General Relativity and Cosmology – Part I

I. Cosmology
  a. Nature of universe – isotropic, homogeneous
  b. Size and age of universe, Milky Way, etc
  c. Content of Universe
  d. Hubble's Law $v = H_0 r$
  e. Redshift $z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \approx \frac{v}{c}$.
  f. CMB
  g. Important players in early cosmology

II. Space, Time and Gravity
  a. Coordinate Transformations – displacement(translation), rotation, uniform motion, Galilean transformation, Lorentz transformation
  b. Rotation of Axes $x' = x\cos\varphi + y\sin\varphi, y' = -x\sin\varphi + y\cos\varphi, z' = z$
  c. Principle of Relativity
  d. Newtonian Gravity
    i. $\nabla^2\Phi = 4\pi G_N \rho$, $\mathbf{F} = -m\nabla\Phi = m\mathbf{g}$
  e. Inertial mass vs gravitational mass, Eötvös
  f. Einstein's Equivalence Principle
  g. Pound, Rebka, Snider
  h. Gravitational redshift, time intervals: $\Delta\tau_B = \Delta\tau_A\left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right)$,

     fractional frequency shift: $\frac{\Delta\omega}{\omega} = \frac{\Delta\Phi}{c^2}$
  i. Light deflection:

III. Special Relativity
  a. Einstein's Postulates and consequences
    i. Simultaneity
    ii. Time Dilation $d\tau = dt\sqrt{1 - V^2/c^2}$ or $\Delta t' = \gamma\Delta t_0$, $\gamma = (1 - v^2/c^2)^{-1/2}$
    iii. Length Contraction $L' = L_0/\gamma$
  b. Line element $ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2$
  c. Spacetime Diagrams and Light Cones
    i. timelike, spacelike, null, worldlines
  d. Lorentz Transformations/Boosts

     $t' = \gamma(t - vx/c^2), x' = \gamma(x - vt), y' = y, z' = z$
  e. Proper time $d\tau^2 = -ds^2/c^2$
  f. Addition of Velocities $u_x' = \frac{dx'}{dt'} = \frac{u_x - v}{1 - vu_x/c^2}$, etc.
  g. Paradoxes

IV. Four Vectors and Dynamics
  a. $\mathbf{a} = a_\alpha \mathbf{e}^\alpha$ for $\alpha = 0,1,2,3$ - using Einstein Summation Convention
  b. $\mathbf{a}\cdot\mathbf{b} = \eta_{\alpha\beta} a^\alpha b^\beta$ for $\eta = diag(-1,1,1,1)$
  c. $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$
  d. $x^\alpha = x^\alpha(\tau), u^\alpha = \frac{dx^\alpha}{d\tau} = (\gamma, \gamma\vec{V})$, $\mathbf{u}\cdot\mathbf{u} = -1$

e. $m\mathbf{a} = m\dfrac{d\mathbf{u}}{d\tau} = \mathbf{f},\ \mathbf{f}\cdot\mathbf{u} = 0$

f. $\mathbf{p} = m\mathbf{u} \Rightarrow \mathbf{p}\cdot\mathbf{p} = -m^2,\ p^\alpha = (E, \vec{p}) = (m\gamma, m\gamma\vec{V})$

V. Geometry

a. Line elements in space – Cartesian, polar, spherical, cylindrical

i. $dS^2 = dx^2 + dy^2,\ dS^2 = dr^2 + (rd\phi)^2,\ dS^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$

b. Metric tensor from $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$

c. Manipulating indices – free and repeated, Einstein summation convention

d. Sphere

i. Sum of Interior Angles of a Triangle = $\pi + \dfrac{A}{a^2}$

ii. Sphere: $\dfrac{C}{r} = 2\pi\dfrac{\sin(r/a)}{r/a}$ and geometries like $ds^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$.

e. Line Elements

i. $dS^2 = a^2(d\theta^2 + f^2(\theta)d\phi^2)$

ii. $C(\theta) = \int_0^{2\pi} af(\theta)\,d\theta,\ d_{pole-pole} = \int_0^{\pi} a\,d\theta$

VI. Geodesics

a. Variational Principle: $s[x(t)] = \int_{t_A}^{t_B} L(\dot{x}(t), x(t))\,dt \quad \delta s = 0 \Rightarrow -\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{x}}\right) + \dfrac{\partial L}{\partial x} = 0$

b. Space: $\delta s = 0 \quad L = \sqrt{g_{ab}\dfrac{dx^a}{d\lambda}\dfrac{dx^b}{d\lambda}}$

c. Spacetime: $\delta\tau = 0 \quad L = \sqrt{-g_{\alpha\beta}\dfrac{dx^\alpha}{d\sigma}\dfrac{dx^\beta}{d\sigma}}$.

d. The Geodesic Equation for Timelike Geodesics

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}, \quad \text{or} \quad \frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$

e. Christoffel Symbols $g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \dfrac{1}{2}\left(\dfrac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \dfrac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \dfrac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$