Spherical Turkeys and Vibrating Balloons

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How long does it take to cook a turkey?

Example 1

If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

A Thanksgiving turkey.

Panofsky Equation

- Pief Panofsky [SLAC Director Emeritus] *SLAC Today*, Nov 26, 2008 http://today.slac.stanford.edu/a/2008/11-26.htm For a stuffed turkey at 325° F

$$t = \frac{W^{2/3}}{1.5}$$

  vs. 30 minutes/lb.
- Also, check out WolframAlpha http://www.wolframalpha.com/input/?i=how+long+should+you+cook+a+turkey
- Musings of an Energy Nerd http://www.greenbuildingadvisor.com/blogs/dept/musings/heat-transfer-when-roasting-turkey

Consider a Spherical Turkey

The depiction of a spherical turkey.

Scaling a Spherically Symmetric Turkey

The baking follows the heat equation in the form

$$u_t = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

Rescale the coordinates (r, t) to (ρ, τ) :

$$r = \beta\rho \text{ and } t = \alpha\tau.$$

Then, the derivatives transform as

$$\begin{aligned}\frac{\partial}{\partial r} &= \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = \frac{1}{\beta}\frac{\partial}{\partial \rho},\\ \frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{1}{\alpha}\frac{\partial}{\partial \tau}.\end{aligned} \tag{1}$$

Scaling

Inserting these transformations into the heat equation, we have

$$u_\tau = \frac{\alpha}{\beta^2}\frac{k}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right).$$

- Invariance of heat equation implies $\alpha = \beta^2$.
- Gives a self-similarity transformation: $r = \beta\rho$, and $t = \beta^2\tau$.
- So, if the radius increases by a factor of β, then the time to cook the turkey increases by β^2.

Problem Solution

Example 1

If it takes 4 hours to cook a 10 pound turkey in a 350^o F oven, then how long would it take to cook a 20 pound turkey at the same conditions?

- The weight of the doubles $\Rightarrow$ the volume doubles. (if density = constant).
- $V \propto r^3 \Rightarrow r$ increases by factor: $2^{1/3}$.
- Therefore, the time increases by a factor of $2^{2/3} \approx 1.587$.
- For a 20 lb turkey:

$$t = 4(2^{2/3}) = 2^{8/3} \approx 6.35 \text{ hours.}$$

Solving the Heat Equation, $u_t = k\nabla^2 u$

Example 2

Find the temperature, $T(\rho, t)$ inside a spherical turkey, initially at 40° F, which is placed in a 350° F oven.

This is a heat equation problem for $T(\rho, t)$:

$$\begin{aligned} T_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right), \quad 0 < \rho < a, t > 0, \\ T(a,t) &= 350, \quad T(\rho,t) \quad \text{finite at } \rho = 0, \quad t > 0, \\ T(\rho,0) &= 40. \end{aligned} \tag{2}$$

Introduce the auxiliary function

$$u(\rho, t) = T(\rho, t) - 350.$$

Homogeneous Boundary Conditions

The problem to be solved becomes

$$\begin{aligned} u_t &= \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad 0<\rho<a, t>0, \\ u(a,t) &= 0, \quad u(\rho,t) \quad \text{finite at } \rho=0, \quad t>0, \\ u(\rho,0) &= T_i - T_a = -310, \end{aligned} \tag{3}$$

where $T_i = 40$, $T_a = 350$.
Solve using Method of Separation of Variables: $u(\rho,t) = R(\rho)G(t)$.

$$u(\rho,t) = \sum_{n=1}^{\infty} A_n \frac{\sin\frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

Particular Solution

The general solution for the temperature

$$T(\rho, t) = T_a + \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

Using the initial condition, $T(\rho, 0) = T_i = 40$, we obtain the Fourier sine series.

$$(T_i - T_a)\rho = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi\rho}{a},$$

where

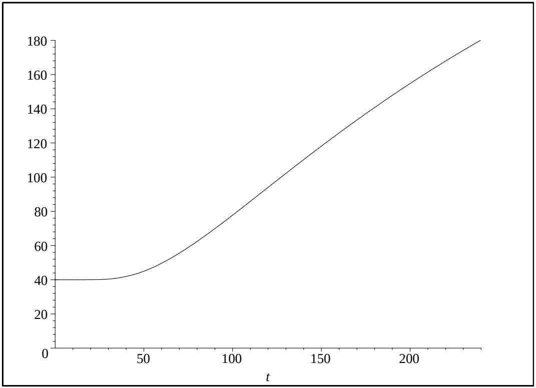
$$A_n = \frac{2}{a}\int_0^a (T_i - T_a)\rho \sin \frac{n\pi\rho}{a}\, d\rho = \frac{2a}{n\pi}(T_i - T_a)(-1)^{n+1}, \qquad (4)$$

The solution:

$$T(\rho, t) = T_a + \frac{2a(T_i - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

Roasting Times for Two Turkeys

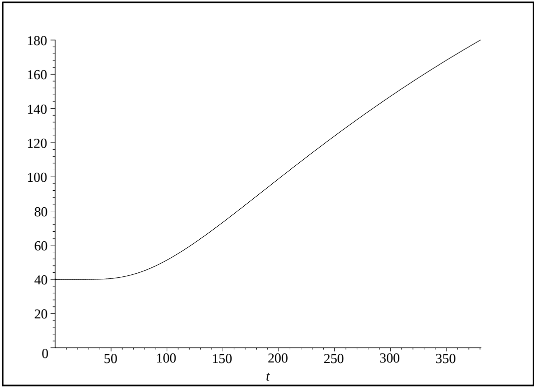
If a 6" radius turkey takes 4 hours to cook (to 180° F), the solution with 400 terms gives $k \approx 0.000089$ and a "baking time" of $t_1 = 239.63$.



The temperature at the center with $a = 0.5$ ft and $k \approx 0.000089$.

Roasting Times for Two Turkeys (cont'd)

Increasing the radius of the turkey to $a = 0.5(2^{1/3})$ ft (doubling the volume), we obtain $t_2 = 380.38$.



The temperature at the center with $a = 0.5(2^{1/3})$ ft.

Comparing Baking Times

Comparing the two temperatures, we find the ratio

$$\frac{t_2}{t_1} = \frac{380.3813709}{239.6252478} \approx 1.587401054$$

as compared to

$$2^{2/3} \approx 1.587401052.$$

But .... there are no spherical, uniform turkeys.

Balloons - Vibrations of a Spherical Surface

Let $u(\theta, \phi, t)$ obey the wave equation, $u_{tt} = c^2 \nabla^2 u$, or

$$u_{tt} = \frac{c^2}{r^2} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 u}{\partial\phi^2} \right), \qquad (5)$$

satisfying the initial conditions

$$u(\theta, \phi, 0) = f(\theta, \phi), \quad u_t(\theta, \phi, 0) = g(\theta, \phi).$$

Assume $u = u(\theta, \phi, t)$ remains bounded and satisfies periodic boundary conditions

$$u(\theta, 0, t) = u(\theta, 2\pi, t), \quad u_\phi(\theta, 0, t) = u_\phi(\theta, 2\pi, t),$$

where $t > 0$ and $\quad 0 < \theta < \pi$.

Solution I

The wave equation takes the form

$$u_{tt} = \frac{c^2}{r^2} Lu, \quad \text{where} \quad LY_{\ell m} = -\ell(\ell+1) Y_{\ell m}$$

for the spherical harmonics $Y_{\ell m}(\theta, \phi) = P_\ell^m(\cos\theta) e^{im\phi}$,
We seek product solutions of the form

$$u_{\ell m}(\theta, \phi, t) = T(t) Y_{\ell m}(\theta, \phi).$$

Inserting this form into the wave equation in spherical coordinates, we find

$$T'' + \ell(\ell+1) \frac{c^2}{r^2} T(t) = 0.$$

Solution II

The solutions of this equation are easily found as

$$T(t) = A\cos\omega_\ell t + B\sin\omega_\ell t, \quad \omega_\ell = \sqrt{\ell(\ell+1)}\frac{c}{r}.$$

Therefore, the product solutions are given by

$$u_{\ell m}(\theta,\phi,t) = [A\cos\omega_\ell t + B\sin\omega_\ell t]\, Y_{\ell m}(\theta,\phi)$$

for $\ell = 0, 1, \ldots,\ m = -\ell, -\ell+1, \ldots, \ell.$
The general solution is found as

$$u(\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} [A_{\ell m}\cos\omega_\ell t + B_{\ell m}\sin\omega_\ell t]\, Y_{\ell m}(\theta,\phi).$$

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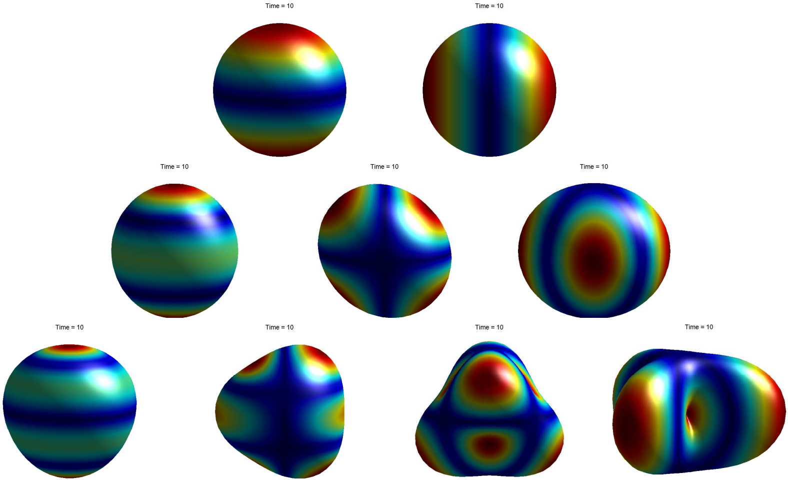
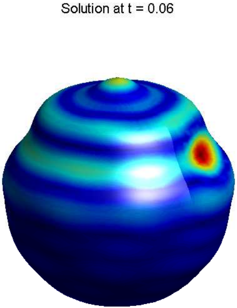


Modes for a vibrating spherical membrane:
`http://people.uncw.edu/hermanr/pde1/sphmem/`
Row 1: $(1,0),(1,1)$; Row 2: $(2,0),(2,1),(2,2)$;
Row 3 $(3,0),(3,1),(3,2),(3,3)$.

An interesting problem is to consider hitting the balloon with a velocity impulse while at rest.



A moment captured from a simulation of a spherical membrane after hit with a velocity impulse.

http://russherman.com/Talks/

Summary

Conclusion

Partial Differential Equations can be fun!

Next - A Vibrating Spherical Turkey.

Separation of Variables

Let $u(\rho, t) = R(\rho)G(t)$. Inserting into the heat equation for u, we have

$$\frac{1}{k}\frac{G'}{G} = \frac{1}{R}\left(R'' + \frac{2}{\rho}R'\right) = -\lambda.$$

This give the two ordinary differential equations, the temporal equation,

$$G' = -k\lambda G, \tag{6}$$

and the radial equation,

$$\rho R'' + 2R' + \lambda\rho R = 0. \tag{7}$$

The temporal equation is easy to solve,

$$G(t) = G_0 e^{-\lambda k t}.$$

Radial Equation, $\rho R'' + 2R' + \lambda \rho R = 0$

Making the substitution $R(\rho) = y(\rho)/\rho$, we obtain

$$y'' + \lambda y = 0.$$

- Boundary conditions for $u(\rho, t) = R(\rho)G(t)$ become
  - $R(a) = 0$ and $R(\rho)$ finite at the origin.
- Then, $y(a) = 0$ and $y(\rho)$ has to vanish near the origin.
  - If $y(\rho)$ does not vanish $\to 0$, then $R(\rho)$ is not finite as $\rho \to 0$.]

So, we need to solve the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(a) = 0.$$

This has the well-known set of eigenfunctions

$$y(\rho) = \sin \frac{n\pi\rho}{a}, \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

General Solution

Therefore, we have found

$$R(\rho) = \frac{\sin \frac{n\pi\rho}{a}}{\rho}, \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, 3, \ldots.$$

The general solution to the auxiliary problem is

$$u(\rho, t) = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

This gives the general solution for the temperature as

$$T(\rho, t) = T_a + \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$

Particular Solution

Using the initial condition, $T(\rho, 0) = 40$, we have

$$T_i - T_a = \sum_{n=1}^{\infty} A_n \frac{\sin \frac{n\pi\rho}{a}}{\rho}.$$

Multiplying by ρ, we have the Fourier sine series.

$$(T_i - T_a)\rho = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi\rho}{a}.$$

Solving for the coefficients,

$$A_n = \frac{2}{a} \int_0^a (T_i - T_a)\rho \sin \frac{n\pi\rho}{a}\, d\rho = \frac{2a}{n\pi}(T_i - T_a)(-1)^{n+1}, \tag{8}$$

gives the solution,

$$T(\rho, t) = T_a + \frac{2a(T_i - T_a)}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{\sin \frac{n\pi\rho}{a}}{\rho} e^{-(n\pi/a)^2 kt}.$$