Instructions:

- Place your name on all of the pages.
- Do all of your work in this booklet. Do not tear off any sheets.
- Be clear and neat in your work. Any illegible work, or scribbling in the margins, will not be graded.
- All short answers and essays should be responded to with full sentences conveying thoughtful responses.
- If you need more space, you may use the **back of a page** and write *On back of page #* in the problem space or the **extra page**. **No other paper is allowed.**

Try to answer as many problems as possible. Provide as much information as possible. Show sufficient rationale for full credit.

Use proper significant digits and units.

Pay attention to the point distribution. Not all problems have the same weight. Pace yourself!

Page	Pts	Score
1	22	
2	20	
3	15	
4	13	
Total	70	

Some Constants - $M_{sun} = 1.989 \times 10^{30}$ kg, $M_{earth} = 5.98 \times 10^{24}$ kg, $G = 6.673 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}

$1\ pc = 3.086 \times 10^{16}$ m $= 3.26$ lyr ,

Bonus: Let $ds^2 = -X^2 dT^2 + dX^2$. What Lagrangian is needed to find the geodesics in this spacetime?

1. (8 pts) Provide the following based on our current observations/theory:

    a. How old is the universe?   _____

    b. How many galaxies are there?   _____

    c. Hubble's constant.   _____

    d. What is the current CMB temperature?   _____

2. (6 pts) M31 is 778 kpc from us.
    a. What is the common name of M31?   _____

    b. How fast should M31 be receding from us according to Hubble's Law?

    c. The redshift parameter for M31 is $z = -0.001001$. What does this tell you about
        the speed of M31 relative to the Earth?

3. (2 pts) Describe the following:
    a. Homogeneous

    b. Isotropic

4. (6 pts) Give exact expressions for the following:
    a. The 2D spatial line element in polar coordinates.

    b. Minkowski spacetime in Cartesian coordinates.

    c. Minkowski spacetime in spherical coordinates.

5. (3 pts) Which of the following are a correct use of indices: _____
    a. $g_{\alpha\beta}dx^{\alpha}dx^{\beta} = g_{\alpha\beta}dx^{\alpha}dx^{\gamma}$.
    b. $g_{\alpha\beta}a^{\alpha}b^{\beta} = g_{\beta\gamma}a^{\beta}b^{\gamma}$.
    c. $\Gamma^{\beta}_{\alpha\beta} = \Gamma^{\beta}_{\beta\beta}$.

6. (4 pts) Let $a^{\alpha} = (-2,0,0,1)$ and $b^{\alpha} = (5,0,3,4)$.
    a. Are these four-vectors timelike, spacelike, or null? a: _____ b: _____

    b. Compute $\mathbf{a} \cdot \mathbf{b}$.

7. (3 pts) At what speed does a clock move if it runs one half the rate of a clock at rest?

8. (3 pts) Two rockets of length L_0 approach each other from opposite directions at speed $c/2$ relative to an observer on Earth. How long (in terms of L_0) does a rocket appear to an observer on the other rocket?

9. (4 pts) A signal of frequency 900 MHz is emitted from the ground floor of the Sears (Willis) Tower in Chicago. An antenna receives it at the top of the tower (530 m). What is the fractional change in the frequency received? Is it blueshifted or redshifted when received? _____

10. (3 pts) A triangle is drawn on a ball of radius 5.0 cm. The sum of the interior angles is $5\pi / 2$. What fraction of the surface area does the triangle take up?

11. (4 pts) An astronaut orbits a $10\,M_{sun}$ black hole at a radius of 35 km for 1 year (on her watch). She then returns to the parent spaceship, located very far from the black hole. How much time elapsed at the spaceship while orbiting the black hole?

12. (4 pts) Let $(x, ct) = (240, 300)$ m in System S. In a system moving at $0.8c$ with respect to S, what are the measured coordinates (x', ct')?

13. (4 pts) Consider the worldline given by $x^{\alpha} = (\sinh \tau, \cosh \tau, 0, 0)$. Determine the four-velocity. Is this path time-like, space-like, or null? _____

14. (3 pts) Consider the metric tensor given by $g_{\mu\nu} = \begin{pmatrix} 0 & f(r) & 0 & 0 \\ f(r) & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$ for a given set of coordinates (U, V, θ, ϕ), Write out the corresponding line element.

15. (3 pts) Prove that the four-velocity of a particle satisfies $\mathbf{u} \cdot \mathbf{u} = -1$.

16. (10 pts) The spacetime diagram shows four supernovae events at A, B, C, and D. These supernovae are observed on the Earth (system S) and on a fast moving spaceship (system S'). Note that the worldlines of the Earth and spaceship start at $(x,t) = (x',t') = (0,0)$.
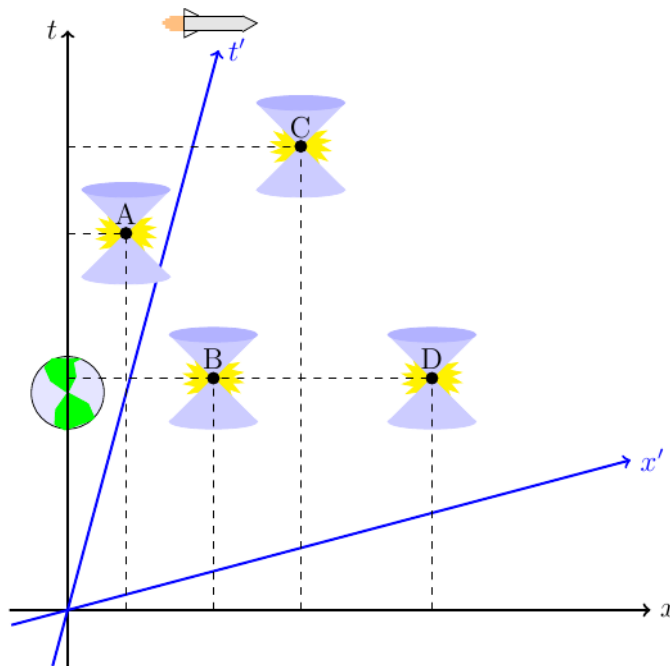
Answer the following questions:
- In which chronological order do the supernovae occur in the Earth frame of reference? _____
- In which chronological order do the supernovae occur in the spaceship frame of reference? _____
- In which chronological order do astronomers on Earth see the supernovae?
  _____
- In which chronological order do observers on the spaceship see the supernovae?
  _____
- Is the chronological order in which supernovae A and B occur the same in all frames of reference? Explain. _____

Extra Page