The Joy of Mathematical Puzzles and Games

Part I. The Logic Puzzles of Raymond M. Smullyan

Fall 2024 - R. L. Herman

Overview

Introduction
Brief History
Famous Puzzlers
Puzzles of Raymond M. Smullyan
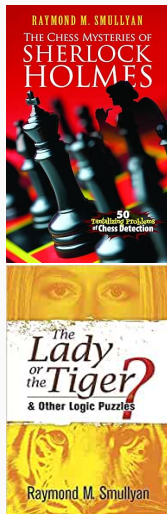   Knights and Knaves
    The Lady or the Tiger
    The Hardest Logic Puzzle Ever
    The Monte Carlo Lock Puzzle
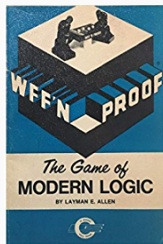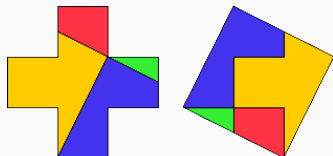   *The Chess Mysteries of Sherlock Holmes*

Puzzles and Games

- Logic puzzles,
- Number games,
- Geometrical puzzles,
- Games of chance
- Network problems, and
- Combinatorial problems.

How is a raven

like a writing desk?

History of Games and Puzzles

- Rhind Papyrus - 1650 BCE

    Long scroll $18' \times 13''$.
    Purchased 1858.
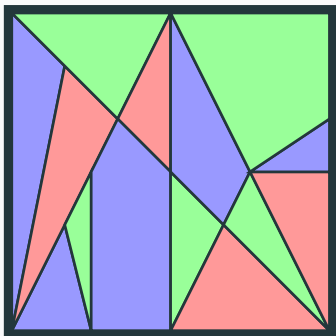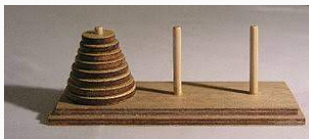
- Problem 79
- Fibonacci's *Liber Abaci*

Seven houses contain seven cats.
Each cat kills seven mice.
Each mouse had eaten seven ears of grain.
Each ear of grain would have produced seven hekats of wheat.
What is the total of all of these?

houses	7
cats	49
mice	343
spelt	2401
hekat	16807
Total	19607

Historical Puzzles

- 1256, Ibn Kallikan, total grains of wheat placed on a chess board: $18, 446, 744, 073, 709, 551, 615$.

- Archimedes 287-212 BCE

    Cattle Problem
    Ostomachion
    (*loculus Archimedius*)

- Leonardo of Pisa 1100 (Fibonacci Sequence - Rabbits)

- Tower of Hanoi, Édouard Lucas, 1883.

- Magic squares
  2500 BCE, Chinese, *Lo-shu*.
  Dürer's *Melancolia*, 1514 - next.

- See more here.
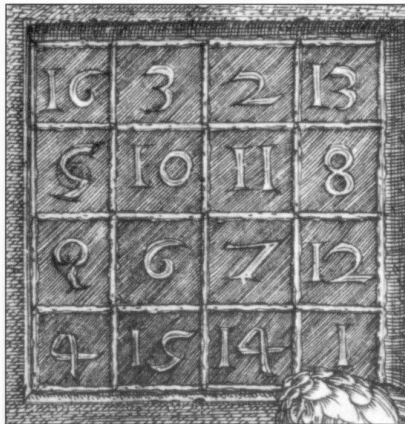
Charles L. Dodgson (1832-1898)

- Mathematical logician, Author.
- Christ Church College Oxford.
    - First Class honours in math.
    - Mathematics lecturer.
- Works - Euclid, Determinants, Logic, Elections.
- Photography
- Alice Liddell, daughter of Dean Henry George Liddell.
- Charles Lutwidge in Latin: Carolus Lodovicus, Anglicise and reverse order - pen name Lewis Carroll.
- Published puzzles and stories.

Sam Loyd (1841-1911) - "America's Greatest Puzzlist"

- 1st chess problem published: NY Saturday Courier 4/14/1855.
- Problem Ed., *Chess Monthly*, 16.
- *Chess Strategy*, 1878.
- Worked with Dudeney, scandal.
- Over 10,000 puzzles, 15 puzzle.
- Popularized tangrams.
- Sam Jr. carried on.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

$\Longrightarrow$ \$1000

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	
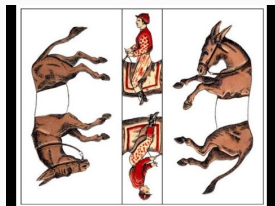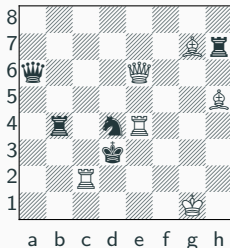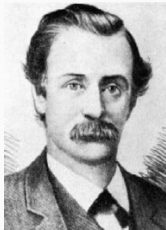
White to play and mate in four.

Henry Dudeney (1857-1930)

- Clerk in the Civil Service
- No college, Family of educators.
- Started with chess, puzzles.
- Pseudonym 'Sphinx'.
- 1893, corresponded with Loyd.
- Contributed to the Strand Magazine for over 30 years,
- British Chess Problem Society, Founding member, 1918.
- Puzzle collections.
- Haberdasher's problem.
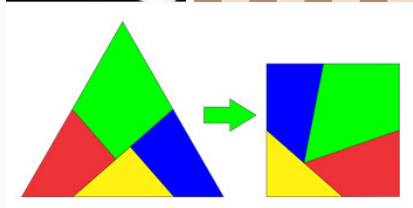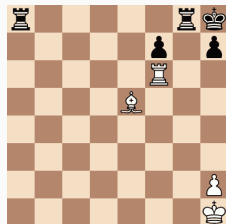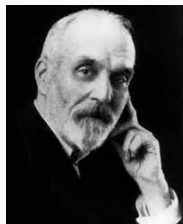
Future Talk on Dissection Problems

With three cuts, dissect an equilateral triangle into a square.
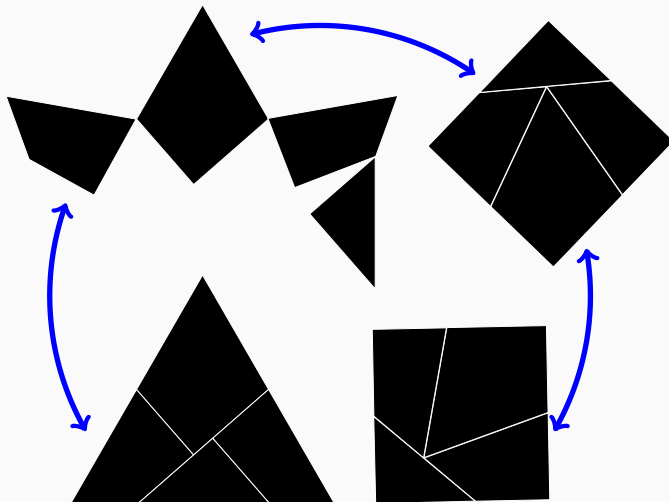
Raymond Smullyan (1919-2017)

- Mathematician, magician, concert pianist, logician and philosopher.
- From Far Rockaway, NYC.
- Composed 1st chess puzzle at 16.
- Retrograde analysis in chess.
- Unusal education.
  - Piano lessons in CA.
  - Studied Logic, Chicago.
  - Taught -Dartmouth, 1954-1956, B.S. from Univ. of Chicago, 1955.
  - Ph.D. Princeton, 1959, advisor: Alonzo Church.
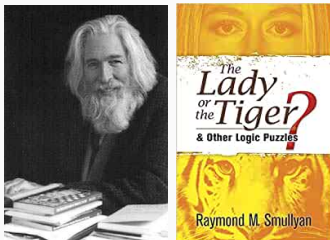
Find the missing piece in the blue space.

Figure 1: https://www.youtube.com/watch?v=E27v83WWiGo (12:21)

Liar's Paradox: "This statement is false."

The key issue is **self-reference**, i.e., it refers to itself leading to a loop of contradictory reasoning.

In formal systems of mathematics or logic, self-reference can introduce similar challenges, like Russell's Paradox.

- Discovered by Bertrand Russell in 1901.

- Exposed a fundamental problem in set theory.
  The paradox: Is the "set of all sets that do not contain themselves" itself a member of this set?

  - If it is, it leads to a contradiction (it must contain itself).
  - If it isn't, it leads to a contradiction (it must not contain itself).

- Led to changes in formal systems and axiomatic set theory

- Hilbert's Program sought a complete and consistent foundation for all of mathematics.

Gödel's Incompleteness Theorems

- Gödel responded Hilbert's efforts, in 1931.
- Results revealed limitations of formal systems (in arithmetic).
- In any formal system, there are statements that:
    - Cannot be proven true or false within the system (First Theorem).
    - If the system is consistent (free of contradictions), it cannot prove its own consistency (Second Theorem).
- Gödel did this by constructing a self-referential statement within the formal system — similar to the Liar Paradox.

Smullyan used accessible logical puzzles to introduce the notions of self-reference, paradox, and formal systems — all central ideas in Gödel's work.

Let the Games Begin

- Ready to try some puzzles?
- From Smullyan's *The Lady or the Tiger?*

    Knights and Knaves
    The Lady or the Tiger
    The Mystery of the Monte Carlo Lock

- The Hardest Logic Puzzle Ever

Knights and Knaves

A special island is inhabited only by knights and knaves. Knights always tell the truth and knaves always lie.
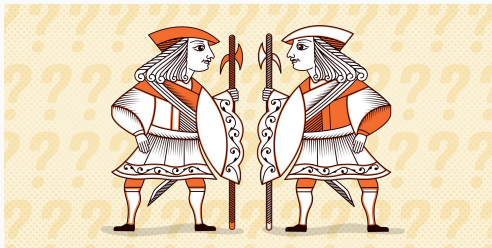
You meet two inhabitants: A and B. A tells you that B is a knave. B says, "Neither A nor I are knaves." Who is a knight and who is a knave? ▶

For more problems: Ch 3 *What Is the Name of This Book?*

A King heard a story where a prisoner had to choose between two doors: behind one there was a princess, behind the other a tiger. If the prisoner chose the princess, he could marry her; if he chose the tiger, he would probably be eaten.
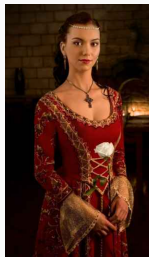
The Challenge

The King liked the idea but didn't want to leave things to chance. So, he decided to post signs on the doors giving clues as to what was behind the doors. Also, it would be possible that there are princesses behind both doors or tigers behind both doors. A clever prisoner who can reason logically would be able to save his own life.

In this room there is a lady, and in the other room there is a tiger.

In one of these rooms there is a lady, and in one of these rooms there is a tiger.

Wait! - What's your answer?

In this room there is a lady, and in the other room there is a tiger.

In one of these rooms there is a lady, and in one of these rooms there is a tiger.

Did you really wait?

Room 1 In this room there is a lady and in the other is a lion.

Room 2 In one of these rooms is a lady and in one there is a tiger.

One of the doors is true. If Door 1 is true, then Door 2 is also true.

Therefore, Door 1 is false and Door 2 is true.

Conclusion: A tiger is behind Door 1 and a Lady behind Door 2. Truth Table

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Trial 2 The signs are changed. They are either both true or both false. Which room would you pick?

Door 1

At least one of these rooms contains a lady.

Door 2

A tiger is in the other room.

Three gods A, B, and C are called, in no particular order, True, False, and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes–no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for yes and no are da and ja, in some order. You do not know which word means which.

Figure 2: Some of Smullyan's many books.

The Monte Carlo Lock Puzzle

From Smullyan's *The Lady of the Tiger & Other Logic Puzzles.*

- Inspector Craig is called to a case.
- The safe combination (on one card) was lost.
- Martin Farkus, a mathematician, left clues about code.
- The safe has to be opened by June 1, or it is blown.
- The code has capital letters of any length and may be repeated.
- Entering a code will

  Open the lock.
  Jam the lock, or
  It does nothing (neutral).

Goal Use Farkus' clues to open the lock.

Farkas' Clues

Denoting xy as the concatenation of the codes x and y, then for any letter combinations x and y, there are "special relations" satisfying the properties below.

Property Q: For any combination x, the combination QxQ is specially related to x.

Property L: If x is specially related to y, then Lx is specially related to Qy.

Property V (the reversal property: If x is specially related to y, then Vx is specially related to the reverse of y.

Property R (the repetition property: If x is specially related to y, then Rx is specially related to yy.

Property Sp: If x is specially related to y, then if x jams the lock, y is neutral, and if x is neutral, then y jams the lock.

Symbolic Summary of Clues

We write $y \rightarrow x$ to mean "x is specially related to y." Then,

- The code contains the letters L, Q, R, V.
- Denote xy as the concatenation of the codes x and y.
- Use $\overleftarrow{y}$ for the reverse order of the code y.

For codes x and y, the following are true:

- Q: $x \rightarrow QxQ$.
- L: If $y \rightarrow x$, then $Qy \rightarrow Lx$,
- V: If $y \rightarrow x$, then $\overleftarrow{y} \rightarrow Vx$,
- R: If $y \rightarrow x$, then $yy \rightarrow Rx$.

Craig surmised that a code x opens the safe if and only if $x \rightarrow x$.[*]
For example: $RVLVQRVLVQ \rightarrow RVLVQRVLVQ$,

[*]From Property Sp, if $x \rightarrow x$, then x both jams the lock and is neutral. This contradiction implies

that x must open the lock. *Math & Stat Club - Smullyan's Logic*   R. L. Herman   Fall 2024   25/30

Raymond Smullyan opens his 1979 book, *The Chess Mysteries of Sherlock Holmes,* with the following chessboard position. He then asks, "Suppose I told you that in the following position no pawn has ever reached the eight square. Would you believe me?"
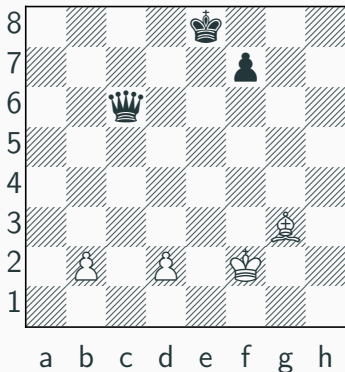
Solutions

Go no further!

Start with the statements (A: "B is a knave." B: "Neither A nor I are knaves.") $A \iff \sim B$ and $B \iff A \wedge B$.

A	B	$\sim B$	$A \iff \sim B$	$A \wedge B$	$B \iff A \wedge B$
T	T	F	F	T	T
T	F	T	T	F	T
F	T	F	T	F	F
F	F	T	F	F	T

Table 1: Truth table.

A	B	$\sim A$	$\sim B$	$A \wedge B$	$\sim (A \wedge B)$	$\sim A \vee \sim B$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 2: de Morgan's Law $\sim (A \wedge B) \iff \sim A \vee \sim B$.
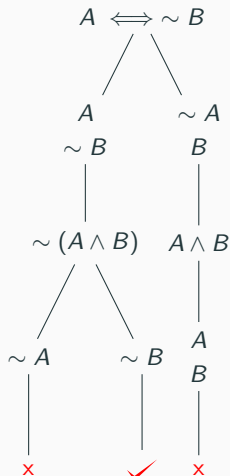
Lady or Lion - a Truth Table.

L_1: Lion behind door 1.

L_2: Lion behind door 2.

Door 1: P_1 and L_2.

P_1: Lady behind door 1.

P_2: Lady behind door 2.

Door 2: P_1 and L_2, or P_2 and L_1.

L_1	L_2	P_1	P_2	$A = P_1 \wedge L_2$	$P_2 \wedge L_1$	$B = A \vee (P_2 \wedge L_1)$	$A \wedge \sim B$	$\sim A \wedge B$
T	T	T	T	T	T	T	F	F
T	T	T	F	T	F	T	F	F
T	T	F	T	F	T	T	F	T
T	T	F	F	F	F	F	F	F
T	F	T	T	F	T	T	F	T
T	F	T	F	F	F	F	F	F
T	F	F	T	F	T	T	F	T
T	F	F	F	F	F	F	F	F
F	T	T	T	T	F	T	F	F
F	T	T	F	T	F	T	F	F
F	T	F	T	F	F	F	F	F
F	T	F	F	F	F	F	F	F
F	F	T	T	F	F	F	F	F
F	F	T	F	F	F	F	F	F
F	F	F	T	F	F	F	F	F
F	F	F	F	F	F	F	F	F

Question: "If I asked you 'Is B Random?'
in your language, would you say 'ja'?"

Response: "ja"

Response: "da"

B is not Random

B might be Random

A is Truth-teller or Liar

A might be Random

C is Random

C might be Random