Fractals in Nature and Mathematics: From Simplicity to Complexity

Dr. R. L. Herman, UNCW Mathematics & Physics
Outline

1. Complexity in Nature
2. What are Fractals?
3. Geometric Fractals
4. Fractal Dimensions
5. Function Iteration
6. Applications
Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow
- Fractals
- Studied “roughness” in nature
- Fractal Geometry
- Mandelbrot Set
- TED Talk link

"Think of color, pitch, loudness, heaviness, and botness. Each is the topic of a branch of physics."

Benoît Mandelbrot
Figure: How would you measure roughness and complexity
Trees

Figure: Self-similarity
Figure: Fractals - what do you see?
Figure: Leaf and rivers

Fractals in Nature and Mathematics

Figure: Lungs

R. L. Herman

OLLI STEM Society, Oct 13, 2017
Ferns

Figure: Do you see similarity at different scales?
Ferns - Example of Self-Similarity
Figure: What is the length of the coastline?
What are Fractals?

- From *fractus*, - “broken”
- Self-similarity
- Fractional Dimension

*Figure: Fractal Fern*
Figure: Fractals - self-similarity, roughness
Figure: Fractals - what do you see?
Step 1 \((L = 1)\)
Geometric Fractals - Koch Curve (1904)

Step 1 ($L = 1$)

Step 2

Step 3 ($L = 4(\frac{1}{3}) = 4\frac{1}{3}$)

Step 4

Step 5 ($L = 16\frac{9}{9}$)
Geometric Fractals - Koch Curve (1904)

Step 1 ($L = 1$)

Step 2

Step 3 ($L = 4(\frac{1}{3}) = \frac{4}{3}$)
Geometric Fractals - Koch Curve (1904)

Step 1 \((L = 1)\)

Step 2

Step 3 \((L = 4(\frac{1}{3}) = \frac{4}{3})\)

Step 4

Step 5 \((L = \frac{16}{9})\)
Geometric Fractals - Koch Curve (1904)

Step 1 \( (L = 1) \)

Step 2

Step 4

Step 5 \( (L = \frac{16}{9}) \)
Koch Curve - Self Similarity \( (L = \frac{4^n}{3^n} \rightarrow \infty) \)
Simple Fractals - Sierpinski Triangle
Simple Fractals - Sierpinski Triangle
Simple Fractals - Sierpinski Triangle
Simple Fractals - Sierpinski Triangle
Simple Fractals - Sierpinski Triangle

Self-similar
Dimensions - \( r = \) magnification, \( n = \) Number of shapes

\[
\begin{align*}
\text{ Dimensions } & \quad n = 2, \quad r = 2 \\
\Rightarrow \ln 2 &= \ln 2. \\
\text{ Dimensions } & \quad n = 4, \quad r = 2 \\
\Rightarrow \ln 4 &= 2 \ln 2. \\
\text{ Dimensions } & \quad n = 8, \quad r = 2 \\
\Rightarrow \ln 8 &= 3 \ln 2.
\end{align*}
\]

\[
D = \frac{\ln n}{\ln r}.
\]
For each step length of line segment is reduced by $r = 3$.

The number of lines increases by factor $n = 4$.

Therefore

$$D = \frac{\ln n}{\ln r} = \frac{\ln 4}{\ln 3} = 1.26.$$
Sierpinski Triangle - Dimension

Self-similar

\[ D = \frac{\ln 3}{\ln 2} = 1.585 \]
Coastlines - Great Britain, $D = 1.25$
What is the fractal dimension of the NC coast?
Iterations

Fractals in Nature and Mathematics  R. L. Herman  OLLI STEM Society, Oct 13, 2017  22/41
Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$

**Figure:** $r = 0.05$
Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$

Figure: $r = 2.0$
Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$

Figure: $r = 3.1$
Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$
Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$

Figure: $r = 3.56$
Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$

Figure: $r = 4.0$
Bifurcations of Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given $x_0$
Iterate complex numbers

\[ z = a + bi, \ i = \sqrt{-1}. \]

\[ z_{n+1} = z_n^2 + c, \ z_0 = 0. \]

Example: \( c = 1 \):

\[ 0, 1, 2, 5, 26 \ldots . \]

Example: \( c = -1 \):

\[ 0, -1, 0, -1, 0, \ldots . \]
Mandelbrot Set - $z_{n+1} = z_n^2 + c, \quad z_0 = 0$.

Example: $c = i$:

\[
\begin{align*}
x_0 & = 0 \\
x_1 & = 0^2 + i = i \\
x_2 & = i^2 + i = -1 + i \\
x_3 & = (-1 + i)^2 + i = -i \\
x_4 & = (-i)^2 + i = -1 + i \\
x_4 & = (-i)^2 + i = -i
\end{align*}
\]

Gives period 2 orbit =
\[
\{ -1 + i, -i, -1 + i, -i, \ldots \}.
\]

- Show more points at Fractals site.
- Mandelbrot Set Zoom online.
Mandelbrot Set - Bulbs

Numbers in yellow indicate the number of dendrites or spiral arms found in each region, and in the corresponding Julia fractals for each region.
Mandelbrot Set - Bulbs
Mandelbrot Set - Plot and Zoom

http://www.flashandmath.com/advanced/mandelbrot/MandelbrotPlot.html
Iterated Function Systems

- Iterated Function Systems
- Scalings, Rotations, and Translations

Fractals in Nature and Mathematics  R. L. Herman  OLLI STEM Society, Oct 13, 2017
The Collage Theorem and Fractal Image Compression.

- Fractal Landscapes
- First completely computer-generated sequence in a film http://design.osu.edu/carlson/history/tree/images/pages/genesis1_jpeg.htm

https://www.youtube.com/watch?v=QXbWCrzWJo4
Mountains

Clouds
1D Midpoint Displacement Algorithm
1D Midpoint Displacement Algorithm
1D Midpoint Displacement Algorithm
1D Midpoint Displacement Algorithm
Diamond - Square Algorithm

- Square of size $2^n + 1$.
- Find Midpoint, adding random small heights.
- Create Diamond.
- Edge midpoints, ...
Diamond - Square Algorithm

- Square of size $2^n + 1$.
- Find Midpoint, adding random small heights.
- Create Diamond.
- Edge midpoints, . . .
Other Applications

- Video Games
- Fracture - link
- Ceramic Material - link
- Biology - wrinkles, lungs, brain, ...
- Astrophysics

Figure: The Sloan Digital Sky Survey
Conclusion

- Fractals - Measure of Roughness
- Fractal Dimension
- Function Iteration - Mandelbrot Set
- Applications