Nonlinear Waves in Physics:

From Solitary Waves and Solitons to Rogue Waves

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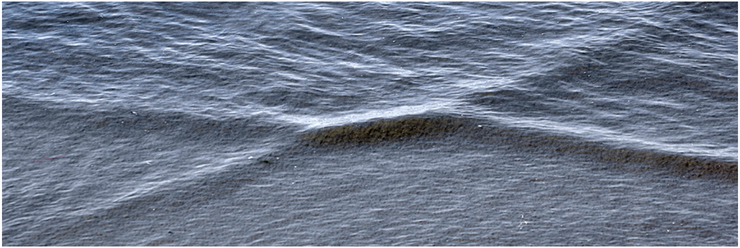


Outline

Soliton History

Korteweg-deVries Equation

Other Nonlinear PDEs

Soliton Perturbations

Integral Modified KdV

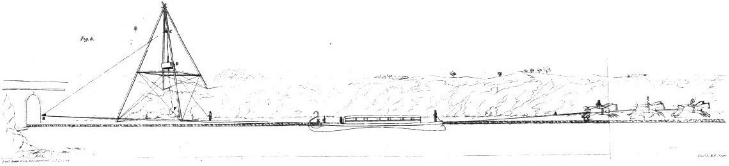
Nonlinear Schrödinger Equation

Rogue Waves

- hydrodynamics,
- plasmas,
- nonlinear optics,
- biology,
- relativistic field theory,
- relativity,
- geometry ...

Great Wave of Translation - 1834

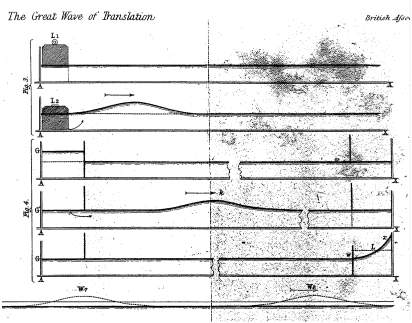
I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when **the boat suddenly stopped** *not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently* **without change of form or diminution of speed.** *I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour [14 km/h], preserving its original figure some thirty feet [9 m] long and a foot to a foot and a half [300-450 mm] in height. Its height gradually diminished, and after a chase of one or two miles [2-3km] I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the* **Wave of Translation.** - John Scott Russell



Union Canal, Hermiston, Scotland, `http://www.ma.hw.ac.uk/~chris/canal.jpg`

John Scott Russell (1808-1882)

- Engineer, Scotland
- Used 30 ft tank
- $v^2 = g(h+a)$
- "Committee on Waves"
- Reports: 1837, 1844



http://www.ma.hw.ac.uk/~chris/scott_russell.html

Years of Controversy

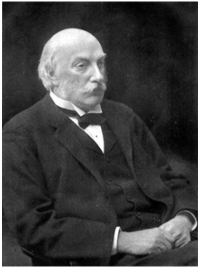
George Biddle Airy (1801-1892).

Sir George Gabriel Stokes (1819-1903).

Publications on water waves: George Green (1839), Philip Kelland (1840), George Biddell Airy (1941- *Tides and Waves*), and Samuel Earnshaw (1847). Tried to do better than Lagrange, Laplace, Cauchy, and Poisson.

1870's - Nonlinear Theory from Euler's Equations

Joseph Valentin Boussinesq (1842-1929).



John William Strutt (Lord Rayleigh) (1842-1919).

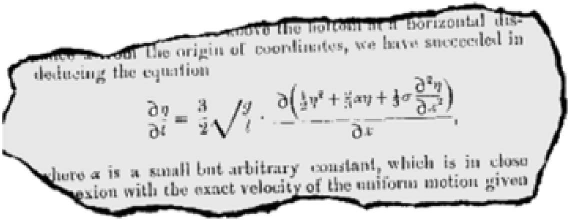
$$u(x,t) = 2\eta^2 \operatorname{sech}^2(\eta(x - 4\eta^2 t)).$$

Rayleigh (1876) derived correct approximate solution, w/dispersion and nonlinearity, later learned of Boussinesq's (1871) work.

Korteweg-de Vries Equation - 1895

Gustav de Vries (1866-1934)

Diederik Johannes Korteweg (1848-1941)



above the bottom at a horizontal dis-
the origin of coordinates, we have succeeded in
deducing the equation

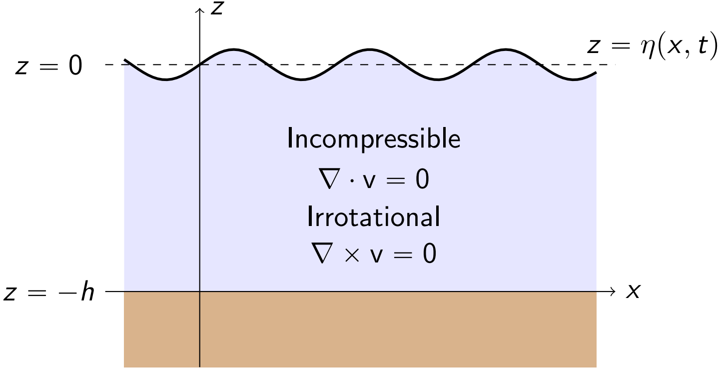
$$\frac{\partial\eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{l}} \cdot \frac{\partial\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right)}{\partial x},$$

here α is a small but arbitrary constant, which is in close
exion with the exact velocity of the uniform motion given

Fluid Equations - Mass and Momentum Conservation

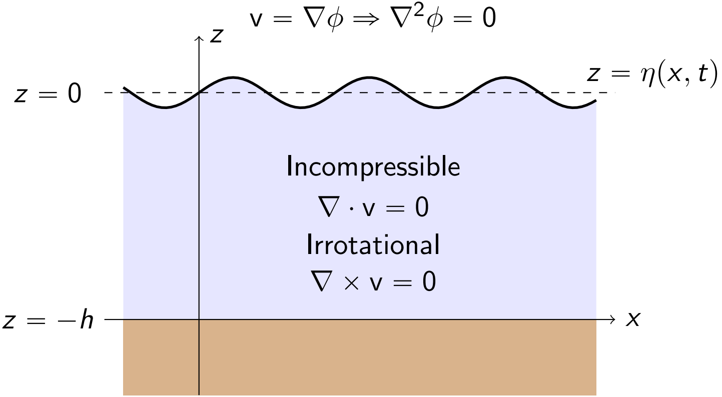
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathrm{v}) = 0$$

$$\frac{\partial \mathrm{v}}{\partial t} + \mathrm{v} \cdot \nabla \cdot \mathrm{v} = \frac{1}{\rho}\left(-\nabla P + \mathrm{f}\right)$$

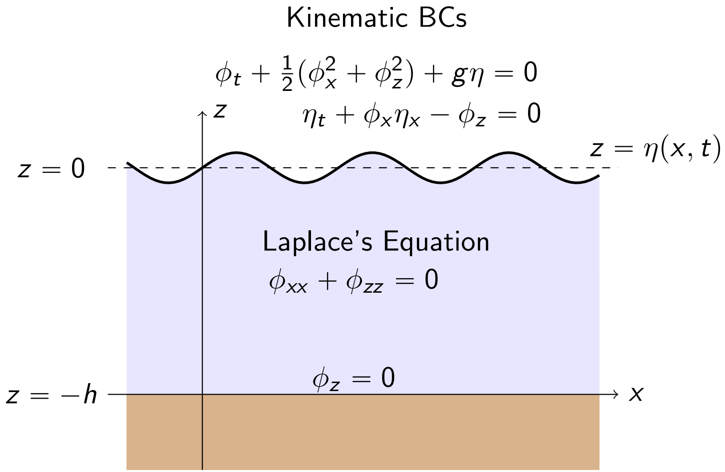


KdV Derivation

Laplace's Equation



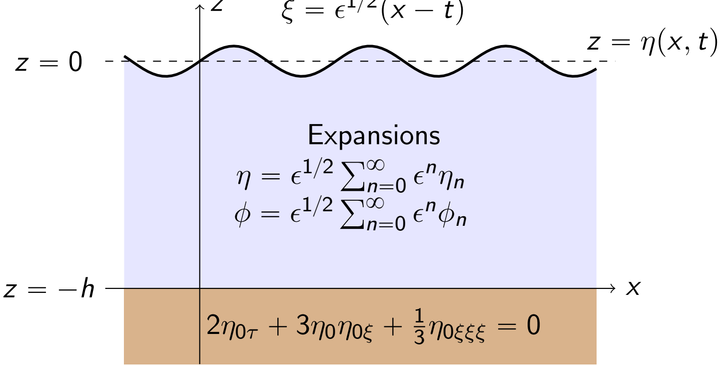
KdV Derivation - Boundary Conditions



KdV Derivation - Reductive Perturbation Theory

Scaling

$$\tau = \epsilon^{3/2} t$$
$$\xi = \epsilon^{1/2}(x - t)$$



KdV Traveling Waves - $u_t + 6uu_x + u_{xxx} = 0$

Seek solutions of form $u(x,t) = f(x - ct) \equiv f(\xi)$,
Noting $u_x = f'(\xi)$ and $u_t = -cf'(\xi)$, we obtain ODE for $f(\xi)$:

$$-cf' + 6ff' + f''' = 0.$$

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$$[-cf + 3f^2 + f'']' = 0.$$

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$$[-cf + 3f^2 + f'']' = 0.$$

$$-cf + 3f^2 + f'' = A.$$

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$$-cff' + 3f^2 f' + f'' f' = Af'.$$

$$-\frac{1}{2}cf^2 + f^3 + \frac{1}{2}f'^2 = Af + B.$$

KdV Traveling Waves - $u_t + 6uu_x + u_{xxx} = 0$

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$$-cf + 3f^2 + f'' = A.$$

$$-cff' + 3f^2f' + f''f' = Af'.$$

$$-\frac{1}{2}cf^2 + f^3 + \frac{1}{2}f'^2 = Af + B.$$

$$\sqrt{\frac{c}{2}}\frac{df}{d\xi} = \sqrt{Af + B + \frac{1}{2}cf^2 - f^3}.$$

KdV Traveling Waves - $u_t + 6uu_x + u_{xxx} = 0$

Traveling wave solution, $u(x,t) = f(x-ct) \equiv f(\xi)$,

$$\sqrt{\frac{c}{2}}\frac{df}{d\xi} = \sqrt{Af + B + \frac{1}{2}cf^2 - f^3}.$$

$$\xi - \xi_0 = \sqrt{\frac{c}{2}}\int \frac{df}{\sqrt{Af + B + \frac{1}{2}cf^2 - f^3}}.$$

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For $A = B = 0$,

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$u(x,t) = 2\eta^2 \operatorname{sech}^2(\eta(x - 4\eta^2 t))$ where $c = 4\eta^2$.

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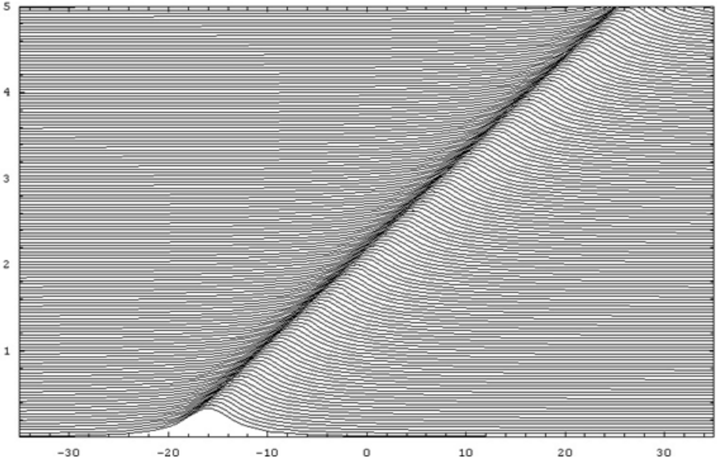
$u(x,t) = 2\eta^2 \operatorname{sech}^2(\eta(x - 4\eta^2 t))$ where $c = 4\eta^2$.

For $A, B \neq 0$, get Jacobi Elliptic functions.

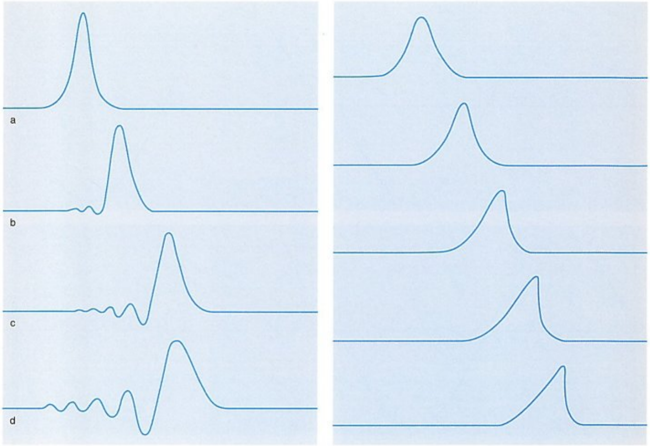
- Leads to (periodic) cnoidal waves.

Soliton Solution - $u(x, t) = 2\eta^2 \operatorname{sech}^2(\eta(x - 4\eta^2 t))$



Time vs position plot of soliton solution, for the KdV $u_t + 6uu_x + u_{xxx} = 0$.

Dispersion vs Nonlinearity - $u_t + 6uu_x + u_{xxx} = 0$



Dispersion (left): Waves spread and amplitude diminishes.

Nonlinearity (right): Width decreases and waves steepen. (Herman 1992)

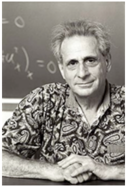
Re-enactment

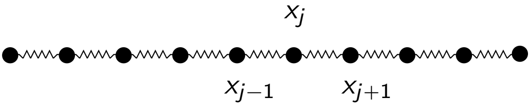
July 12, 1995 Union Canal, Scott Russell Aqueduct,

http://apachepersonal.miun.se/~tomnil/solitoner/solipic.htm

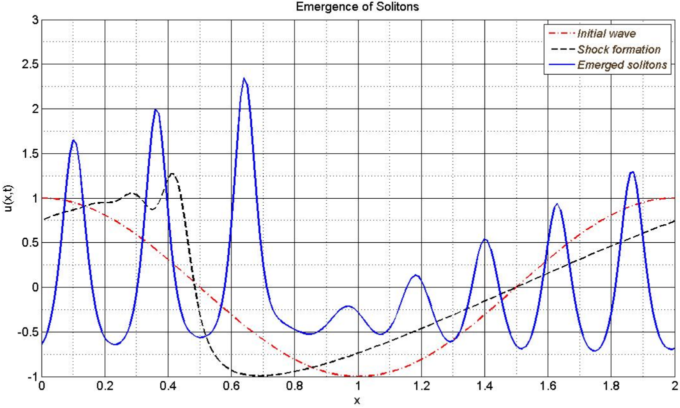
The KdV Resurgence 1960's

- Fermi, Pasta, Ulam (FPU) Problem (1953-4)
  - 1D chain of masses linked by nonlinear springs.
  - Observed - Energy does not go to equipartition, but periodically returns to original mode.
- Kruskal and Zabusky 1965
  - FPU Problem → KdV.
  - Discrete to continuous
  - Observed emergence of energy modes and then recurrence.
  - Coined term "soliton."





Zabusky-Kruskal (1965) - Recurrence (Recreated)



Soliton Research Begins - Inverse Scattering

- Gardner, Greene, Kruskal, Miura 1967
  - Inverse Scattering Transform (IST)
  - Nonlinear Fourier Transform
  - Sparks a NLEE Revolution

Transform

$$u(x,0) \longrightarrow A(k,0)$$

$$u_t + 6uu_x + u_{xxx} = 0$$

KdV Equation

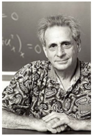
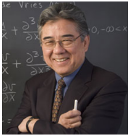
$$A_t = ik^3 A$$

Evolve Scattering Data

$$u(x,t) \longleftarrow A(k,t)$$

Inverse Scattering Transform



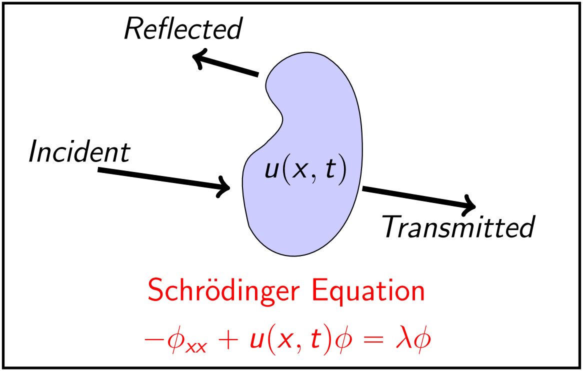



Lax Pair L, B for KdV

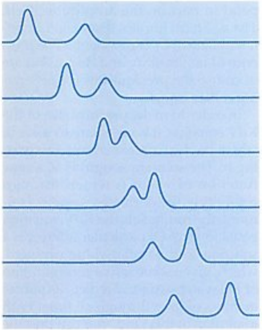
$L\phi = \lambda\phi$ and $\phi_t = B\phi$, where

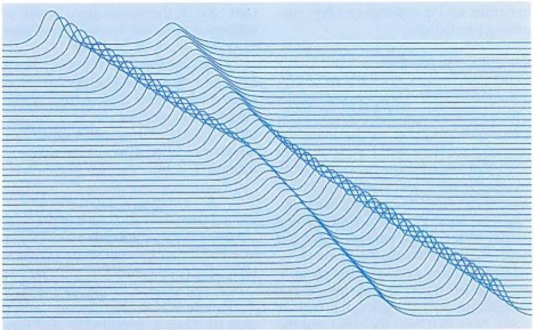
$$\begin{aligned} L\phi &= -\phi_{xx} + u(x,t)\phi \\ B\phi &= u_x\phi + (4\lambda - 2u)\phi_x \end{aligned} \tag{1}$$

gives evolution equation, $u_t = [L, B] = -uu_x - u_{xxx}$,



Two Soliton Solution of KdV

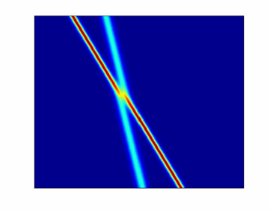


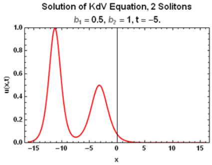


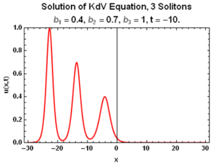
$$u(x,t) = 12\frac{3 + 4\cosh(2\xi + 24t) + \cosh(4\xi)}{[3\cosh(\xi - 12t) + \cosh(3\xi + 12t)]^2},$$

where $\xi = x - 16t$.

Solitons Elastic Collisions



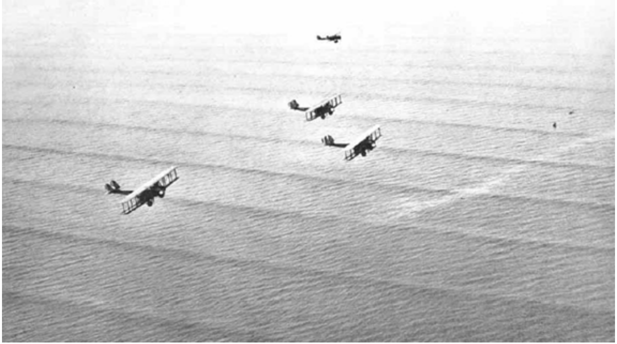




http://www.scholarpedia.org/article/Soliton

Cnoidal Waves

$\eta(x,t) = \eta_2 + H\mathrm{cn}^2\left(\frac{K(m)}{\lambda}(x-ct)|m\right)$ where $cn(x|m)$ is Jacobi elliptic function and $K(m)$ is complete elliptic integral.

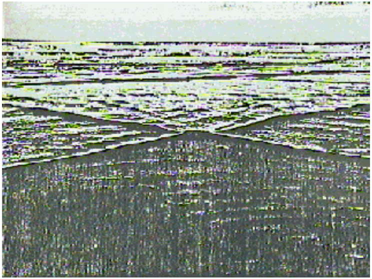


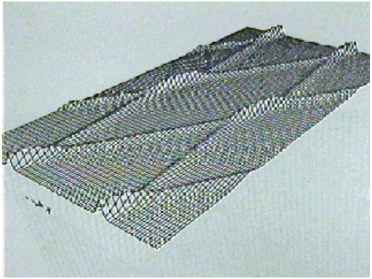
US Army bombers flying close to the Panama coast (1933).

Two-Dimensional Waves

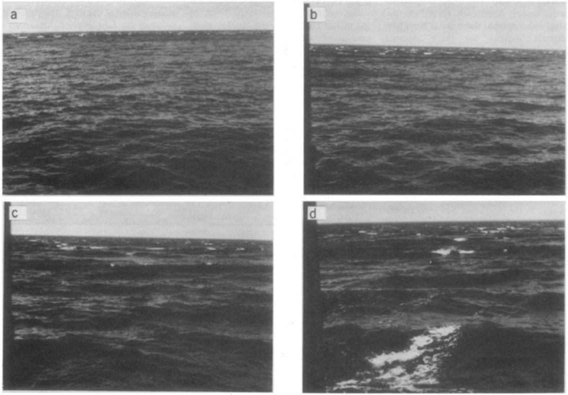
Kadomtsev-Petviashvili equation (1970),

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0.$$





Internal Waves



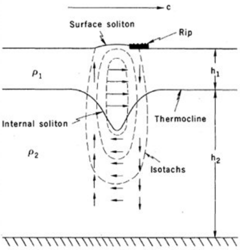
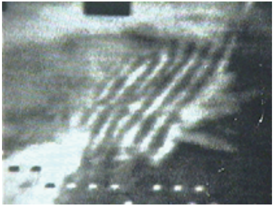
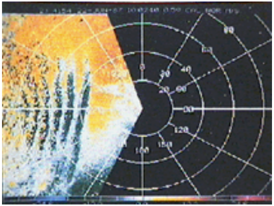


Fig. 3. An internal soliton in a two-layer fluid of finite depth is a wave of depression when $h_1 < h_2$. Dashed lines are lines of constant water particle speed (isotachs); the arrows indicate the magnitude and direction of the water particles. A small surface soliton of amplitude $\sim (\rho_2 - \rho_1)\eta_0$ accompanies the internal soliton (35). The approximate location of the surface rips, as observed in our Andaman Sea measurements, is also shown.

Rip Waves, Osborne and Burch - *Science*, Vol. 208, No. 4443 (May 2, 1980), pp. 451-460

Atmospheric Solitons

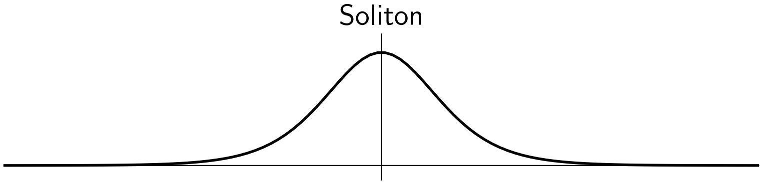


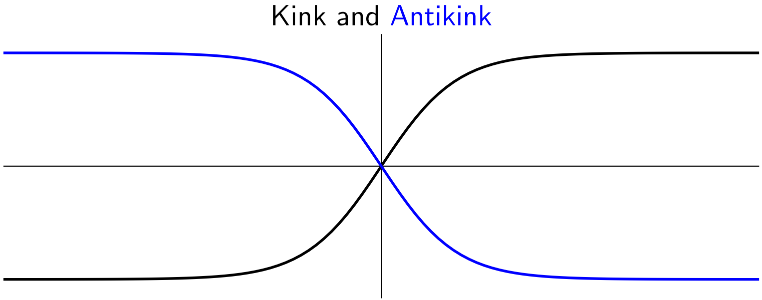


Solutions of Integrable Equations

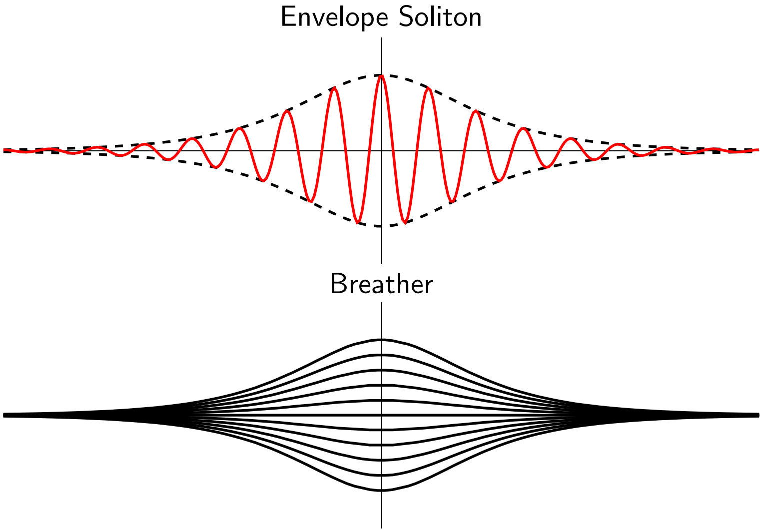
- Extending IST
  - Lax - 1968 $L\phi = \lambda\phi, \phi_t = B\phi \Rightarrow L_t = [L, B]$.
  - Zakharov and Shabat - 1973
  - Ablowitz, Kaup, Newell, Segur - 1974
- Research extended to other equations, dimensions
  - Modified KdV, $u_t + u^2 u_x + u_{xxx} = 0$.
  - sine-Gordon, $u_{tt} - u_{xx} = \sin u$.
    [Edmond Bour (1862), surfaces of constant negative curvature, later Frenkel and Kontorova (1939) - crystal dislocations]
  - Nonlinear Schrödinger, $i\psi_t = -\frac{1}{2}\psi_{xx} + \alpha\psi|\psi|^2$.
    [1973, optics, 1924/1995 Bose–Einstein condensates]
  - Toda Lattice,
    [1967, chain of particles with nearest neighbor interactions].
  - Coupled Systems [e.g., birefringence]
- Types: Solitons, Kinks, Breathers, Loop solitons, ...

Solitons, Kinks, ...





... Envelope Solitons, Breathers, ...



Cuspons, Compactons, and Loop Solitons

From Boussinesq-like Equations [Zhang and Chen],

$$u_{tt} + \left(u^2\right)_{xx} + \left(u^2\right)_{xxxx} = 0.$$

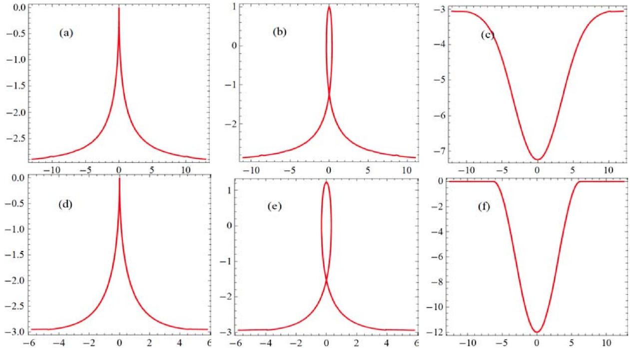


Fig. 2 – Soliton solutions; a) cuspon; b) loop soliton; c) smooth soliton; d) cuspon; e) loop soliton; f) compacton.

Soliton Perturbation Theory

Add Perturbation $\epsilon R[u]$, ϵ small.

$$u_T + \mathcal{N}[u] = \epsilon R[u], \qquad 0 < \epsilon \ll 1. \tag{2}$$

Multiple time scales

$$\partial_T = \partial_t + \epsilon \partial_\tau$$

Expansion about soliton solution $u_0 = u_0(z, \tau)$, $z = x - vt$

$$u(x, T) = u_0(z, \tau) + \epsilon u_1(z, t, \tau) + \dots. \tag{3}$$

Linearized Equation

$$u_{1t} + \hat{L}[u_1] = R[u_0] - u_{0\tau} = F(z). \tag{4}$$

Eigenfunction Expansion Method for $u_{1t} + \hat{L}[u_1] = F(z)$

Eigenfunctions and Adjoint Functions

$$\hat{L}\phi = \lambda\phi, \qquad \hat{L}^{\dagger}\psi = \lambda'\psi. \tag{5}$$

Expansion in Eigenfunctions

$$u_1(z,t) = \int U(t,\lambda)\phi(z,\lambda)\,d\lambda + \sum_j U_j(t)\phi_j(z). \tag{6}$$

$$U_t + \lambda U = \int_{-\infty}^{\infty} F(z)\psi(z,\lambda)\,dz, \quad U(0,\lambda) = 0, \tag{7}$$

$$U_{it} + \lambda_i U_i = \int_{-\infty}^{\infty} F(z)\psi_i(z,\lambda_i)\,dz, \quad U_i(0,\lambda_i) = 0. \tag{8}$$

Example: Perturbed KdV Equation

$$u_T + 6uu_x + u_{xxx} = \epsilon R[u]. \tag{9}$$

Leading Order: $[\partial_T = \partial_t + \epsilon\partial_\tau,\ \eta = \eta(\tau),\ \xi = \xi(t,\tau)]$

$$u_0(z) = 2\eta^2 \operatorname{sech}^2 z, \qquad z = \eta(x - \xi), \text{ and } \xi_t = 4\eta^2.$$

First Order:

$$u_{1t} + \eta^3 \hat{L} u_1 = R[u_0] - 4\eta\eta_\tau \phi_1(z) - 4\eta^3 \xi_\tau \phi_2(z) \equiv F(z), \tag{10}$$

where

$$\hat{L} = \frac{d^3}{dz^3} + (12\operatorname{sech}^2 z - 4)\frac{d}{dz} - 24\operatorname{sech}^2 z \tanh z, \tag{11}$$

$$\phi_1(z) = (1 - z\tanh z)\operatorname{sech}^2 z,$$

$$\phi_2(z) = \operatorname{sech}^2 z \tanh z. \tag{12}$$

Eigenfunctions

Eigenfunctions:

$$\hat{L}\phi = \lambda\phi, \qquad \lambda = -ik(k^2+4)$$
$$\hat{L}^\dagger\psi = \lambda'\psi, \qquad \lambda' = ik(k^2+4) \tag{13}$$

Continuous States

$$\phi(z,k) = \frac{k(k^2+4)+4i(k^2+2)\tanh z - 8k\tanh^2 z - 8i\tanh^3 z}{\sqrt{2\pi}k(k^2+4)}e^{ikz}$$
$$\psi(z,k) = \frac{k^2-4ik\tanh z - 4\tanh^2 z}{\sqrt{2\pi}(k^2+4)}e^{-ikz}. \tag{14}$$

Bound (discrete) states

$$\phi_1(z) = (1 - z\tanh z)\,\mathrm{sech}^2 z, \qquad \phi_2(z) = \tanh z\,\mathrm{sech}^2 z,$$
$$\psi_1(z) = \mathrm{sech}^2 z, \qquad \psi_2(z) = \tanh z + z\,\mathrm{sech}^2 z. \tag{15}$$

Completeness Relations and Orthogonality

Completeness

$$P\int_{-\infty}^{\infty} \phi(z,k)\psi(z',k)\,dk + \sum_{j=1}^{2} \phi_j(z)\psi_j(z') = \delta(z-z'), \qquad (16)$$

Orthogonality

$$\int_{-\infty}^{\infty} \phi(z,k)\psi(z,k')\,dz = \delta(k-k'),$$
$$\int_{-\infty}^{\infty} \phi_j(z)\psi_\ell(z)\,dz = \delta_{j,\ell}, \qquad j,k = 1,2 \qquad (17)$$

Perturbation Expansion for $u_{1t} + \eta^3 \hat{L} u_1 = F$

Expand u_1 and F :

$$
\begin{aligned}
u_1(z,t) &= P\int_{-\infty}^{\infty} U(t,k)\phi(z,k)\,dk + \sum_{j=1}^{2} U_j(t)\phi_j(z), \\
F(z) &= P\int_{-\infty}^{\infty} f(k)\phi(z,k)\,dk + \sum_{j=1}^{2} f_j\phi_j(z), \qquad (18)
\end{aligned}
$$

where

$$
f(k) = \int_{-\infty}^{\infty} F(z)\psi(z,k)\,dz, \qquad f_j = \int_{-\infty}^{\infty} F(z)\psi_j(z)\,dz, \qquad j = 1,2
$$

Then, solve for expansion coefficients

$$
\boxed{
\begin{aligned}
U_t + \eta^3\lambda(k)U, &= f(k), \quad U(0,k) = 0, \\
U_{1t} &= f_1, \quad U_1(0) = 0, \\
U_{2t} - 8\eta^3 U_1 &= f_2, \quad U_2(0) = 0. \qquad (19)
\end{aligned}
}
$$

Damped KdV: $u_t + 6uu_x + u_{xxx} = -\gamma u$,

First Order Equation:

$$u_{1t} + \eta^3 \hat{L} u_1 = -2\eta^2\gamma \operatorname{sech}^2 z - 4\eta\eta_\tau\phi_1(z) - 4\eta^3\xi_\tau\phi_2(z), \quad (20)$$

Expand in Basis

$$u_1(z) = P\int_{-\infty}^{\infty} U(k)\phi(z,k)\,dk + U_1\phi_1(z) + U_2\phi_2(z) \quad (21)$$

Solve for coefficients

$$\begin{aligned} U_t + \eta^3\lambda(k)U, &= \int_{-\infty}^{\infty} F(z)\psi(z,k)\,dx = \frac{\sqrt{2\pi}}{3}\frac{\gamma\eta^2 k}{\sinh\frac{\pi k}{2}}, \\ U_{1t} &= \int_{-\infty}^{\infty} F(z)\operatorname{sech}^2 z\,dz = -\frac{8}{3}\gamma\eta^2 - 4\eta\eta_\tau, \\ U_{2t} - 8\eta^3 U_1 &= \int_{-\infty}^{\infty} F(z)[\tanh z + z\operatorname{sech}^2 z]\,dz = -4\eta^3\xi_\tau. \end{aligned}$$

Damped KdV - Results

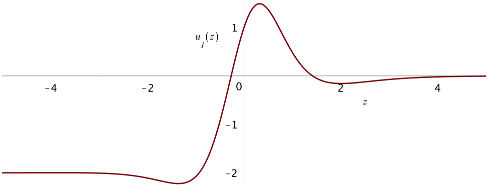
Impose secularity conditions, $f_1 = 0$, $f_2 = 0$,

$$\eta_\tau = -\frac{2}{3}\gamma\eta \Rightarrow \eta = \eta_0 e^{-2\gamma\tau/3}$$

$$\xi_\tau = -\frac{\gamma}{3\eta} \Rightarrow \xi = \xi_0 - \frac{1}{2} e^{2\gamma\tau/3}$$

First order solution develops a shelf.

$$u_1(z) = \frac{\gamma}{6\eta}\left[-1 + \tanh z + 2(1 - z\tanh z)\operatorname{sech}^2 z + z(2 - z\tanh z)\operatorname{sech}^2 z\right].$$



Other Applications: Stochastic KdV, Numerical truncation error.

Modified Integral mKdV Equation - with M. Saravanan

Study [?] of EM wave propagation in ferrites leads to:

$$u_\tau + \frac{3}{2}(A - \alpha f)u^2 u_\zeta + (A - \alpha f)u_{\zeta\zeta\zeta} = \alpha P, \qquad (22)$$

$$P = f_{\zeta\zeta}u_\zeta + 3f_\zeta u_{\zeta\zeta} - f_{\zeta\zeta\zeta}u + f_\zeta u^3 + \frac{1}{2}u_\zeta \int_{-\infty}^{\zeta} f_{\zeta'} u^2 \, d\zeta'. \qquad (23)$$

Setting $\alpha = 0$, one obtains

$$u_\tau + \frac{3}{2}Au^2 u_\zeta + Au_{\zeta\zeta\zeta} = 0. \qquad (24)$$

Letting $t = A\tau, x = \zeta, u = 2v$, we obtain the mKdV equation,

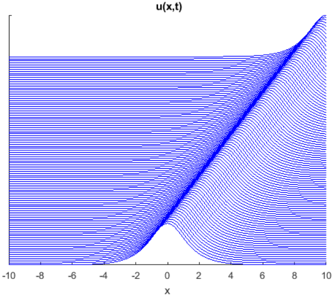
$$\boxed{v_t + 6v^2 v_x + v_{xxx} = 0. \qquad (25)}$$

Modified KdV Equation, $v_t + 6v^2 v_x + v_{xxx} = 0$

one soliton solution of the mKdV Equation,

$$v(x,t) = \eta \operatorname{sech} \eta (x - \eta^2 t - x_0),$$

with amplitude η and speed η^2.



The unperturbed solution with $\eta = 1$ for $t \in [0,5]$.

Perturbed mKdV Equation, $v_t + 6v^2 v_x + v_{xxx} = \epsilon F[u]$.

We seek a solution close to the mKdV soliton solution,

$$v_0(z) = \eta \operatorname{sech} z, \quad \text{where} \quad z = \eta(x - \xi) \quad \text{and} \quad \xi_t = \eta^2.$$

1) Introduce **multiple time scales**,

$$\partial_t = \partial_{t_1} + \epsilon \partial_{t_2}. \tag{26}$$

2) We **expand** $v(x,t)$ as

$$v(x,t) = v_0(x, t_1, t_2) + \epsilon v_1(x, t_1, t_2) + \dots, \tag{27}$$

assuming $\eta = \eta(t_2)$ and $\xi = \xi(t_1, t_2)$.
3) Obtain

$$\begin{aligned} v_{0t_1} + 6{v_0}^2 v_{0x} + v_{0xxx} &= 0, & (28)\\ v_{1t_1} + (6{v_0}^2 v_1)_x + v_{1xxx} &= F[v_0] - v_{0t_2}. & (29) \end{aligned}$$

Eigenvalue Problem for $\hat{L} = \frac{d^3}{dz^3} + \frac{d}{dz}(6\,\text{sech}^2 z - 1)$

Eigenvalues

: $\lambda = \lambda' = (ik)^3 - ik = -ik(k^2+1)$. The **continuous eigenfunctions** are given by

$$\phi(z,k) = C\left(-1 - k^2 - 2ik\tanh z + 2\tanh^2 z\right)e^{ikz}, \quad (30)$$

$$\psi(z,k) = C'\left(1 - k^2 - 2ik\tanh z\right)e^{ikz}, \quad (31)$$

where $2\pi(1+k^2)^2 C\,\bar{C}' = 1$ and $\bar{\psi}(z,k) = \psi(z,-k)$.
The **discrete states** are

$$\psi_1(z) = \text{sech}\, z, \qquad \psi_2(z) = z\,\text{sech}\, z, \quad (32)$$

$$\phi_1(z) = (1 - z\tanh z)\,\text{sech}\, z, \qquad \phi_2(z) = \text{sech}\, z\tanh z. \quad (33)$$

$$\hat{L}\phi_1 = -2\phi_2, \qquad \hat{L}\phi_2 = 0,$$
$$\hat{L}^\dagger\psi_1 = 0, \qquad \hat{L}^\dagger\psi_2 = 2\psi_1. \quad (34)$$

First Order Solution

$v_1(x, t_1, t_2)$ satisfies a **linearized mKdV equation**:

$$v_{1\,t_1} + (6{v_0}^2 v_1)_x + v_{1\,xxx} = F[v_0] - v_{0\,t_2}.$$

Since $v_0(x, t_1) = \eta \operatorname{sech} z$ with $z = \eta(x - \xi)$ and $\xi_{t_1} = \eta^2$,

$$\begin{aligned} v_{0\,t_2} &= \eta_{t_2}(\operatorname{sech} z - z \operatorname{sech} z \tanh z) + \eta^2 \xi_{t_2} \operatorname{sech} z \tanh z \\ &\equiv \eta_{t_2}\phi_1(z) + \eta^2 \xi_{t_2}\phi_2(z), \end{aligned} \tag{35}$$

we have

$$v_{1\,t_1} + \eta^3 \hat{L} v_1 = F[v_0] - \eta_{t_2}\phi_1(z) - \eta^2 \xi_{t_2}\phi_2(z) \equiv \mathcal{F}(z), \tag{36}$$

where

$$\hat{L} = \frac{d^3}{dz^3} + \frac{d}{dz}(6 \operatorname{sech}^2 z - 1).$$

Solution of the Eigenvalue Problem for $\hat{L}$

Goal: Express v_1 as a **perturbation expansion** over the eigenstates of $\hat{L}$,

$$v_1(z,t) = \int_{-\infty}^{\infty} U(t,k)\phi(z,k)\,dk + \sum_{j=1}^{2} U_j(t)\phi_j(z). \qquad (37)$$

Require **orthogonality conditions**

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(z,k)\bar{\psi}(z,k')\,dz &= \delta(k-k'), \\ \int_{-\infty}^{\infty} \phi_j(z)\bar{\psi}_\ell(z)\,dz &= \delta_{j,\ell}, \qquad j,k = 1,2, \end{aligned} \qquad (38)$$

and a **completeness relation**,

$$P\int_{-\infty}^{\infty} \phi(z,k)\bar{\psi}(z',k)\,dk + \sum_{j=1}^{2} \phi_j(z)\bar{\psi}_j(z') = \delta(z-z').$$

Solving for Coefficients

Expanding $\mathcal{F}(z)$ in the basis of eigenfunctions,

$$\mathcal{F}(z) = \int_{-\infty}^{\infty} \hat{f}(t_1, k)\phi(z, k)\, dk + \sum_{j=1}^{2} f_j(t_1)\phi_j(z), \qquad (39)$$

we have the set of equations

$$\boxed{\begin{aligned} U_{t_1}(t_1, k) - ik(1 + k^2)\eta^3 U(t_1, k) &= \hat{f}(t_1, k), \\ U_1'(t_1) &= f_1(t_1), \\ U_2'(t_1) - 2\eta^3 U_1(t_1) &= f_2(t_1). \qquad (40) \end{aligned}}$$

Solve subject to $U(0, k) = 0$, $U_1(0) = 0$, $U_2(0) = 0$. Then,

$$\hat{f}(t_1, k) = \int_{-\infty}^{\infty} \mathcal{F}(z)\bar{\psi}(z, k)\, dz \equiv \langle \mathcal{F}, \bar{\psi} \rangle, \qquad (41)$$

$$f_j(t_1) = \int_{-\infty}^{\infty} \mathcal{F}(z)\psi_j(z)\, dz \equiv \langle \mathcal{F}, \psi_j \rangle, \quad j = 1, 2. \qquad (42)$$

Example: $f(\zeta) = \operatorname{sech} z,\ z = \eta(\zeta - \xi)$.

We want to solve

$$u_{1\,t_1} + \eta^3 A\hat{L}u_1 = F_1(z) + P[u_0]$$

where

$$\begin{aligned} F_1(z) &= -2\eta_{t_2}\phi_1(z) - 2\eta^2\xi_{t_2}\phi_2(z) - 2\eta^4 \operatorname{sech}^2 z \tanh z \qquad (43) \\ P[u_0] &= 6\eta^3 f_\zeta \operatorname{sech} z - 2\eta^2 f_{\zeta\zeta} \operatorname{sech} z \tanh z - 2\eta f_{\zeta\zeta\zeta} \operatorname{sech} z \\ &\quad -4\eta^3 f_\zeta \operatorname{sech}^3 z - 4\eta^4 \operatorname{sech} z \tanh z \int_{-\infty}^{\zeta} f_{\zeta'} \operatorname{sech}^2 z'\, d\zeta' \\ P[u_0] &= -6\eta^4 \operatorname{sech}^2 z \tanh z - \frac{16}{3}\eta^4 \operatorname{sech}^4 z \tanh z. \qquad (44) \end{aligned}$$

Example - Expansion Coefficients

The inner products with the discrete states, ψ_1 and ψ_2, are

$$\begin{aligned} f_1 &= \langle F_1 + P[u_0], \psi_1 \rangle = -2\eta_{t_2}. \\ f_2 &= \langle F_1 + P[u_0], \psi_2 \rangle = -2\eta^2 \xi_{t_2} - \frac{26}{15}\pi\eta^4. \end{aligned}$$

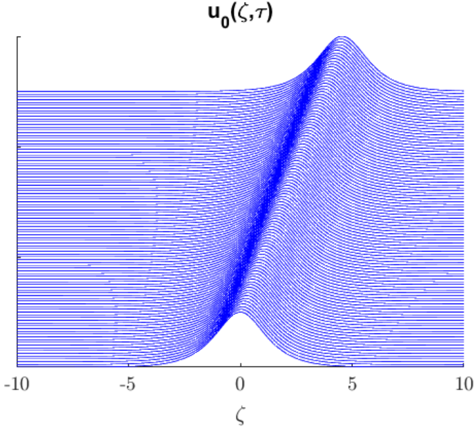
Leads to growth in time unless $f_j = 0$, so

$$\eta_{t_2} = 0, \quad \xi_{t_2} = -\frac{13}{15}\pi\eta^2. \tag{45}$$

Continuous inner products give $\hat{f}(k) = \langle P[u_0], \bar{\psi} \rangle$

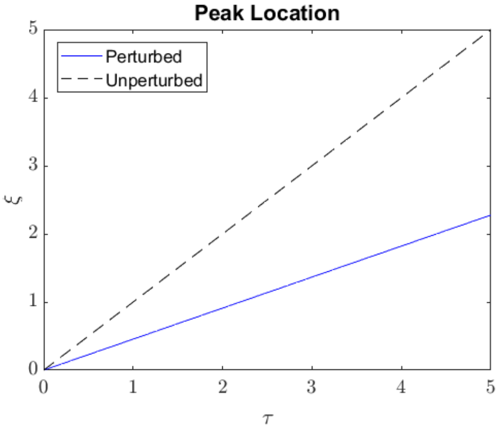
$$\hat{f}(k) = -\frac{\sqrt{2\pi}}{15} \frac{i\eta^4 k^2 (k^2 + 14)}{\sinh \frac{\pi k}{2}}.$$

The Zeroth Order Solution



The zeroth order solution with $\eta = 1$, $A = 1$, and $\alpha = 0.2$ for $\tau \in [0, 10]$.

Location of the Zeroth Order Solution



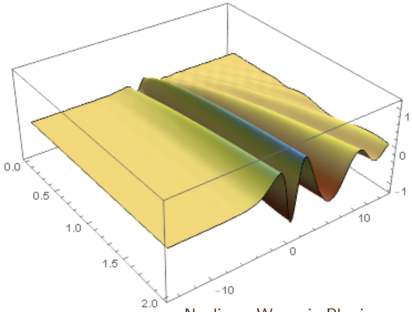
The location of the peak for the zeroth order solution (blue) with $\eta = 1$, $A = 1$, and $\alpha = 0.2$ for $t \in [0, 5]$ as compared to the unperturbed solution (black).

First Order Correction, $u(\zeta,\tau) \approx \eta \mathrm{sech} \eta z + \alpha u_1(\zeta,\tau)$.

$$u_1(z,t_1) = \frac{\eta}{15A}\int_{-\infty}^{\infty}\left(\frac{k(k^2+14)}{(1+k^2)^2 \sinh\frac{\pi k}{2}}\right)\frac{\phi(z,k)}{C}\left(1-e^{-ik(k^2+1)\eta^3 A t_1}\right)dk,$$

where

$$\frac{\phi(z,k)}{C} = \left(-1-k^2-2ik\tanh z + 2\tanh^2 z\right)e^{ikz}.$$



Optical Solitons

- 1973 Hasegawa, Tappert predicted optical solitons in communications
- 1987, First experimental observation in an optical fiber.
- 1988, Mollenauer, et al. transmitted pulses 4,000 km.
- 1991, Bell Labs - transmitted solitons error-free, 2.5 Gb, >14,000 km.
- 1998, Georges, et al. - data transmission of 1 Tb/s (10^{12} bits of information per sec).
- 2001, Algety Telecom deployed submarine telecom equipment in Europe, carrying real information using John Scott Russell's solitary wave.

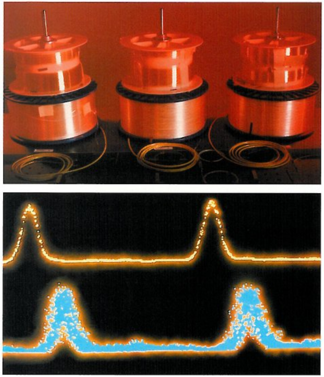


Figure 10. Fiber-optic communications could be enhanced by encoding information in solitons. Investigators at AT&T Bell Laboratories have been experimenting with soliton propagation since they first transmitted one through an optical fiber in 1980. Here the oscilloscope trace of a series of solitons is shown before *(yellow trace)* and after *(blue trace)* traveling 10,000 kilometers in an optical fiber. The pulses show little tendency to dispersion. The transmission rate was five billion bits per second, which would be equivalent to about 100,000 digitized voice channels. (Photograph courtesy of AT&T Bell Laboratories.)

NLS in optics

- Spatial or temporal solitons
- Spatial - balance between diffraction and refraction

$$\nabla^2 \mathrm{E} - \frac{1}{c^2}\frac{\partial^2 \mathrm{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\frac{\partial^2 \mathrm{P}}{\partial t^2}$$

$$\mathrm{E}(\mathrm{r}, t) = \frac{1}{2}\hat{x}\left(A(\mathrm{r})e^{i\beta_0 z - i\omega t} + c.c.\right)$$

$$\mathrm{P} = \mathrm{P}_L + \mathrm{P}_{NL} = \mathrm{P}_L + \alpha A|A|^2 e^{i\beta_0 z}$$

$$u(x, y, z) \propto A(x, y, z)$$

$$iu_z + \frac{1}{2}(u_{xx} + u_{yy}) \pm u|u|^2 = 0,$$

$$u(x, z) = a \,\mathrm{sech}\, ax \exp(ia^2 z/2).$$

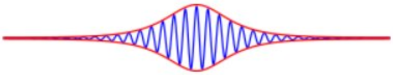
NLS Equation

The nonlinear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \psi|\psi|^2.$$

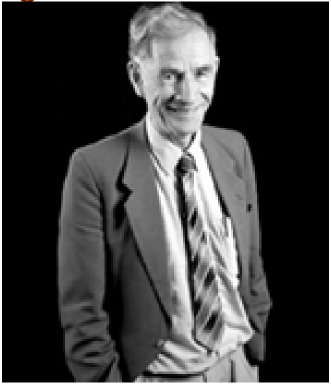
For water waves, $\eta(x,t) = a(x,t)\cos(kx - \omega t - \theta)$, where $\psi = ae^{i\theta}$, or $a = |\psi|$.

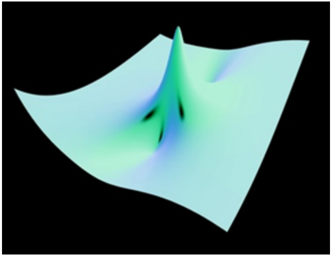
$$\psi(x,t) = 2\beta e^{2\alpha(x+4\alpha t-4(\alpha^2+\beta^2))}\,\text{sech}(2\beta(x+4\alpha t+\delta))$$



Modulational wave solution of NLS.

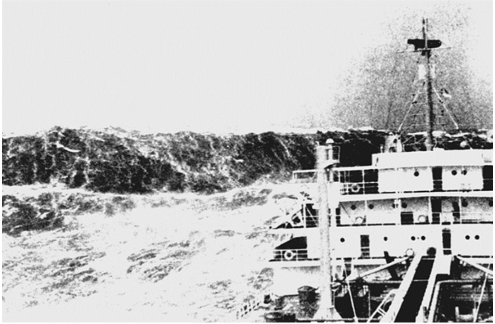
Peregrine Soliton





- Howell Peregrine (1938-2007)
- 1983 Peregrine predicted spatio-temporal evolution of an NLS soliton
- 20 years later used as prototypical example of rogue waves in water and in optics.

Bay of Biscay, France - 1940



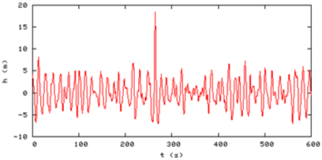
Merchant ship laboring in heavy seas as a huge wave looms astern. Huge waves are common near the 100-fathom line in the Bay of Biscay. Published in Fall 1993 issue of Mariner's Weather Log. http://en.wikipedia.org/wiki/File:Wea00800,1.jpg

Bering Sea - 1979

Ship Discoverer gets pounded by monster wave in the Bering Sea.

Draupner Wave Jan 1, 1995

- Oil platform in the central North Sea
- Minor damage
- Read by a laser sensor.
- During wave heights of 12 m (39ft),
  - Freak wave - max height of 25.6 m (84ft)
  - (peak elevation was 18.5 m (61ft)).



Oil platform and time series.

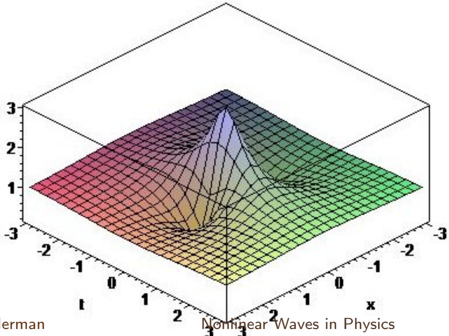
Rational Solutions

NLS Equation,

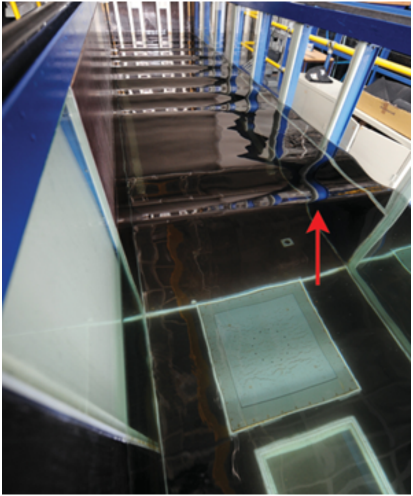
$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \psi|\psi|^2 = 0.$$

Peregrine solution,

$$\psi(x,t) = \left(1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right)e^{it}$$



Rogue Waves 2010



Home / News / June 18th, 2011; Vol.179 #13 / Article

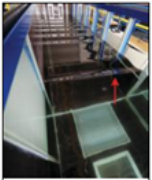
Rogue waves captured

Re-creating monster swells in a tank helps explain their origin

By Devin Powell
June 18th, 2011; Vol.179 #13 (p. 12)

Text Size

Freak waves that swallow ships whole have been re-created in a tank of water. Though these tiny terrors are only centimeters high, a devilishly difficult mathematical equation describing their shape may help to explain the origins of massive rogue waves at sea..



ENLARGE

Wave gauges in a water tank spot the peak of a tiny rogue wave.
Amin Chabchoub

Sailors have long swapped stories about walls of water leaping up in the open ocean — even in calm water — without warning or obvious cause. But for centuries, rogue waves were little more than talk; no one had ever measured one with scientific instruments.

Then on New Year's Eve of 1995, a laser on an oil rig off Norway's coast recorded one of these rare events: a wave 26 meters from bottom to top, flanked by deep troughs on either side.

This wave and others measured since look like breather waves, says Amin Chabchoub, a mathematician at the Hamburg University of Technology in Germany. A breather wave...

The Research Continues

2784 A. Ankiewicz et al. / Physics Letters A 375 (2011) 2782–2785

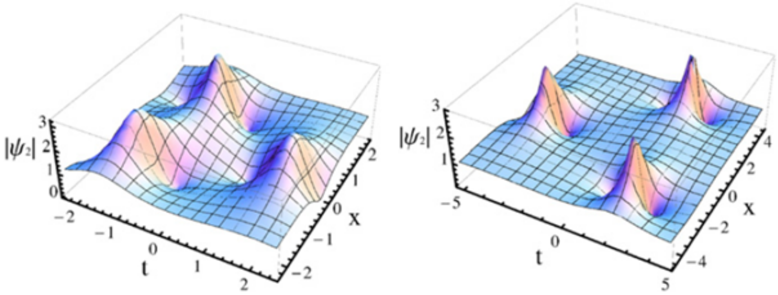


Fig. 3. Rogue wave triplets. Parameters (a) $\gamma = 20$ and $\beta = 40$; (b) $\gamma = 100$ and $\beta = -400$.

Mathematics Research

RESEARCH Open Access

Darboux transformation of the general Hirota equation: multisoliton solutions, breather solutions, and rogue wave solutions

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Abstract

In this paper, we investigate the exact solutions and conservation laws of a general Hirota equation. Firstly, the *N*-fold Darboux transformation of this equation is proposed. Then by choosing three kinds of seed solutions, the multisoliton solutions, breather solutions, and rogue wave solutions of the general Hirota equation are obtained based on the Darboux transformation. Finally, the conservation laws of this equation are derived by using its linear spectral problem. The results in this paper may be useful in the study of ultrashort optical solitons in optical fibers.

Keywords: Darboux transformation; multisoliton solutions; breather solutions; rogue wave solutions; optical fibers

Recent Research

Physics

VIEWPOINT

A Unifying Framework for Describing Rogue Waves

A theory for rogue waves based on instantons—a mathematical concept developed in quantum chromodynamics—has been successfully tested in controlled laboratory experiments.

by Stefano Trillo* and Amin Chabchoub†

In oceanography, rogue or freak waves (Fig. 1) are defined as waves that are abnormally large compared with the average waves for a given sea state [1]. With heights exceeding 30 meters, these statistically rare waves pose severe threats even to the largest ships. Unlike tsunamis caused by earthquakes, rogue waves are, so far, unpredictable and localized in space and time—they are often said "to appear from nowhere and disappear without a trace" [2]. Understanding the mechanisms of their formation would be essential to develop tools for predicting their occurrence. However, such mechanisms remain disputed, with two competing schools of thought: one arguing that their origin is due to linear interference [3], the other saying that nonlinear phenomena are the key [4]. Now, Giovanni Dematteis from the Polytechnical University of Turin in Italy and coworkers have shown that a theory for rogue waves based on mathematical entities known as instantons successfully describes controlled experiments performed in a large-scale water tank [5]. The instanton theory, which can tackle regimes of propagation ranging from nearly linear to fully nonlinear, could pave the way for a universal description of rogue waves that is applicable to many real-world situations.

Figure 1: Photograph of a rogue wave in the western North Atlantic. (M. A. Donelan and A. K. Magnusson, Sci. Rep. 7, 44124 (2017))

The reliable recording of rogue waves, starting from the well-known Draupner wave observed in 1995 around a North Sea oil platform [6], brought these waves out of a mythical status, turning them into a fascinating area of rigorous experimental and theoretical research. Work performed in the last two decades suggested that rogue waves are ubiquitous phenomena with clear manifestations in optics, plasmas, economics, and climatology [7]. Water waves and optics, in particular, have offered the possibility to investigate rogue waves in the lab by performing experiments that are controllable but also able to simulate the random nature of wave superposition that is at the basis of rogue wave formation.

In spite of significant theoretical, numerical, and experimental progress, there is no consensus on how rogue waves form. Researchers in the linear camp attribute rogue wave events to constructive interference between waves, which can accidentally pile up at some location. Conversely, supporters of a nonlinear origin argue that rogue waves could be produced by mechanisms that amplify and focus long-wavelength fluctuations [8, 9]. These nonlinear effects can lead to so-called Peregrine solitons, or breathers, which are waves characterized by an isolated high peak that first grows and then dies out [9]. Some features of Peregrine solitons are consistent with the properties of rogue waves, making them an attractive hypothesis for explaining their formation.

Summary

- History of integrable PDEs (KdV, NLS, etc)
  - Solitons, Kinks, Breathers, Loop Solitons
  - Rational solutions - rogue waves
- Solution Techniques
  - Inverse Scattering
  - Lie Symmetries
  - New Solution Methods
  - Darboux Transformations
  - Perturbation Theory
- Rogue waves exist! - 1995 Draupner data
- Active area of rogue wave research
  - Analytical - new methods of solution generation
  - Numerical - robustness of solutions
  - Experimental [optics, hydrodynamics, plasmas]

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