kind).
> tensorsGR(coord,g,contra_metric,det_met, C1, C2, Rm, Rc, R, G, C):
> displayGR(Christoffel2,C2);
```

The Christoffel Symbols of the Second Kind

non-zero components :

$\{2,33\} = -r$

$\{2,44\} = -r\sin(\theta)^2$

$\{3,23\} = \dfrac{r}{r^2+b^2}$

$\{3,44\} = -\sin(\theta)\cos(\theta)$

$\{4,24\} = \dfrac{r}{r^2+b^2}$

$\{4,34\} = \dfrac{\cos(\theta)}{\sin(\theta)}$